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AN ANALYTICAL APPROACH TO THE NON-OSCILLATORY NONLINEAR MECHANICAL SYSTEMS HAVING INTEGRAL MULTIPLE ROOTS AND STRONG NON-LINEARITY

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Abstract

The existence of over-damped nonlinear differential equations results from a variety of engineering conundrums and physical natural occurrences. Non-oscillatory dynamics with forced over-damping are used in the simulation of nonlinear differential systems. For non-oscillatory nonlinear differential systems, it is possible to derive approximations of solutions using a variety of analytical methods, both with and without external forcing. This paper introduces a novel method for estimating solutions for highly nonlinear damped vibration systems subject to parameterized external forcing. The extended Krylov-Bogoliubov-Mitropolsky (KBM) technique and harmonic equilibrium (HM), which have both been previously developed in the literature, are the foundation of the suggested method. This method was initially created by Krylov-Bogoliubov to discover periodic details in second-order nonlinear differential equations. Several examples are provided to show how the suggested technique is applied. The process is fairly simple and straightforward, and using this formula, the result can be found with very marginal errors from the previous

citations. The primary significance of this approach is in its ability to provide approximate analytical solutions of the first order that closely align with the findings obtained by numerical methods. These solutions are applicable to a variety of beginning scenarios and are distinct from those presented in earlier literature. Also, we illustrated the two-dimensional graph of all the solutions that we got in this article by using the data from the mentioned table. The results that we obtained from this method are effective and reliable for better measurements of strong nonlinearities.

Keywords: Nonlinear non-autonomous system, Damped nonlinear system, External force Vary with time, Perturbation equation.

I. Introduction

Over-damp nonlinear differential equations are caused by a variety of engineering issues and natural physical phenomena. It is vital in the study of nonlinear oscillation design, quantum mechanics, fluid dynamics, and so on. Several analytical methods have been developed to find approximation solutions for non-oscillatory nonlinear differential systems with or without force. To avoid algebraic complexity, the approximate lower-order solution is typically found using perturbation methods. The Krylov-Bogoliubov-Mitropolsky method (KBM) [VII, XIX, XVI] is a widely used technique to obtain approximate solutions to nonlinear systems characterized by a low degree of nonlinearity. The harmonic balancing (HB) technique is a widely used [I, X, IV, XXVIII, XXVII, XX, XV, XVIX] approach for efficiently managing highly nonlinear systems.

Krylov and Bogolyubov [VII] initially established this technique to identify periodic solutions for second-order nonlinear differential equations. The approach was expanded and quantitatively validated by Bogolyubov and Mitropolsky [XIX, XVI]. Popov [XXV] expanded the methodology to include a damped oscillatory process in the presence of a strong linear damping force. Krisna, Dekshatulu, and Murty [VI] expanded upon the method to include over-damped nonlinear systems. Several authors [XII, XXVI, XIV, XIII, VIII, III, IX, XI, XXX] studied over-damped, damped, and critical damped nonlinear systems and as an outcome created perturbation methods. Pinakee et al. [XXIV, XXII, V] have devised an improved KBM (Krylov-Bogoliubov-Mitropolsky) method for mitigating vibrations in systems that exhibit under-damping, damping, and over-damping, especially when the coefficients of these systems undergo gradual changes over time. Recently, Pinakee et.al [II, XXI] developed an improved technique for analyzing damped-driven nonlinear systems that have coefficients that change. The objective of this study is to determine a resolution for a non-oscillatory forced nonlinear vibrating system that exhibits gradual changes over time, whereby one of the eigenvalues is a multiple of the other eigenvalues (more than 250 times), and measure the best result for strong non-linearity's (even if $\varepsilon \geq 1.5$) but in this situation, the unified KBM method does not give the expected results. To the best of our acknowledgements this method has not been used in the previously known references, and in the graphical segment, we illustrated the two-dimensional figure of the present solution with the aid of MATLAB. At last, we compare our present method to the previous method to argue which one provides the better result.

II. Methodology

Consider a nonlinear vibrating system that is not autonomous and is governed by

$$\ddot{x} + 2\xi_1(\tau)\dot{x} + (\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 \sin 2\tau)x = -\varepsilon f(x, \dot{x}, \nu t, \tau,), \tau = \varepsilon t \quad (1)$$

The presence of over-dots indicates differentiation with respect to the variable t, which happens to be a minor parameter.

