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QUALITATIVE BEHAVIOR OF THIRD-ORDER DAMPED NONLINEAR DIFFERENTIAL EQUATIONS WITH SEVERAL DELAYS

M. Sathish Kumar[1], G. Veeramalai[2], S. Janaki[3], V. Ganesan[4]

[1] Department of Mathematics, Paavai Engineering College, Namakkal-637018, Tamil Nadu, India

[2] Department of Mathematics, M. Kumarasamy College of Engineering, Karur-639113, Tamil Nadu, India

[3,4] PG and Research Department of Mathematics, Aringar Anna Government Arts College, Namakkal - 637002, Tamil Nadu, India

[1]msksjv@gmail.com, [2]veeramusiri@gmail.com, [3]janakisms@gmail.com

Corresponding Author: **M. Sathish Kumar**

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Abstract

In this article, we examine the oscillation of a class of third-order damped nonlinear differential equations with multiple delays. Using the integral average and generalized Riccati techniques, new necessary criteria for the oscillation of equation solutions are established. The major effect is exemplified by an example.

Keywords: Oscillation; nonlinear differential equations; third-order; delay arguments; damping.

I. Introduction

Differential equations are a modeling technique used in many different domains, and most of them have general solutions that are difficult to come up with. As a result, researchers have been very interested in the qualitative components of differential equations. Asymptotic characteristics and oscillation are two important elements of qualitative research that are still in trend. It is not difficult to discover during problem-solving that the past and present states have an impact on the future state as well. As a result, the problem description should incorporate some time delay in the equation. Readers who are interested in learning more about the theoretical and practical significance of qualitative research on neutral equations should read O. Arino et al. [XIII], G. S. Ladde et al. [III], and J. K. Hale [IV]. Real-world applications cover a range of models that involve oscillation phenomena. In the context of mathematical biology, certain models can describe oscillation and/or delay behaviors using cross-diffusion expressions. For more information on this topic, refer to papers [II, VIII, IX, X].

Extensive research has been conducted on the subject of oscillation in third or higher-order differential equations, resulting in the development of various strategies to establish criteria for oscillation. Several publications, cited as [I, V, VI, VII, XI, XIV, XV, XVI, XVII, XVIII], give interesting developments about the oscillatory characteristics of solutions to differential equations and damped delay differential equations, with or without distributed deviating arguments.

This work is concerned with the oscillatory and asymptotic behaviour of third-order damped nonlinear differential equations with many delays of the following form:

$$\left(m_1(r)([m_2(r)y'(r)]')^{\beta}\right)' + p(r)([m_2(r)y'(r)]')^{\beta} + \sum_{i=1}^{n} q_i(r)f\left(y\big(\theta_i(r)\big)\right) = 0 \qquad \text{(E)}$$

for $r \geq r_0$, where $m_1, m_2, p, q_i \in C_{rd}(I, \mathbb{R}_+), f \in C(\mathbb{R}, \mathbb{R})$ satisfying $yf(y) > 0, \frac{f(y)}{y^{\beta}} \geq L > 0$ for $y \neq 0, \theta_i \in C(\mathbb{R}, \mathbb{R})$ satisfying $\theta_i(r) \leq r, \theta_i'(r) \geq 0$ and $\lim_{r\to\infty} \theta_i(r) = \infty, \beta \geq 1$ is a quotient of two odd positive integers.

A solution of Eq.(E) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise, it is nonoscillatory. Eq.(E) is said to be oscillatory in case all its solutions are oscillatory. In Section 3, we will use a generalized Riccati function and inequality technique to develop some new oscillatory and asymptotic conditions for Eq. (E). In Section 4, we will give some applications for a result.

In the follow-up, we outline the prerequisites that ensure the following results:

(C) then every solution of Eq. (E) is oscillatory or tends to zero.

II. Main Results

While illustrating our primary results, we define a few of the auxiliary functions and lemmas in this section. For the sake of convenience, in the rest of the paper, set

$$\mathcal{A}(r) = m_1(r)([m_2(r)y'(r)]')^{\beta}$$

$$\psi_1(r, m_1) = \int_{m_1}^{r} \frac{\left[\exp\left(-\int_{r_0}^{s} \frac{p(s)}{m_1(s)} ds\right)\right]^{\frac{1}{\beta}}}{m_1^{\frac{1}{\beta}}(s)} ds$$

$$\psi_2(r, m_1) = \int_{m_1}^{r} \frac{\psi_1(s, m_1)}{m_2(s)} ds$$

and we always assume $r_i \in \mathbb{T}, i = 1,2, \ldots, 6$.

Lemma 2.1. Assume that

$$\int_{r_0}^{\infty}\left(\frac{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)}{m_1(s)}\right)^{\frac{1}{\beta}}ds=\infty \tag{1}$$

$$\int_{r_0}^{\infty}\frac{ds}{m_2(s)}=\infty \tag{2}$$

and y is eventually a positive solution of Eq. (E). Then, one of the following two cases holds.

$(C_I) y(r)>0, y'(r)>0, [m_2(r)y'(r)]'>0$

$(C_{II}) y(r)>0, y'(r)<0, [m_2(r)y'(r)]'>0.$

Proof. Let (E) have a positive solution $y(r)$ on $[r_0,\infty)$, say $y(r)>0, y(\theta_i(r))>0$ for $i=1,2\cdots,n$.

$$\begin{aligned}\left[\frac{\mathcal{A}(r)}{\exp\left(-\int_{r_0}^{r}\frac{p(s)}{m_1(s)}ds\right)}\right]' &= \frac{\exp\left(-\int_{r_0}^{r}\frac{p(s)}{m_1(s)}ds\right)\mathcal{A}'(r)-\left(\exp\left(-\int_{r_0}^{r}\frac{p(s)}{m_1(s)}ds\right)\right)'\mathcal{A}(r)}{\exp\left(-\int_{r_0}^{r}\frac{p(s)}{m_1(s)}ds\right)^2}\\ &= \frac{\mathcal{A}'(r)+p(r)\left(\left([m_2(r)y'(r)]'\right)^{\beta}\right)}{\exp\left(-\int_{r_0}^{r}\frac{p(s)}{m_1(s)}ds\right)}\\ &= -\frac{\sum_{i=1}^{n}q_i(r)f\left(y(\theta_i(r))\right)}{\exp\left(-\int_{r_0}^{r}\frac{p(s)}{m_1(s)}ds\right)}<0.\end{aligned} \tag{3}$$

Then $\frac{\mathcal{A}(r)}{\exp\left(-\int_{r_0}^{r}\frac{p(s)}{m_1(s)}ds\right)}$ is strictly decreasing on $[r_1,\infty)$, and together with $m_1(r)>0, \exp\left(-\int_{r_0}^{r}\frac{p(s)}{m_1(s)}ds\right)>0$, we deduce that $[m_2(r)y'(r)]'$ is eventually of on sign. We claim $[m_2(r)y'(r)]'>0$ on $[r_2,\infty)$, where $r_2>r_1$ is sufficiently large. Otherwise, assume there exists a sufficiently large $r_3>r_2$ such that $[m_2(r)y'(r)]'<0$ on $[r_3,\infty)$. Then

$$\begin{aligned}m_2(r)y'(r)-m_2(r_3)y'(r) &= \int_{r_3}^{r}\frac{\left[\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)\right]^{1/\beta}[m_2(s)y'(s)]'}{\left[\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)m_1(s)\right]^{1/\beta}}ds\\ &\le \frac{\mathcal{A}^{\frac{1}{\beta}}(r)}{\left[\exp\left(-\int_{r_0}^{r_3}\frac{p(s)}{m_1(s)}ds\right)\right]^{\frac{1}{\beta}}}\int_{r_3}^{r}\frac{\left[\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)\right]^{\frac{1}{\beta}}}{m_1^{\frac{1}{\beta}}(s)}ds.\end{aligned} \tag{4}$$

By (1), we have $\lim_{r\to\infty}m_2(r)y'(r)=-\infty$, and thus there exists a sufficiently large $r_4\in[r_3,\infty)$ such that $m_2(r)y'(r)<0$ on $[r_4,\infty)$. By the assumption $[m_2(r)y'(r)]'<0$ one can see $m_2(r)y'(r)$ is strictly decreasing on $[r_4,\infty)$, and then

$$y(r)-y(r_4)=\int_{r_4}^{r}\frac{m_2(s)y'(s)}{m_2(s)}ds\le m_2(r_4)y'(r_4)\int_{r_4}^{r}\frac{1}{m_2(s)}ds.$$

Using (2), we have $\lim_{r\to\infty}y(r)=-\infty$, which leads to a contradiction. So $[m_2(r)y'(r)]'<0$ on $[r_2,\infty)$.

Lemma 2.2. Let $y(r)$ satisfies case (C_{II}). Suppose

$$\limsup_{r\to\infty}\int_{r_0}^{r}\left(\frac{1}{m_2(\xi)}\int_{\xi}^{\infty}\left(\frac{\exp\left(\int_{r_0}^{\tau}\frac{-p(s)}{m_1(s)}ds\right)}{m_1(\tau)}\int_{\tau}^{\infty}\frac{\sum_{i=1}^{n}q_i(s)ds}{\exp\left(\int_{r_0}^{s}\frac{-p(s)}{m_1(s)}ds\right)}\right)^{\frac{1}{\beta}}d\tau\right)d\xi=\infty \quad (5)$$

and then $\lim_{r\to\infty}y(r)=0$.

Proof. By Lemma 2.1, we deduce that $y'(r)$ is eventually of one sign. So there exists a sufficiently large $r_5>r_4$ such that either $y'(r)>0$ or $y'(r)<0$ on $[r_5,\infty)$, where r_4 is defined as in Lemma 2.1. If $y'(r)<0$, together with $y(r)$ is eventually a positive solution of Eq. (E), we obtain $\lim_{r\to\infty}y(r)=\alpha\ge 0$ and $\lim_{r\to\infty}m_2(r)y'(r)=\varrho\le 0$.

We claim $\alpha=0$. Otherwise, assume$\alpha>0$. Then $y(r)\ge\alpha$ on$[r_5,\infty)$. Since $\lim_{r\to\infty}\theta_i(r)=\infty$, there exists $r_6>r_5$ such that $\theta_i(r)>r_5$ on $[r_6,\infty)$, and then $y(\theta_i(r))\ge\alpha$ on $[r_6,\infty)$. On the other hand, for $r\in[r_6,\infty)$, an integration for (3) from r to ∞ yields

$$\begin{aligned}
-\frac{\mathcal{A}(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}&=-\lim_{r\to\infty}\frac{\mathcal{A}(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}+\int_{r}^{\infty}\frac{-\sum_{i=1}^{n}q_i(s)f\big(y(\theta_i(s))\big)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)}ds\\
&\le-\lim_{r\to\infty}\frac{\mathcal{A}(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}+\int_{r}^{\infty}\frac{-L\sum_{i=1}^{n}q_i(s)y^{\beta}(\theta_i(s))}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)}ds\\
&\le-L\int_{r}^{\infty}\frac{\sum_{i=1}^{n}q_i(s)y^{\beta}(\theta_i(s))}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)}ds\\
&\le-L\alpha^{\beta}\int_{r}^{\infty}\frac{\sum_{i=1}^{n}q_i(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)}ds
\end{aligned}$$

which is followed by

$$-[m_2(r)y'(r)]'\le-\left[L\alpha^{\beta}\left(\frac{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}{m_1(r)}\int_{r}^{\infty}\frac{\sum_{i=1}^{n}q_i(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)}ds\right)\right]^{\frac{1}{\beta}} \quad (6)$$

Substituting r with τ in (6), an integration for (6) with respect to τ from r to ∞ yields

$$m_2(r)y'(r) = \lim_{r\to\infty} m_2(r)y'(r) - \alpha L^{\frac{1}{\beta}} \int_r^\infty \left(\frac{\exp\left(-\int_{r_0}^{\tau} \frac{p(s)}{m_1(s)} ds\right)}{m_1(\tau)} \right.$$

$$\left. \times \int_\tau^\infty \frac{\sum_{i=1}^n q_i(s)}{\exp\left(-\int_{r_0}^{s} \frac{p(s)}{m_1(s)} ds\right)} ds \right)^{\frac{1}{\beta}} d\tau$$

$$= \varrho - \alpha L^{\frac{1}{\beta}} \int_r^\infty \left(\frac{e_{-\frac{p}{m_1}}^{m_1}(\tau, r_0)}{m_1(\tau)} \int_\tau^\infty \frac{\sum_{i=1}^n q_i(s)}{\exp\left(-\int_{r_0}^{s} \frac{p(s)}{m_1(s)} ds\right)} ds \right)^{\frac{1}{\beta}} d\tau$$

$$\leq -\alpha L^{\frac{1}{\beta}} \int_r^\infty \left(\frac{\exp\left(-\int_{r_0}^{\tau} \frac{p(s)}{m_1(s)} ds\right)}{m_1(\tau)} \int_\tau^\infty \frac{\sum_{i=1}^n q_i(s)}{\exp\left(-\int_{r_0}^{s} \frac{p(s)}{m_1(s)} ds\right)} ds \right)^{\frac{1}{\beta}} d\tau$$

which implies

$$y'(r) \leq -\alpha L^{\frac{1}{\beta}} \frac{1}{m_2(r)} \int_r^\infty \left(\frac{\exp\left(-\int_{r_0}^{\tau} \frac{p(s)}{m_1(s)} ds\right)}{m_1(\tau)} \int_\tau^\infty \frac{\sum_{i=1}^n q_i(s)}{\exp\left(-\int_{r_0}^{s} \frac{p(s)}{m_1(s)} ds\right)} ds \right)^{\frac{1}{\beta}} d\tau \quad (7)$$

Substituting r with ξ in (7), an integration for (7) with respect to ξ from r_6 to r yields

$$y(r) - y(r_6) \leq -\alpha L^{\frac{1}{\beta}} \int_{r_6}^{r} \left[\frac{1}{m_2(\xi)} \int_\xi^\infty \left(\frac{\exp\left(-\int_{r_0}^{\tau} \frac{p(s)}{m_1(s)} ds\right)}{m_1(\tau)} \right. \right.$$

$$\left. \left. \times \int_\tau^\infty \frac{\sum_{i=1}^n q_i(s)}{\exp\left(-\int_{r_0}^{s} \frac{p(s)}{m_1(s)} ds\right)} ds \right)^{\frac{1}{\beta}} d\tau \right] d\xi. \quad (8)$$

By (8) and (5) we have $\lim_{r\to\infty} y(r) = -\infty$, which leads to a contradiction. So we have $\alpha = 0$, and the proof is complete.