$\zeta_1 = \zeta_2 = 0(\varepsilon) = \zeta_3$, $\tau = \varepsilon t$ is the slowly varying time, $\xi(\tau) \geq 0$, f is a specified nonlinear function. We set $\omega^2(\tau) = (\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 \sin 2\tau)$, where the function $\omega(\tau)$ is referred to as the inner frequency, whereas ν represents the frequency produced by the externally exerting force.

Putting $\varepsilon = 0$ and $\tau = \tau_0$= constant, into Eq. (1), The unperturbed result of Eq. (1) is obtained in the following manner.

$$x(t, 0) = x_0 e^{\lambda_1(\tau_0)t} + y_0 e^{\lambda_2(\tau_0)t}, \quad (2)$$

We assume Eq. (1) contains two eigenvalues, $\lambda_1(\tau_0)$ and $\lambda_2(\tau_0)$ which acts as constants, but in the case of $\varepsilon \neq 0, \lambda_1(\tau_0)$ and $\lambda_2(\tau_0)$ change slowly over time. Suppose that $|\lambda_2(\tau_0)| \gg |\lambda_1(\tau_0)|$. When $\varepsilon \neq 0$, we seek a solution to the problem (1) in the specified format.

$$x(t, \varepsilon) = x_{1,0}(t, \tau) + y_{1,0}(t, \tau) + \varepsilon u_1(x, y, t, \tau) + \varepsilon^2 u_2(x, y, t, \tau) + \cdots \quad (3)$$

Where x_1 and y_1 satisfy the equations

$$\dot{x}_1 = \lambda_1(\tau)x_1 + \varepsilon X_1(x_1, y_1, \tau) + \varepsilon^2 X_2(x_1, y_1, \tau),$$
$$\dot{y}_1 = \lambda_2(\tau)y_1 + \varepsilon Y_1(x_1, y_1, \tau) + \varepsilon^2 Y_2(x_1, y_1, \tau)\ldots \quad (4)$$

We will focus on the initial terms in the series expansions of Eqs. (3) and (4), which range from 1 to 2 up to a maximum of 3, we aim to find the functions $u_1, u_2, \ldots,$ etc., $X_1, X_2, \ldots Y_1, Y_2$ ..., etc. These functions are included in Eqs. (3) and (4) and fulfill the specified differential Eq. (1), along with a high precision of ε^{m+1}. We rely on the work of Murty et al. [VI] to identify these unknown characteristics. It's important to note that the functions under consideration do not contain terms involving $\varepsilon^{\lambda_j t}$, where j takes values of 1 or 2. This omission is because these terms are already accounted for in the series expansion (3) at the ε^0 level. As we determine these unknown functions, we can assume that the functions $u_1, u_2, \ldots,$ etc., do not include secular-type terms like $te.^{-t}$. We obtain

$$(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_1)X_1 + (\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_2)Y_1 + \lambda_1' x_1 + \lambda_2' y_1 + (\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_1)(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_2)u_1 = -f^{(0)}(x_1, y_1, \nu t, \tau) \quad (5)$$

where $\lambda_1' = d\lambda_1/d\tau$, $\lambda_2' = d\lambda_2/d\tau$, $\Omega x_1 = \partial/\partial x_1$, $\Omega y_1 = \partial/\partial y_1$, $f^{(0)} = f(x_0, \dot{x}_0, \tau)$

Here, it is considered that $f^{(0)}$can be enlarged in the Fourier series as

$$f^{(0)} = \sum_{i_1, i_2=0}^{\infty} F_{i_1, i_2}(\tau)\, x^{i_1} y^{i_2} \quad (6)$$

The equations are created by equating the coefficients of equivalent harmonic terms on both sides. To get a solution (1) with over-damping, apply constraint that $u_1, u_2, \ldots,$ exclude the terms $x^{i_1}y^{i_2}, i_1\lambda_1 + i_2\lambda_2 < (i_1 + i_2)\xi(\tau_0), i_1, i_2 = 0,1,2\cdots$. The assumption affirms the absence of secular terms $te^{-\lambda_1 t}$. Alam. [XII, XIV, XIII, XVII], Dey et.al.[XXIV, XXII, II, XXI, XXIII], Murty et. al. [VI]. Given that, this study aims to identify the unknown functions u_1 and X_1, Y_1, the first order solution has been fully determined in (1).

Early on, Krylov and Bogoliubov [XIX] imposed that u_1 excludes secular terms (*e.g.*, $t\cos t$ and $t\sin t$) to obtain the periodic solution of (1) in which $\xi = 0$, Popov [XXV] further extended this method to the case of insufficient damping, when $\sqrt{\omega^2} > \xi_1 > 0$. Murty, Deekshatulu, and Krishna [VI] extend the same method in the case of over-damped *i.e.*, for$\xi_1 > \sqrt{\omega^2}$. The solution of [VI] by Murty, Deekshatulu, and Krisna gives wrong results when one root is a multiple of the other or one root becomes much smaller than the other. In this situation, Alam [XII, XIV, XIII] found some special types of over-damped solutions (i) $\lambda_1 \approx 3\lambda_2$ (ii) $\lambda_1 \approx 2\lambda_2$ and (iii) $\lambda_1 \approx 10\lambda_2$ provided that u_1 exclude terms $x^{i_1}y^{i_2}, i_1\lambda_1 + i_2\lambda_2 < (i_1 + i_2)\xi(\tau_0), i_1, i_2 = 0,1,2\cdots. i_1, i_2 = 0, 1, 2,$. The assumption affirms the absence of secular terms of the type $te^{-\lambda_1 t}$.

III. Example

III.i. Assume a non-autonomous, non-linear vibrating system that is

$$\ddot{x} + 2\xi_1(\tau)\dot{x} + (\zeta_1^2 + \zeta_2\, cos\,\tau + \zeta_3^2\, sin\, 2\,\tau)x = -\varepsilon x^3 + \varepsilon E e^{-(.03)t} \qquad (7)$$

Using Eq. (2), Eq. (6) becomes

$$(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_1)X_1 + (\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_2)Y_1 + \lambda_1' x_1 + \lambda_2' y_1 \; = -(x_1^3 + y_1^3 + 3\, x_1 y_1^2) + E e^{-(.03)t} \qquad (8)$$

and

$$(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_1)(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_2)u_1 = -(3\, x_1^2 y_1) \qquad (9)$$

The particular solution of (8) is

$$u_1 = \alpha_1 x_1^2 y_1 \qquad (10)$$

where $\alpha_1 = -3/2(\lambda_2^2 + \lambda_2\lambda_1)$

We now resolve the two functions of (7) X and Y(explained in part 2)

The precise answers are

$$(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_2)X_1 + \lambda_1' x_1 = -x_1^3 + E e^{-(.03)t} \qquad (11)$$
$$(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_1)Y_1 + \lambda_2' y_1 = -y_1^3 - 3x_1 y_1^2$$

The particular solution of (11) and replace the functional values of $X_1, Y_1,$ into (4) and rearrange, we obtain

$$\dot{x}_1 = \lambda_1 x_1 + \varepsilon(\lambda_1' x_1 \beta_1 + \beta_2 x_1^3 + \beta_3 E)$$
$$\dot{y}_1 = \lambda_2 y_1 + \varepsilon(\lambda_2' y_1 \chi_1 + \chi_2 x_1 y_1^2 + \chi_3 y_1^3), \quad (12)$$

where, $\beta_1 = -1/(\lambda_1 - \lambda_2), \beta_2 = -1/(3\lambda_1 - \lambda_2)\,, \beta_3 = 1/(.03 - \lambda_2)$

$$\chi_1 = -1/(\lambda_1 - \lambda_2), \chi_2 = -3/(2\lambda_2), \chi_3 = -1/(3\lambda_2 - \lambda_1$$

In the general case, Eq. (11) does not have exact solutions. To solve them, normally numerical steps are used. We applied the Runge-Kutta (4th order) technique in this paper. It is numerically preferable to solve the converted equations (11) rather than the original equations (6) since the integration can use a large step (see [XVIII] for detail).

Consequently, the first-order answer to equation (7) is

$$x(t, \varepsilon) = x_1 + y_1 + \varepsilon u_1, \quad (13)$$

III.ii. Consider a nonlinear nonautonomous third-order vibrating system

$$\dddot{x} + \xi_1(\tau)\ddot{x} + \xi_2\dot{x} + (\zeta_1^2 + \zeta_2\, cos\, \tau + \zeta_3\, sin\, 2\,\tau)x = -\varepsilon x^3 + \varepsilon E e^{-(.03)t} \quad (14)$$

Put $\varepsilon = 0$ and $\tau = \tau_0 =$ constant, in Eq. (22), we obtain

$$x(t, 0) = x_0 e^{\lambda_1(\tau_0)t} + y_0 e^{\lambda_2(\tau_0)t} + z_0 e^{\lambda_3(\tau_0)t} \quad (15)$$