Lemma 2.3. Assume (1) and (2) hold. If $y(r)$ is eventually a positive solution of Eq. (E) such that with case (C_I) for $r \geq r_1 \geq r_0$, where r_1 is sufficiently large. Then for $r \in [r_1, \infty)$, we have

$$y'(r) \geq \frac{\psi_1(r, T_3^*)}{m_2(r)} \left(\frac{\mathcal{A}^{\frac{1}{\beta}}(r)}{\left[\exp\left(\int_{r_0}^{r} \frac{p(r)}{m_1(r)} dt\right)\right]^{\frac{1}{\beta}}} \right)$$

and

$$y(r) \geq \psi_2(r, T_3^*) \left(\frac{\mathcal{A}^{\frac{1}{\beta}}(r)}{\left[\exp\left(\int_{r_0}^{r} \frac{p(r)}{m_1(r)} dt\right)\right]^{\frac{1}{\beta}}} \right).$$

Proof. By Lemma 2.1 we have $\frac{\mathcal{A}(r)}{\exp\left(\int_{r_0}^{r} \frac{p(r)}{m_1(r)} dt\right)}$ is strictly decreasing on $[T_3^*, \infty)$. So

$$\begin{aligned} m_2(r)y'(r) &\geq m_2(r)y'(r) - m_2(T_3^*)y'(T_3^*) \\ &= \int_{T_3^*}^{r} \frac{\left[\exp\left(-\int_{r_0}^{s} \frac{p(s)}{m_1(s)} ds\right) m_1(s)\right]^{\frac{1}{\beta}} [m_2(s)y'(s)]'}{\left[\exp\left(-\int_{r_0}^{s} \frac{p(s)}{m_1(s)} ds\right) m_1(s)\right]^{\frac{1}{\beta}}} ds \\ &\geq \frac{\mathcal{A}^{\frac{1}{\beta}}(r)}{\left[\exp\left(\int_{r_0}^{r} \frac{p(r)}{m_1(r)} dt\right)\right]^{\frac{1}{\beta}}} \int_{T_3^*}^{r} \frac{\left[\exp\left(-\int_{r_0}^{s} \frac{p(s)}{m_1(s)} ds\right)\right]^{\frac{1}{\beta}}}{m_1^{\frac{1}{\beta}}(s)} ds \\ &= \psi_1(r, T_3^*) \frac{\mathcal{A}^{\frac{1}{\beta}}(r)}{\left[\exp\left(\int_{r_0}^{r} \frac{p(r)}{m_1(r)} dt\right)\right]^{\frac{1}{\beta}}}, \end{aligned}$$

and then

$$y'(r) \geq \frac{\psi_1(r, T_3^*)}{m_2(r)} \left(\frac{\mathcal{A}^{\frac{1}{\beta}}(r)}{\left[\exp\left(\int_{r_0}^{r} \frac{p(r)}{m_1(r)} dt\right)\right]^{\frac{1}{\beta}}} \right)$$

Furthermore,

$$
\begin{aligned}
y(r) \geq y(r) - y(T_3^*) &= \int_{T_3^*}^{r} y'(s)ds \\
&\geq \int_{T_3^*}^{r} \frac{\psi_1(s,T_3^*)}{m_2(s)} \left(\frac{\mathcal{A}^{\frac{1}{\beta}}(s)}{\left[\exp\left(-\int_{r_0}^{s} \frac{p(s)}{m_1(s)} ds\right)\right]^{\frac{1}{\beta}}} \right) ds \\
&\geq \left(\frac{\mathcal{A}^{\frac{1}{\beta}}(r)}{\left[\exp\left(\int_{r_0}^{r} \frac{p(r)}{m_1(r)} dt\right)\right]^{\frac{1}{\beta}}} \right) \int_{T_3^*}^{r} \frac{\psi_1(s,T_3^*)}{m_2(s)} ds \\
&= \psi_2(r,T_3^*) \left(\frac{\mathcal{A}^{\frac{1}{\beta}}(r)}{\left[\exp\left(\int_{r_0}^{r} \frac{p(r)}{m_1(r)} dt\right)\right]^{\frac{1}{\beta}}} \right)
\end{aligned}
$$

which is the desired result.

Lemma 2.4. If $X > 0$ and $Y > 0$, and $\lambda > 1$, then

$$\lambda X Y^{\lambda-1} - X^{\lambda} \leq (\lambda - 1) Y^{\lambda}.$$

Theorem 2.1. Assume that (1), (2), (5) holds. Assume that there exist two nonnegative functions $\zeta_1(r), \zeta_2(r) \in C^1([r_0,\infty),\mathbb{R})$ with $\zeta_1(r) > 0$, such that

$$
\begin{aligned}
&\lim_{r\to\infty} \sup \int_{T}^{r} \left\{ L \frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s} \frac{p(s)}{m_1(s)} ds\right)} - \zeta_1(s)[m_1(s)\zeta_2(s)]' \right. \\
&+ \frac{\zeta_1(s)\theta'(s)\psi_1(\theta(s),T)[m_1(s)\zeta_2(s)]^{1+\frac{1}{\beta}}}{m_2(\theta(s))} \\
&\left. - \frac{m_2(\theta(s))\zeta_1'(s) + (\beta+1)\zeta_1(s)\theta'(s)\psi_1(\theta(s),T)[m_1(s)\zeta_2(s)]^{\frac{1}{\beta}}}{(\beta+1) m_2^{\frac{1}{\beta+1}}(\theta(s))\zeta_1^{\frac{\beta}{\beta+1}}(s)(\theta'(s))^{\frac{\beta}{\beta+1}}\psi_1^{\frac{\beta}{\beta+1}}(\theta(s),T)} \right\} ds = \infty. \qquad (9)
\end{aligned}
$$

Then conclusion $(\boldsymbol{C})$ holds.

Proof. Assume (E) has a nonoscillatory solution y on I. WLOG, we may assume $y(r) > 0, y(\theta(r)) > 0$ on $[r_1,\infty)$. By Lemmas 2.1 and 2.2, there exists sufficiently large r_2 such that $[m_2(r)y'(r)]' > 0$ on $[r_2,\infty)$, and either $y'(r) > 0$ on $[r_2,\infty)$ or $\lim_{r\to\infty} y(r) = 0$.

Now we assume $y'(r) > 0$ on $[r_2,\infty)$. Since $\lim_{r\to\infty}\theta(r) = \infty$, there exists $r_3 > r_2$ such that $\theta(r) > r_2$ on $[r_3,\infty)$. So $y'(\theta(r)) > 0$ on $[r_3,\infty)$.