Let Eq. (1) have three eigenvalues, $\lambda_1(\tau_0)\,, \lambda_2(\tau_0)$ and $\lambda_3(\tau_0)$ are constants, then when $\varepsilon \neq 0$, $\lambda_1(\tau_0)\,, \lambda_2(\tau_0)$ and $\lambda_3(\tau_0)$ changes slowly over time. Using Eq. (2), Eq. (15) becomes

$$(\lambda_1 x\Omega x + \lambda_2 y\Omega y + \lambda_3 z\Omega z - \lambda_2)(\lambda_1 x\Omega x + \lambda_2 y\Omega y + \lambda_3 z\Omega z - \lambda_3)X +$$
$$(\lambda_1 x\Omega x + \lambda_2 y\Omega y + \lambda_3 z\Omega z - \lambda_1)(\lambda_1 x\Omega x + \lambda_2 y\Omega y + \lambda_3 z\Omega z - \lambda_3) +$$
$$(\lambda_1 x\Omega x + \lambda_2 y\Omega y + \lambda_3 z\Omega z - \lambda_1)(\lambda_1 x\Omega x + \lambda_2 y\Omega y + \lambda_3 z\Omega z - \lambda_2)Z +$$
$$\acute{\lambda_1} x + \acute{\lambda_2} y + \acute{\lambda_3} z = -(3yz^2 + z^3 + y^3 + 3y^2 z + 6xyz + 3x^2 z + Ee^{-(.03)t}) \quad (16)$$

and

$$(\lambda_1 x\Omega x + \lambda_2 y\Omega y + \lambda_3 z\Omega z - \lambda_1)(\lambda_1 x\Omega x + \lambda_2 y\Omega y + \lambda_3 z\Omega z - \lambda_2)(\lambda_1 x\Omega x + \lambda_2 y\Omega y + \lambda_3 z\Omega z - \lambda_3)u_1 = -(x^3 + 3x^2 y + 3x^2 z + 3xy^2) \quad (17)$$

The particular solution of (17) is

$$u_1 = \alpha_1 x^3 + \alpha_2 3x^2 y + \alpha_3 3x^2 z + \alpha_4 3xy^2 \quad (18)$$

Where $\alpha_1 = -1/2\lambda_1(3\lambda_1 - \lambda_2)(3\lambda_1 - \lambda_3)$
$\alpha_2 = -3/(2\lambda_1 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)(2\lambda_1 - \lambda_2)$
$\alpha_3 = -3/(\lambda_1 + \lambda_3)(2\lambda_1)(2\lambda_1 + \lambda_3 - \lambda_2)(2\lambda_1)$
$\alpha_4 = -3/(\lambda_1 + \lambda_3)(2\lambda_2)(\lambda_1 + 2\lambda_2 - \lambda_3)$

We now resolve (16) for three functions $X, Y$and Z(described in part 2)

The particular solutions are

$$(\lambda_1 x\Omega x + \lambda_2 y\Omega y + \lambda_3 z\Omega z - \lambda_2)(\lambda_1 x\Omega x + \lambda_2 y\Omega y + \lambda_3 z\Omega z - \lambda_3)X + \acute{\lambda_1} = -\left(3yz^2 + z^3 + Ee^{-(.03)t}\right)$$

$$(\lambda_1 x\Omega x + \lambda_2 y\Omega y + \lambda_3 z\Omega z - \lambda_1)(\lambda_1 x\Omega x + \lambda_2 y\Omega y + \lambda_3 z\Omega z - \lambda_3)Y + \acute{\lambda_2} y = -(y^3 + 3y^2 z)$$

$$(\lambda_1 x\Omega x + \lambda_2 y\Omega y + \lambda_3 z\Omega z - \lambda_1)(\lambda_1 x\Omega x + \lambda_2 y\Omega y + \lambda_3 z\Omega z - \lambda_3)Y + \acute{\lambda_2} y = -(y^3 + 3y^2 z) \quad (19)$$

The specified solution of (15) substituting the functional values of $X, Y$and $Z$and rearranging, we obtain

$$\dot{x} = \lambda_1 x + \epsilon(\acute{\lambda_1} x\beta_1 + \beta_2 yz^2 + \beta_3 z^3 + \beta_4 E)$$

$$\dot{y} = \lambda_2 y + \epsilon(\acute{\lambda_2} y\chi_1 + \chi_2 y^3 + \chi_3 y^3 z)$$

$$\dot{z} = \lambda_3 z + \epsilon\left(\acute{\lambda_3} z\delta_1 + \delta_2 xyz + \delta_3 x^2 z\right) \quad (20)$$

Where, $\beta_1 = -1/(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)$

$$\beta_2 = -3/2\lambda_3(2\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3),$$

$$\beta_3 = -3/2\lambda_3(3\lambda_3 - \lambda_2)$$

$$\beta_4 = -1/(.03 - \lambda_2)(.03 - \lambda_3)$$

$$\chi_1 = -1/(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)$$

$$\chi_2 = -1/2\lambda_3(2\lambda_1 - \lambda_1)(3\lambda_2 - \lambda_3)$$

$$\chi_3 = -3/2\lambda_2(2\lambda_2 - \lambda_{3\_}\lambda_1)$$

$$\delta_1 = -1/(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)$$

$$\delta_2 = -6/(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_3)$$

$$\delta_3 = -3/2\lambda_3(2\lambda_3 - \lambda_2 + \lambda_1)$$

The first-order solution to equation (14) is

$$x(t, 0) = x + y + z + \varepsilon u_1, \quad (21)$$

Where $x, y,$ and z are given by (19) and $u_1$is given by (20).

III. Result and Discussion

This paper describes a novel methodology that combines Popov's expanded Krylov-Bolyubov-Mitropolsky (KBM) technique with the harmonic balance method. Moreover, the technique is intended to solve severely nonlinear, over-damped forced oscillatory systems. The goal is to produce more accurate solutions for such systems, especially those with significant nonlinearities.

To assess the accuracy of the perturbation technique approximation, a comparison is done with a numerical solution, which is considered the accuracy benchmark. The enhanced KBM method given in this paper is compared to the harmonic balancing (HB) method discussed in the article. MATLAB is used to construct all visual representations in the form of two-dimensional graphs. Figure 1(A) depicts the existing perturbation solution (represented by the dotted line) and the equivalent numerical solution (shown by the solid line) and they have been drawn using initial conditions $x(0) = 1.000, \dot{x}(0) = 0.000$ or $x_1 = 1.0000, \dot{y}_1 = -0.000416$ for $\varepsilon = 1.0$, $\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 sin2\tau)}$. $\lambda_1 = -.03$, $\lambda_2 = -8.\nu = -.03$, $E = 1.25$ and figure 1(B) shows previous solutions [VI] (A.7) (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $x(0) = 1.000, \dot{x}(0) = 0.000$ or $x_1 = 3.107038, \dot{y}_1 = -0.062795$ for $\varepsilon = 1.0$, $\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 sin2\tau)}$. $\lambda_1 = -.03$, $\lambda_2 = -8.\nu = -.03$, $E = 1.25$. Wherein, figure 1(Ai) illustrates the contrast between the error line and time for the current second-order technique when $\varepsilon = 1$. Figure 1(Bi) shows the comparison of the line of error vs. time of the previous method of second order when $\varepsilon = 1$.

The next figure, labeled as 2(A), displays the current perturbation solution (represented by a dotted line) and the matching numerical solution (shown by a solid line) and it is drawn using the initial conditions $x(0) = 1.000, \dot{x}(0) = 0.000$ or $x_1 = 1.0000, \dot{y}_1 = -0.006499$ for $\varepsilon = 1.2$, $\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 sin2\tau)}$. $\lambda_1 = -.03$, $\lambda_2 = -8,\nu = -.03$, $E = 1.25$, and figure 2(B) displays the preceding solutions (A.9) (represented by a dotted line) together with the matching numerical solution (shown by a solid line), both displayed with their respective initial conditions $x(0) = 1.000, \dot{x}(0) = 0.000$ or $x_1 = 3.528445, \dot{y}_1 = -0.069354$ for $\varepsilon = 1.2$, $\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 sin2\tau)}$. $\lambda_1 = -.03$, $\lambda_2 = -8.\nu = -.03$, $E = 1.25$. Figure 1(Ai) shows the comparison of the line of error vs. time of the present method of second order when $\varepsilon = 1.2$, and figure 2(Bi) the comparison of the line of error vs. time of the previous method of second order when $\varepsilon = 1.2$.