Define a generalized Riccati function:

$$\omega(r) = \zeta_1(r)m_1(r)\left[\frac{([m_2(r)y'(r)]')^\beta}{y^\beta(\theta(r))\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)} + \zeta_2(r)\right]$$

Then for $r \in [r_3,\infty)$, we have

$$\omega'(r) = \frac{\zeta_1(r)}{y^\beta(\theta(r))}\left[\frac{\mathcal{A}(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}\right]' + \left[\frac{\zeta_1(r)}{y^\beta(\theta(r))}\right]'\left(\frac{\mathcal{A}(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}\right)$$
$$+\zeta_1(r)[m_1(r)\zeta_2(r)]' + \zeta_1'(r)m_1(r)\zeta_2(r)$$
$$= \frac{\zeta_1(r)}{y^\beta(\theta(r))}\left\{\frac{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)(\mathcal{A}(r))' - \left(\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)\right)'\mathcal{A}(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)^2}\right\}$$
$$+\left[\frac{y^\beta(\theta(r))\zeta_1'(r) - \left(y^\beta(\theta(r))\right)'\zeta_1(r)}{y^\beta(\theta(r))y^\beta(\theta(r))}\right]\left(\frac{\mathcal{A}(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}\right)$$
$$+\zeta_1(r)[m_1(r)\zeta_2(r)]' + \zeta_1'(r)m_1(r)\zeta_2(r)$$
$$= \frac{\zeta_1(r)}{y^\beta(\theta(r))}\left[\frac{(\mathcal{A}(r))' + p(r)([m_2(r)y'(r)]')^\beta}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}\right] + \frac{\zeta_1'(r)}{\zeta_1(r)}\omega(r)$$
$$-\left[\frac{\zeta_1(r)\left(y^\beta(\theta(r))\right)'}{y^\beta(\theta(r))}\right]\frac{\mathcal{A}(r)}{y^\beta(\theta(r))\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)} + \zeta_1(r)[m_1(r)\zeta_2(r)]'$$
$$= -\frac{\zeta_1(r)}{y^\beta(\theta(r))}\left[\frac{\sum_{i=1}^{n} q_i(r)f\left(y(\theta_i(r))\right)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}\right] + \frac{\zeta_1'(r)}{\zeta_1(r)}\omega(r)$$
$$-\left[\frac{\zeta_1(r)\left(y^\beta(\theta(r))\right)'}{y^\beta(\theta(r))}\right]\frac{\mathcal{A}(r)}{y^\beta(\theta(r))e_{-\frac{p}{m_1}}^{m_1}(r,r_0)} + \zeta_1(r)[m_1(r)\zeta_2(r)]'$$

$$\le -L\frac{\sum_{i=1}^{n} q_i(r)\zeta_1(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}+\frac{\zeta_1'(r)}{\zeta_1(r)}\omega(r) - \left[\frac{\zeta_1(r)\left(y^{\beta}(\theta(r))\right)'}{y^{\beta}(\theta(r))}\right]\frac{\mathcal{A}(r)}{y^{\beta}(\theta(r))\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)} + \zeta_1(r)[m_1(r)\zeta_2(r)]'.$$

then

$$\omega'(r)\le -L\frac{\sum_{i=1}^{n} q_i(r)\zeta_1(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}+\frac{\zeta_1'(r)}{\zeta_1(r)}\omega(r)+\zeta_1(r)[m_1(r)\zeta_2(r)]' -\zeta_1(r)\left[\frac{\beta y^{\beta-1}(\theta(r))y'(\theta(r))\theta'(r)}{y^{\beta}(\theta(r))}\right]\frac{\mathcal{A}(r)}{y^{\beta}(\theta(r))\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)} \tag{10}$$

By Lemma 2.3 and $y'(r) > 0$, we have

$$\omega'(r)\le -L\frac{\sum_{i=1}^{n} q_i(r)\zeta_1(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}+\frac{\zeta_1'(r)}{\zeta_1(r)}\omega(r)+\zeta_1(r)[m_1(r)\zeta_2(r)]' -\left[\frac{\beta\zeta_1(r)\theta'(r)}{y(\theta(r))}\right]\left\{\frac{\psi_1(\theta(r),r_3)}{m_2(\theta(r))}\frac{m_1^{\frac{1}{\beta}}(\theta(r))[m_2(\theta(r))y'(\theta(r))]'}{\left[e_{-\frac{p}{m_1}}(\theta(r),r_0)\right]^{\frac{1}{\beta}}}\right\} \times\frac{\mathcal{A}(r)}{y^{\beta}(\theta(r))\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}$$

By Lemma 2.1, $\frac{\mathcal{A}(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}$ is strictly decreasing on $[r_2,\infty)$. So

$$\frac{(m_1(\theta(r)))^{\frac{1}{\beta}}[m_2(\theta(r))y'(\theta(r))]'}{\left[e_{-\frac{p}{m_1}}(\theta(r),r_0)\right]^{\frac{1}{\beta}}} > \frac{(m_1(r))^{\frac{1}{\beta}}[m_2(r)y'(r)]'}{\left[\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)\right]^{\frac{1}{\beta}}}$$

for $r\in[r_3,\infty)$, and we get

$$\omega'(r) \leq -L\frac{\sum_{i=1}^{n} q_i(r)\zeta_1(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)} + \frac{\zeta_1'(r)}{\zeta_1(r)}\omega(r) + \zeta_1(r)[m_1(r)\zeta_2(r)]' \tag{11}$$
$$-\beta\frac{\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)}{m_2(\theta(r))}\left[\frac{\omega(r)}{\zeta_1(r)} - m_1(r)\zeta_2(r)\right]^{1+\frac{1}{\beta}}.$$

Using the following inequality

$$(u-v)^{1+\frac{1}{\beta}} \geq u^{1+\frac{1}{\beta}} + \frac{1}{\beta}v^{1+\frac{1}{\beta}} - \left(1+\frac{1}{\beta}\right)v^{\frac{1}{\beta}}u$$

we obtain

$$\left[\frac{\omega(r)}{\zeta_1(r)} - m_1(r)\zeta_2(r)\right]^{1+\frac{1}{\beta}} \geq \frac{\omega^{1+\frac{1}{\beta}}(r)}{\zeta_1^{1+\frac{1}{\beta}}(r)} + \frac{1}{\beta}[m_1(r)\zeta_2(r)]^{1+\frac{1}{\beta}} \tag{12}$$
$$-\left(1+\frac{1}{\beta}\right)\frac{[m_1(r)\zeta_2(r)]^{\frac{1}{\beta}}\omega(r)}{\zeta_1(r)}.$$

A combination of (11) and (12) yields:

$$\omega'(r) \leq -L\frac{\sum_{i=1}^{n} q_i(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}\zeta_1(r) + \zeta_1(r)[m_1(r)\zeta_2(r)]'$$
$$-\frac{\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)[m_1(r)\zeta_2(r)]^{1+\frac{1}{\beta}}}{m_2(\theta(r))}$$
$$+\frac{m_2(\theta(r))\zeta_1'(r) + (\beta+1)\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)[m_1(r)\zeta_2(r)]^{\frac{1}{\beta}}}{m_2(\theta(r))\zeta_1(r)}\omega(r) \tag{13}$$
$$-\beta\frac{\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)}{m_2(\theta(r))}\frac{\omega^{1+\frac{1}{\beta}}(r)}{\zeta_1^{1+\frac{1}{\beta}}(r)}$$

Setting

$$\lambda = 1+\frac{1}{\beta},\ X^{\lambda} = \beta\frac{\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)}{m_2(\theta(r))}\frac{\omega^{1+\frac{1}{\beta}}(r)}{\zeta_1^{1+\frac{1}{\beta}}(r)}$$

$$Y^{\lambda-1} = \beta^{\frac{1}{\beta+1}}\left[\frac{m_2\big(\theta(r)\big)\zeta_1'(r) + (\beta+1)\zeta_1(r)\theta'(r)\psi_1\big(\theta(r),r_3\big)[m_1(r)\zeta_2(r)]^{\frac{1}{\beta}}}{(\beta+1)m_2^{\frac{1}{\beta+1}}\big(\theta(r)\big)\zeta_1^{\frac{\beta}{\beta+1}}(r)\big(\theta'(r)\big)^{\frac{\beta}{\beta+1}}\psi_1^{\frac{\beta}{\beta+1}}\big(\theta(r),r_2\big)}\right]$$

Using Lemma 2.4 in (12) we get that

$$\omega'(r) \leq -L\frac{\sum_{i=1}^{n} q_i(r)\zeta_1(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)} + \zeta_1(r)[m_1(r)\zeta_2(r)]'$$
$$-\frac{\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)[m_1(r)\zeta_2(r)]^{1+\frac{1}{\beta}}}{m_2(\theta(r))}$$
$$+\left[\frac{m_2(\theta(r))\zeta_1'(r) + (\beta+1)\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)[m_1(r)\zeta_2(r)]^{\frac{1}{\beta}}}{(\beta+1)m_2\frac{1}{\beta+1}(\theta(r))\zeta_1^{\frac{\beta}{\beta+1}}(r)(\theta'(r))^{\frac{\beta}{\beta+1}}\psi_1^{\frac{\beta}{\beta+1}}(\theta(r),r_2)}\right] \quad (14)$$

Substituting r with s in (14), an integration for (14) with respect to s from r_3 to r yields

$$\int_{r_3}^{r}\left\{L\frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)} - \zeta_1(s)[m_1(s)\zeta_2(s)]'\right.$$
$$+\frac{\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)[m_1(s)\zeta_2(s)]^{1+\frac{1}{\beta}}}{m_2(\theta(s))}$$
$$\left.-\frac{m_2(\theta(s))\zeta_1'(s) + (\beta+1)\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)[m_1(s)\zeta_2(s)]^{\frac{1}{\beta}}}{(\beta+1)m_2^{\frac{1}{\beta+1}}(\theta(s))\zeta_1^{\frac{\beta}{\beta+1}}(s)(\theta'(s))^{\frac{\beta}{\beta+1}}\psi_1^{\frac{\beta}{\beta+1}}(\theta(s),r_2)}\right\}ds$$
$$\leq \omega(r_3) - \omega(r) \leq \omega(r_3) < \infty,$$

which contradicts (9).

Theorem 2.2. Assume that (1), (2), (5) holds. Assume that there exist two nonnegative functions $\zeta_1(r), \zeta_2(r) \in C^1([r_0,\infty),\mathbb{R})$ with $\zeta_1(r) > 0$, such that

$$\limsup_{r\to\infty}\int_T^r \left\{ L\frac{\sum_{i=1}^n q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^s \frac{p(s)}{m_1(s)}ds\right)} - \zeta_1(s)[m_1(s)\zeta_2(s)]' + \frac{\beta\zeta_1(s)\theta'(s)\psi_1(\theta(s),T)\psi_2^{\beta-1}(\theta(s,T))m_1^2(s)\zeta_2^2(s)}{m_2(\theta(s))} - \frac{\left[m_2(\theta(s))\zeta_1'(s)+2\beta\zeta_1(s)\theta'(s)\psi_1(\theta(s),T)\psi_2^{\beta-1}(\theta(s,T))m_1(s)\zeta_2(s)\right]^2}{4\beta m_2(\theta(s))\zeta_1(s)\theta'(s)\psi_1(\theta(s),T)\psi_2^{\beta-1}(\theta(s,T))} \right\} ds = \infty. \quad (15)$$

Then conclusion ($\boldsymbol{C}$) holds.

Proof. Assume (E) has a nonoscillatory solution y on I. Similar to Theorem 2.1, we may assume $y(r) > 0$ on $[r_1,\infty)$. By Lemmas 2.1 and 2.2, there exists sufficiently large r_2 such that $[m_2(r)y'(r)]' > 0$ on $[r_2,\infty)$, and either $y'(r) > 0$ on $[r_2,\infty)$ or $\lim_{r\to\infty} y(r) = 0$. Now we assume $y'(r) > 0, y'(\theta(r)) > 0$ on $[r_3,\infty)$, where $r_3 > r_2$ is sufficiently large. Let $\omega(r)$ be defined as in Theorem 2.1 By Lemma (3), for $r \in [r_3,\infty)$, we have the following observations:

$$\frac{y'(\theta(r))}{y(\theta(r))} \geq \frac{y'(\theta(r))}{y(\theta(r))} = \frac{y'(\theta(r))}{y^\beta(\theta(r))} y^{\beta-1}(\theta(r))$$

$$\geq \frac{\psi_1(\theta(r),r_3)}{m_2(\theta(r))y^\beta(\theta(r))}\left\{\frac{m_1^{\frac{1}{\beta}}(\theta(r))[m_2(\theta(r))y'(\theta(r))]'}{\left[e_{-\frac{p}{m_1}}(\theta(r),r_0)\right]^{\frac{1}{\beta}}}\right\} y^{\beta-1}(\theta(r))$$

$$\geq \frac{\psi_1(\theta(r),r_3)}{m_2(\theta(r))y^\beta(\theta(r))}\left\{\frac{m_1^{\frac{1}{\beta}}(\theta(r))[m_2(\theta(r))y'(\theta(r))]'}{\left[e_{-\frac{p}{m_1}}(\theta(r),r_0)\right]^{\frac{1}{\beta}}}\right\}$$

$$\times \frac{\psi_2^{\beta-1}(\theta(r),r_3)\left\{\frac{m_1^{\frac{1}{\beta}}(\theta(r))[m_2(\theta(r))y'(\theta(r))]'}{m_2(\theta(r))y^\beta(\theta(r))}\right\}^{\beta-1}}{\left[e_{-\frac{p}{m_1}}(\theta(r),r_0)\right]^{\frac{1}{\beta}}}\left\{\frac{m_1^{\frac{1}{\beta}}(\theta(r))[m_2(\theta(r))y'(\theta(r))]'}{\left[e^{m_1}_{-\frac{p}{m_1}}(\theta(r),r_0)\right]^{\frac{1}{\beta}}}\right\}$$

$$\times \psi_2^{\beta-1}(\theta(r), r_3)\left\{\frac{\mathcal{A}^{\frac{1}{\beta}}(r)}{\left[\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)\right]^{\frac{1}{\beta}}}\right\}$$

$$\geq \frac{\psi_1(\theta(r), r_3)\psi_2^{\beta-1}(\theta(r), r_3)}{m_2\big(\theta(r)\big)}\left\{\frac{\mathcal{A}(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)y^{\beta}\big(\theta(r)\big)}\right\}. \qquad (16)$$