Figure 3(A) depicts the current perturbation solution (13) as a dotted line, together with the matching numerical solution as a solid line. These lines are shown based on the given initial conditions $x(0) = 1.000, \dot{x}(0) = 0.000$ or $x_1 = 1.0000, \dot{y}_1 = -0.012582$ for $\varepsilon = 1.4$, $\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 sin2\tau)}$. $\lambda_1 = -.03$, $\lambda_2 = -8,\nu = -.03$, $E = 1.25$ and the plot displays the prior solutions (A.9) represented by a dotted line and the equivalent numerical solution represented by a solid line. The map includes the initial conditions $x(0) = 1.000, \dot{x}(0) = 0.000$ or $x_1 = 3.949852, \dot{y}_1 = -0.075913$ for $\varepsilon = 1.4$, $\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 sin2\tau)}$. $\lambda_1 = -.03$, $\lambda_2 = -8.\nu = -.03$, $E = 1.25$. Figure 3(Ai) shows the comparison of the line of error vs. time of the present method of second order when $\varepsilon = 1.4$ and figure 3(Bi) the comparison of the line of error vs. time of the previous method of second order when $\varepsilon = 1.4$.

Figure 4(A) displays the perturbation solution (21) (represented by the dotted line) and the equivalent numerical solution (represented by the solid line) with their respective initial conditions $x(0) = 1.000,$ or $\dot{x}(0) = \dot{0}.000\ddot{x}(0) = \dot{-}0.1000x_1 = 0.87317, y_1 = -\dot{0}.678472, z_1 = -\dot{0}.441201$for $\varepsilon = 1.0$, $\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2\cos\tau + \zeta_3^2 sin2\tau)}$, $\lambda_1 = -.100,\ \lambda_2 = -8.00.\lambda_3 = -0.700.\nu = -.03,$ $E = 0.5$ and figure 4(B) displays the prior solutions (A.9) represented by a dotted line, together with the matching numerical solution represented by a solid line. These solutions are depicted with their respective initial conditions $xx(0) = 1.000,$ or $\dot{x}(0) = \dot{0}.000\ddot{x}(0) = \dot{-}0.1000x_1 = 0.873170, y_1 = -\dot{0}.174143, z_1 = -\dot{0}.307677$for $\varepsilon = 1.0$, $\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2\cos\tau + \zeta_3^2 sin2\tau)}$, $\lambda_1 = -.100,\ \lambda_2 = -8.00.\lambda_3 = -0.700.\nu = -.03,$ $E = 0.5$. Figure 4(Ai) shows the comparison of the line of error vs. time of the present method of third order when $\varepsilon = 1$, and figure 4(Bi) the comparison of the line of error vs. time of the previous method of third order when $\varepsilon = 1$.

Finally, figure 5(A) displays the perturbation solution (21) (represented by the dotted line) and the equivalent numerical solution (represented by the solid line) with their respective initial conditions $x(0) = 1.000,$ or $\dot{x}(0) = \dot{0}.000\ddot{x}(0) = \dot{-}0.1000x_1 = 0.881031, y_1 = -\dot{0}.737872, z_1 = -\dot{0}.561487$for $\varepsilon = 1.4$, $\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2\cos\tau + \zeta_3^2 sin2\tau)}$, $\lambda_1 = -.100,\ \lambda_2 = -10.00.\lambda_3 = -0.700.\nu = -.03,$ $E = 0.5$ and figure 5(B) displays the current solutions (A.9) with the dotted line representing the present solutions and the solid line representing the equivalent numerical solution. These solutions are presented with their respective initial conditions $x(0) = 1.000,$ or $\dot{x}(0) = \dot{0}.000\ddot{x}(0) = \dot{-}0.1000x_1 = 0.881031, y_1 = -\dot{0}.550949, z_1 = -\dot{0}.421256$for $\varepsilon = 1.0$, $\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2\cos\tau + \zeta_3^2 sin2\tau)}$, $\lambda_1 = -.100,\ \lambda_2 = -8.00.\lambda_3 = -0.700.\nu = -.03,$ $E = 0.5$. Figure 5(Ai) shows the comparison of the line of error vs. time of the present method of third order when $\varepsilon = 1.4$, and figure 5(Bi) the comparison of the line of error vs. time of the previous method of third order when $\varepsilon = 1.4$. From Figures 1(A), 2(A), 3(A), 4(A), and 5(A) we can see that the perturbation solutions agree very well with the numerical result, but in these situations figures 1(B), 2(B), 3(B), 4(B) and 5(B) disagree, and the proposed solution does not obtain the expected outcome.

With the use of all these figures, we can also demonstrate that the current approach yields results that are superior to those of the previous approach.