Using (16) in (10) we get that

$$\omega'(r) \leq -L\frac{\sum_{i=1}^{n} q_i(r)\zeta_1(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)} + \frac{\zeta_1'(r)}{\zeta_1(r)}\omega(r) + \zeta_1(r)[m_1(r)\zeta_2(r)]'$$

$$-\zeta_1(r)\left[\frac{\beta y^{\beta-1}(\theta(r))y'(\theta(r))\theta'(r)}{y^{\beta}(\theta(r))}\right]\frac{\mathcal{A}(r)}{y^{\beta}(\theta(r))\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}$$

$$\leq -L\frac{\sum_{i=1}^{n} q_i(r)\zeta_1(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)} + \frac{\zeta_1'(r)}{\zeta_1(r)}\omega(r) + \zeta_1(r)[m_1(r)\zeta_2(r)]'$$

$$-\beta\zeta_1(r)\theta'(r)\frac{\psi_1(\theta(r), r_3)\psi_2^{\beta-1}(\theta(r), r_3)}{m_2(\theta(r))}\left\{\frac{\mathcal{A}(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)y^{\beta}(\theta(r))}\right\}^2$$

$$= -L\frac{\sum_{i=1}^{n} q_i(r)\zeta_1(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)} + \frac{\zeta_1'(r)}{\zeta_1(r)}\omega(r) + \zeta_1(r)[m_1(r)\zeta_2(r)]'$$

$$-\beta\zeta_1(r)\theta'(r)\frac{\psi_1(\theta(r), r_3)\psi_2^{\beta-1}(\theta(r), r_3)}{m_2\big(\theta(r)\big)}\left[\frac{\omega(r)}{\zeta_1(r)} - m_1(r)\zeta_2(r)\right]^2$$

$$= -L\frac{\sum_{i=1}^{n} q_i(r)\zeta_1(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)} + \zeta_1(r)[m_1(r)\zeta_2(r)]'$$

$$-\frac{\beta\zeta_1(r)\theta'(r)\psi_1(\theta(r), r_3)\psi_2^{\beta-1}(\theta(r), r_3)m_1^2(r)\zeta_2^2(r)}{m_2\big(\theta(r)\big)}$$

$$+\left[\frac{m_2\big(\theta(r)\big)\zeta_1'(r)+2\beta\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)\psi_2^{\beta-1}(\theta(r),r_3)m_1(r)\zeta_2(r)}{m_2\big(\theta(r)\big)\zeta_1(r)}\right]\omega(r)$$

$$-\frac{\beta\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)\psi_2^{\beta-1}(\theta(r),r_3)}{m_2\big(\theta(r)\big)\zeta_1^2(r)}\omega^2(r)$$

$$\leq -L\frac{\sum_{i=1}^{n} q_i(r)\zeta_1(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)}+\zeta_1(r)[m_1(r)\zeta_2(r)]'$$

$$-\frac{\beta\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)\psi_2^{\beta-1}(\theta(r),r_3)m_1^2(r)\zeta_2^2(r)}{m_2\big(\theta(r)\big)}$$

$$+\frac{\left[m_2\big(\theta(r)\big)\zeta_1'(r)+2\beta\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)\psi_2^{\beta-1}(\theta(r),r_3)m_1(r)\zeta_2(r)\right]^2}{4\beta m_2\big(\theta(r)\big)\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)\psi_2^{\beta-1}(\theta(r),r_3)} \quad (17)$$

Substituting r with s in (17), an integration for (17) with respect to s from r_3 to r yields

$$\int_{r_3}^{r}\left\{L\frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)}-\zeta_1(s)[m_1(s)\zeta_2(s)]'\right.$$

$$+\frac{\beta\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)\psi_2^{\beta-1}(\theta(s),r_3)m_1^2(s)\zeta_2^2(s)}{m_2(\theta(s))}$$

$$\left.-\frac{\left[m_2(\theta(s))\zeta_1'(s)+2\beta\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)\psi_2^{\beta-1}(\theta(s),r_3)m_1(s)\zeta_2(s)\right]^2}{4\beta m_2(\theta(s))\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)\psi_2^{\beta-1}(\theta(s),r_3)}\right\}ds$$

$$<\omega(r_3)-\omega(r)\leq\omega(r_3)<\infty,$$

which contradicts (15).

Theorem 2.3. Let (1), (2), (5) hold and define

$$\mathbb{D}=\{(r,s)\mid r\geq s\geq r_0, r,s\in\mathbb{T}\}$$

If there exists a function $H\in C(\mathbb{D},\mathbb{R})$ such that

$$H(r,r)=0,\text{ for } r\geq r_0,\ H(r,s)>0$$

for $r>s\geq r_0$, and H has a nonpositive partial derivative $H'(r,s)$ with respect to the second variable, and for all sufficiently large t,

$$\limsup_{r\to\infty}\frac{1}{H(r,r_0)}\int_{r_0}^{r} H(r,s)\Bigg\{L\frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)} - \zeta_1(s)[m_1(s)\zeta_2(s)]' + \frac{\zeta_1(s)\theta'(s)\psi_1(\theta(s),T)[m_1(s)\zeta_2(s)]^{1+\frac{1}{\beta}}}{m_2(\theta(s))}$$

$$-\left[\frac{m_2(\theta(s))\zeta_1'(s)+(\beta+1)\zeta_1(s)\theta'(s)\psi_1(\theta(s),T)[m_1(s)\zeta_2(s)]^{\frac{1}{\beta}}}{(\beta+1)m_2^{\frac{1}{\beta+1}}(\theta(s))\zeta_1^{\frac{\beta}{\beta+1}}(s)(\theta'(s))^{\frac{\beta}{\beta+1}}\psi_1^{\frac{\beta}{\beta+1}}(\theta(s),T)}\right\} ds = \infty. \quad (18)$$

Then conclusion $(\boldsymbol{C})$ holds.

Proof. Assume (E) has a nonoscillatory solution y on I. Similar to Theorem 2.1, we may assume $y(r) > 0$ on $[r_1,\infty)$. By Lemmas 2.1 and 2.2, there exists sufficiently large r_2 such that $[m_2(r)y'(r)]' > 0$ on $[r_2,\infty)$, and either $y'(r) > 0$ on $[r_2,\infty)$ or $\lim_{r\to\infty} y(r) = 0$. Now we assume $y'(r) > 0, y'(\theta(r)) > 0$ on $[r_3,\infty)$, where $r_3 > r_2$ is sufficiently large. Let $\omega(r)$ be defined as in Theorem 2.1. Then by (14), for $r \in [r_3,\infty)$, we have

$$L\frac{\sum_{i=1}^{n} q_i(r)\zeta_1(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)} - \zeta_1(r)[m_1(r)\zeta_2(r)]' + \frac{\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)[m_1(r)\zeta_2(r)]^{1+1/\gamma}}{m_2(\theta(r))}$$

$$-\left[\frac{m_2(\theta(r))\zeta_1'(r)+(\beta+1)\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)[m_1(r)\zeta_2(r)]^{\frac{1}{\beta}}}{(\beta+1)m_2\frac{1}{\beta+1}}(\theta(r))\zeta_1^{\frac{\beta}{\beta+1}}(r) \quad (19)$$

$$\times(\theta'(r))^{\frac{\beta}{\beta+1}}\psi_1^{\frac{\beta}{\beta+1}}(\theta(r),r_2)\right] \le -\omega'(r).$$

Substituting r with s in (19), multiplying above by $H(r,s)$ and then integrating with respect to s from r_3 to r yields

$$\int_{r_3}^{r} H(r,s)\left\{L\frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)} - \zeta_1(s)[m_1(s)\zeta_2(s)]'\right.$$

$$+\frac{\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)[m_1(s)\zeta_2(s)]^{1+\frac{1}{\beta}}}{m_2(\theta(s))}$$

$$\left.-\left[\frac{m_2(\theta(s))\zeta_1'(s)+(\beta+1)\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)[m_1(s)\zeta_2(s)]^{\frac{1}{\beta}}}{(\beta+1)m_2^{\frac{1}{\beta+1}}(\theta(s))\zeta_1^{\frac{\beta}{\beta+1}}(s)(\theta'(s))^{\frac{\beta}{\beta+1}\frac{\beta}{\beta+1}}(\theta(s),r_2)}\right]\right\}$$

$$\leq -\int_{r_3}^{r} H(r,s)\omega'(s)ds$$

$$=H(r,r_3)\omega(r_3)+\int_{r_3}^{r} H^{ds_s}(r,s)\omega(s)ds$$

$$\leq H(r,r_3)\omega(r_3)\leq H(r,r_0)\omega(r_3).$$

Then

$$\int_{r_0}^{r} H(r,s)\left\{L\frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)} - \zeta_1(s)[m_1(s)\zeta_2(s)]'\right.$$

$$+\frac{\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)[m_1(s)\zeta_2(s)]^{1+\frac{1}{\beta}}}{m_2\big(\theta(s)\big)}$$

$$=\int_{r_0}^{r_3} H(r,s)\left\{L\frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)} - \zeta_1(s)[m_1(s)\zeta_2(s)]'\right.$$

$$+\frac{\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)[m_1(s)\zeta_2(s)]^{1+\frac{1}{\beta}}}{m_2(\theta(s))}$$

$$-\left[\frac{m_2(\theta(s))\zeta_1'(s)+(\beta+1)\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)[m_1(s)\zeta_2(s)]^{\frac{1}{\beta}}}{(\beta+1)m_2^{\frac{1}{\beta+1}}(\theta(s))\zeta_1^{\frac{\beta}{\beta+1}}(s)(\theta'(s))^{\frac{\beta}{\beta+1}}\psi_1^{\frac{\beta}{\beta+1}}(\theta(s),r_2)}\right\}ds$$

$$+\int_{r_3}^{r} H(r,s)\left\{L\frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)}-\zeta_1(s)[m_1(s)\zeta_2(s)]'\right.$$

$$+\frac{\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)[m_1(s)\zeta_2(s)]^{1+\frac{1}{\beta}}}{m_2(\theta(s))}$$

$$-\left[\frac{m_2(\theta(s))\zeta_1'(s)+(\beta+1)\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)[m_1(s)\zeta_2(s)]^{\frac{1}{\beta}}}{(\beta+1)m_2\frac{1}{\beta+1}(\theta(s))\zeta_1^{\frac{\beta}{\beta+1}}(s)(\theta'(s))^{\frac{\beta}{\beta+1}}\psi_1^{\frac{\beta}{\beta+1}}(\theta(s),r_2)}\right\}ds$$

$$\leq H(r,r_0)\omega(r_3)+H(r,r_0)\int_{r_0}^{r_3}\left|L\frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)}-\zeta_1(s)[m_1(s)\zeta_2(s)]'\right.$$

$$+\frac{\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)[m_1(s)\zeta_2(s)]^{1+\frac{1}{\beta}}}{m_2(\theta(s))}$$

$$\left.-\left[\frac{m_2(\theta(s))\zeta_1'(s)+(\beta+1)\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)[m_1(s)\zeta_2(s)]^{\frac{1}{\beta}}}{(\beta+1)m_2^{\frac{1}{\beta+1}}(\theta(s))\zeta_1^{\frac{\beta}{\beta+1}}(s)(\theta'(s))^{\frac{\beta}{\beta+1}}\psi_1^{\frac{\beta}{\beta+1}}(\theta(s),r_2)}\right]\right|ds$$

So

$$\limsup_{r\to\infty}\frac{1}{H(r,r_0)}\left\{\int_{r_0}^{r} H(r,s)\left\{L\frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)}-\zeta_1(s)[m_1(s)\zeta_2(s)]'\right.\right.$$
$$+\frac{\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)[m_1(s)\zeta_2(s)]^{1+\frac{1}{\beta}}}{m_2(\theta(s))}$$
$$\left.\left.-\left[\frac{m_2(\theta(s))\zeta_1'(s)+(\beta+1)\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)[m_1(s)\zeta_2(s)]^{\frac{1}{\beta}}}{(\beta+1)m_2^{\frac{1}{\beta+1}}(\theta(s))\zeta_1^{\frac{\beta}{\beta+1}}(s)(\theta'(s))^{\frac{\beta}{\beta+1}}\psi_1^{\frac{\beta}{\beta+1}}(\theta(s),r_2)}\right\}ds\right\}$$
$$\leq \omega(r_3)+\int_{r_0}^{r_3}\left|L\frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)}-\zeta_1(s)[m_1(s)\zeta_2(s)]'\right.$$
$$+\frac{\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)[m_1(s)\zeta_2(s)]^{1+\frac{1}{\beta}}}{m_2(\theta(s))}$$
$$-\left[\frac{m_2(\theta(s))\zeta_1'(s)+(\beta+1)\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)[m_1(s)\zeta_2(s)]^{\frac{1}{\beta}}}{(\beta+1)m_2^{\frac{1}{\beta+1}}(\theta(s))\zeta_1^{\frac{\beta}{\beta+1}}(s)(\theta'(s))^{\frac{\beta}{\beta+1}}\psi_1^{\frac{\beta}{\beta+1}}(\theta(s),r_2)}ds.\right.$$
$$<\infty$$

which contradicts (18).

Theorem 2.4. Assume that (1), (2), (5) hold and let H be defined as in Theorem 2.3, such that

$$\limsup_{r\to\infty}\frac{1}{H(r,r_0)}\left\{\int_{r_0}^{r} H(r,s)\left\{L\frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)}-\zeta_1(s)[m_1(s)\zeta_2(s)]'\right.\right.$$
$$+\frac{\beta\zeta_1(s)\theta'(s)\psi_1(\theta(s),T)\psi_2^{\beta-1}(\theta(s,T))m_1^2(s)\zeta_2^2(s)}{m_2(\theta(s))}$$
$$\left.\left.-\frac{\left[m_2(\theta(s))\zeta_1'(s)+2\beta\zeta_1(s)\theta'(s)\psi_1(\theta(s),T)\psi_2^{\beta-1}(\theta(s,T))m_1(s)\zeta_2(s)\right]^2}{4\beta m_2(\theta(s))\zeta_1(s)\theta'(s)\psi_1(\theta(s),T)\psi_2^{\beta-1}(\theta(s,T))}\right\}ds\right\}=\infty. \quad (20)$$

Then conclusion ($\boldsymbol{C}$) holds.