Table 1: Error analysis evaluation of numerical data (exact) with the present method and the unified method is given several times.

t	x_{Exact}	$x_{present\ method}$	$Er\%$	$x_{unified\ method}$	$Er\%$
0	1	1	0	3.107038	0
.1	1.000062	1.000018	0.0044	3.2006	-6.56402
10	0.922487	0.920351	0.232085	16.41663	-94.3189
20	0.82549	0.823658	0.222422	26.92653	-96.9329
30	0.736552	0.734962	0.216338	28.61504	-97.4259
40	0.655338	0.653967	0.209644	24.3449	-97.3081
50	0.58116	0.579987	0.202246	18.15376	-96.7987
70	0.45155	0.450713	0.185706	8.117482	-94.4373
90	0.343859	0.343286	0.166916	3.224452	-89.3359
100	0.297364	0.296899	0.156619	2.038033	2.038033

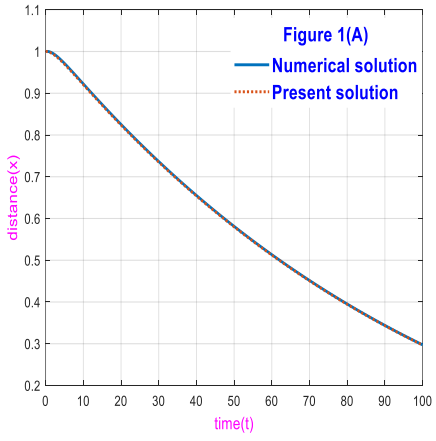

Fig. 1(A). The comparison of the line of the numerical solution which is represented by the solid line and the present solution, which is represented by the dotted line when $\varepsilon = 1$.

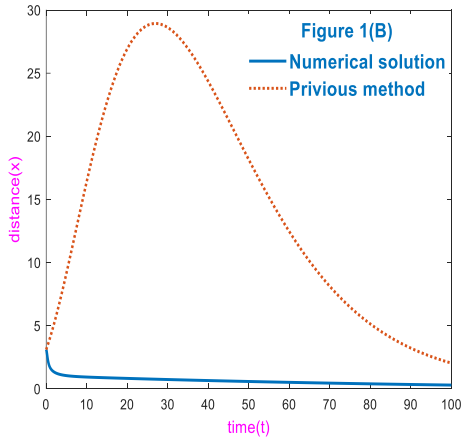

Fig. 1(B). The comparison of the line of the numerical solution, which is represented by the solid line, and the previous solution which is represented by the dotted line when $\varepsilon = 1$.

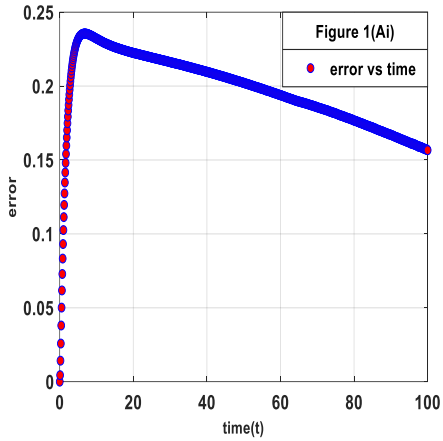

Fig. 1(Ai). The proposed approach involves comparing the line of error concerning the time when $\varepsilon = 1$.

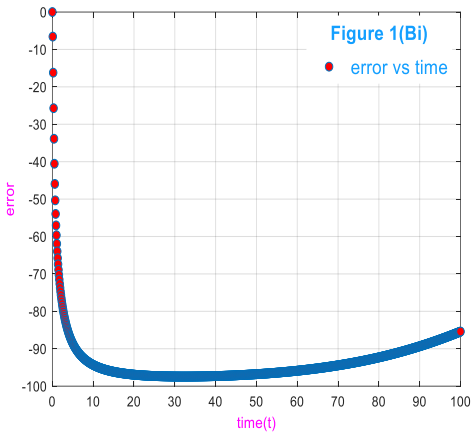

Fig. 1(Bi). The proposed approach involves comparing the line of error concerning the time when $\varepsilon = 1$.