Proof. Assume (E) has a nonoscillatory solution y on $\mathbb{T}_0$. Similar to Theorem 2.1, we may assume $y(r)>0$ on $[r_1,\infty)$, where r_1 is sufficiently large. By Lemmas 2.1 and

2.2, there exists sufficiently large r_2 such that $[m_2(r)y'(r)]' > 0$ on $[r_2, \infty)$, and either $y'(r) > 0$ on $[r_2, \infty)$ or $\lim_{r\to\infty} y(r) = 0$. Now we assume $y'(r) > 0, y'(\theta(r)) > 0$ on $[r_3, \infty)$, where $r_3 > r_2$ is sufficiently large. Let $\omega(r)$ be defined as in Theorem 2.1. Then by (17), for $r \in [r_3, \infty)$, we have

$$L\frac{\sum_{i=1}^{n} q_i(r)\zeta_1(r)}{\exp\left(\int_{r_0}^{r}\frac{p(r)}{m_1(r)}dt\right)} - \zeta_1(r)[m_1(r)\zeta_2(r)]' + \frac{\beta\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)\psi_2^{\beta-1}(\theta(r),r_3)m_1^2(r)\zeta_2^2(r)}{m_2(\theta(r))} - \frac{\left[m_2(\theta(r))\zeta_1'(r) + 2\beta\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)\psi_2^{\beta-1}(\theta(r),r_3)m_1(r)\zeta_2(r)\right]^2}{4\beta m_2(\theta(r))\zeta_1(r)\theta'(r)\psi_1(\theta(r),r_3)\psi_2^{\beta-1}(\theta(r),r_3)} \leq -\omega'(r) \tag{21}$$

Substituting r with s in (17), multiplying both sides by $H(r,s)$, and then integrating with respect to s from r_3 to r yields

$$\int_{r_3}^{r} H(r,s)\left\{L\frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)} - \zeta_1(s)[m_1(s)\zeta_2(s)]' + \frac{\beta\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)\psi_2^{\beta-1}(\theta(s),r_3)m_1^2(s)\zeta_2^2(s)}{m_2(\theta(s))} - \frac{\left[m_2(\theta(s))\zeta_1'(s) + 2\beta\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)\psi_2^{\beta-1}(\theta(s),r_3)m_1(s)\zeta_2(s)\right]^2}{4\beta m_2(\theta(s))\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)\psi_2^{\beta-1}(\theta(s),r_3)}\right\}ds$$

$$\leq -\int_{r_3}^{r} H(r,s)\omega'(s)ds$$

$$= H(r,r_3)\omega(r_3) + \int_{r_3}^{r} H'(r,s)\omega(s)ds$$

$$\leq H(r,r_3)\omega(r_3) \leq H(r,r_0)\omega(r_3).$$

Then similar to Theorem 2.3, we obtain

$$\limsup_{r\to\infty}\frac{1}{H(r,r_0)}\left\{\int_{r_0}^{r} H(r,s)\left\{L\frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)}-\zeta_1(s)[m_1(s)\zeta_2(s)]'\right.\right.$$

$$+\frac{\beta\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)\psi_2^{\beta-1}(\theta(s),r_3)m_1^2(s)\zeta_2^2(s)}{m_2(\theta(s))}$$

$$\left.\left.-\frac{\left[m_2(\theta(s))\zeta_1'(s)+2\beta\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)\psi_2^{\beta-1}(\theta(s),r_3)m_1(s)\zeta_2(s)\right]^2}{4\beta m_2(\theta(s))\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)\psi_2^{\beta-1}(\theta(s),r_3)}\right\}ds\right\}$$

$$\leq \omega(r_3)+\int_{r_0}^{r_3}\left|L\frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)}-\zeta_1(s)[m_1(s)\zeta_2(s)]'\right.$$

$$+\frac{\beta\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)\psi_2^{\beta-1}(\theta(s),r_3)m_1^2(s)\zeta_2^2(s)}{m_2(\theta(s))}$$

$$\left.-\frac{\left[m_2(\theta(s))\zeta_1'(s)+2\beta\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)\psi_2^{\beta-1}(\theta(s),r_3)m_1(s)\zeta_2(s)\right]^2}{4\beta m_2(\theta(s))\zeta_1(s)\theta'(s)\psi_1(\theta(s),r_3)\psi_2^{\beta-1}(\theta(s),r_3)}\right| ds$$

$$<\infty,$$

which contradicts (20).

By considering $H(r,s)$ as a selection of specific functions, such as $(r-s)^m$, in Theorems 2.3 and 2.4, we can derive several corollaries. For instance, if we let

$$H(r,s)=(r-s)^m,\ m\geq 1$$

then we have the following corollaries.

Corollary 2.1. Assume that (1), (2), (5) holds. If

$$\limsup_{r\to\infty}\frac{1}{(r-r_0)^m}\left\{\int_{r_0}^{r}(r-s)^m\left\{L\frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)}-\zeta_1(s)[m_1(s)\zeta_2(s)]'\right.\right.$$

$$+\frac{\zeta_1(s)\theta'(s)\psi_1(\theta(s),T)[m_1(s)\zeta_2(s)]^{1+\frac{1}{\beta}}}{m_2(\theta(s))}$$

$$\left.\left.-\left[\frac{m_2(\theta(s))\zeta_1'(s)+(\beta+1)\zeta_1(s)\theta'(s)\psi_1(\theta(s),T)[m_1(s)\zeta_2(s)]^{\frac{1}{\beta}}}{(\beta+1)m_2^{\frac{1}{\beta+1}}(\theta(s))\zeta_1^{\frac{\beta}{\beta+1}}(s)(\theta'(s))^{\frac{\beta}{\beta+1}}\psi_1^{\frac{\beta}{\beta+1}}(\theta(s),T)}\right\}ds\right\}=\infty.$$

Then conclusion $(\boldsymbol{C})$ holds.

Corollary 2.2. Assume that (1), (2), (5) holds. If

$$\limsup_{r\to\infty}\frac{1}{(r-r_0)^m}\left\{\int_{r_0}^{r}(r-s)^m\left\{L\frac{\sum_{i=1}^{n} q_i(s)\zeta_1(s)}{\exp\left(-\int_{r_0}^{s}\frac{p(s)}{m_1(s)}ds\right)}-\zeta_1(s)[m_1(s)\zeta_2(s)]'\right.\right.$$
$$+\frac{\beta\zeta_1(s)\theta'(s)\psi_1(\theta(s),T)\psi_2^{\beta-1}(\theta(s,T))m_1^2(s)\zeta_2^2(s)}{m_2(\theta(s))}$$
$$\left.\left.-\frac{\left[m_2(\theta(s))\zeta_1'(s)+2\beta\zeta_1(s)\theta'(s)\psi_1(\theta(s),T)\psi_2^{\beta-1}(\theta(s,T))m_1(s)\zeta_2(s)\right]^2}{4\beta m_2(\theta(s))\zeta_1(s)\theta'(s)\psi_1(\theta(s),T)\psi_2^{\beta-1}(\theta(s,T))}\right\}ds\right\}=\infty.$$

Then conclusion $(\boldsymbol{C})$ holds.

III. Applications

Example 3.1. Consider the 3rd-order nonlinear delay differential equation with a damping term:

$$\left[(ry''(r))^{\beta}\right]'+\frac{1}{r^{\beta+1}}(y''(r))^{\beta}+\sum_{i=1}^{n} q_i(t)f\big(y(\theta_i(r))\big)=0,\ r\in[2,\infty)$$

where $\beta\geq 1$. By Corollary 2.1 we deduce that every solution of Eq. (E) is oscillatory or tends to zero.

IV. Conclusion

This paper conducted a thorough investigation into the oscillatory behavior of a specific subclass of the third-order nonlinear delay differential equation with a damping term. We focused our investigation specifically on their canonical form, which was enhanced by the inclusion of damping terms. This study utilized Riccati's technique and the comparison method to establish rigorous criteria that guaranteed the existence of oscillatory behavior in the solutions.

Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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