Table 2: Error analysis evaluation of numerical data (exact) with the present method and the unified method is given several times.

t	x_{Exact}	$x_{present\ method}$	$Er\%$	$x_{unified\ method}$	$Er\%$
0	1	1	0	3.528445	0
.1	1.000062	1.000018	0.0044	3.2006	-6.56402
10	0.922487	0.920351	0.232085	16.41663	-94.3189
20	0.82549	0.823658	0.222422	26.92653	-96.9329
30	0.736552	0.734962	0.216338	28.61504	-97.4259
40	0.655338	0.653967	0.209644	24.3449	-97.3081
50	0.58116	0.579987	0.202246	18.15376	-96.7987
70	0.45155	0.450713	0.185706	8.117482	-94.4373
90	0.343859	0.343286	0.166916	3.224452	-89.3359
100	0.297364	0.296899	0.156619	2.038033	-85.4093

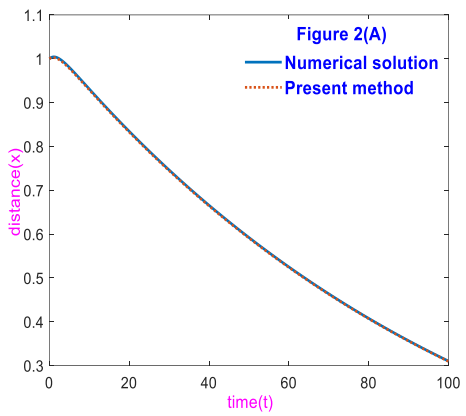

Fig. 2(A). The comparison of the line of the numerical solution, which is represented by the solid line, and the present solution, which is represented by the dotted line when $\varepsilon = 1.2$

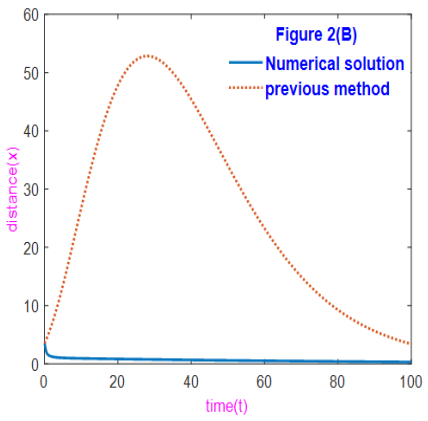

Fig. 2(B). The comparison of the line of the numerical solution, which is represented by the solid line, and the previous solution, which is represented by the dotted line when $\varepsilon = 1.2$

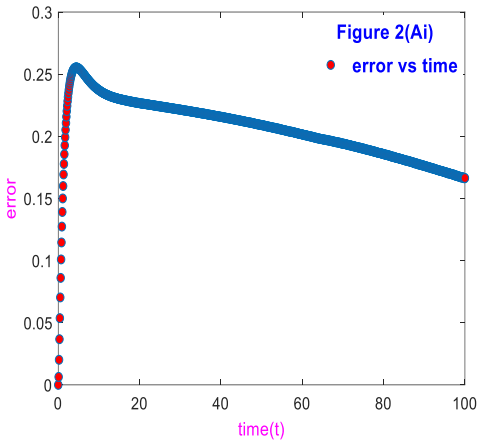

Fig. 2(Ai). The comparison of the line of error vs. time of the present method when $\varepsilon = 1.2$

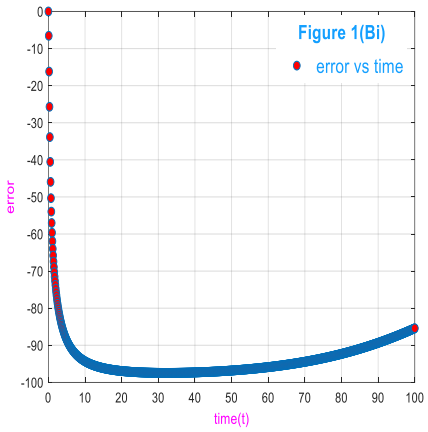

Fig. 2(Bi). The comparison of the line of error vs time of the previous method when $\varepsilon = 1.2$

Table 3: Error analysis evaluation of numerical data (exact) with the present method and the unified method is given several times.

t	x_{Exact}	$x_{present\ method}$	$Er\%$	$x_{unified\ method}$	$Er\%$
0	1	1	0	3.949852	0
.1	1.001279	1.001191	0.00879	4.139151	-12.2676
10	0.937069	0.934838	0.238651	41.50998	-97.7389
20	0.840857	0.838929	0.229817	79.37677	-98.9407
30	0.753088	0.751395	0.225314	89.67317	-99.1602
40	0.673055	0.671577	0.220079	78.41473	-99.1417
50	0.600021	0.598739	0.214117	59.00064	-98.983
70	0.472415	0.47147	0.200437	25.76858	-98.1667
90	0.366001	0.36533	0.18367	9.407728	-96.1096
100	0.319738	0.319182	0.174195	5.547051	-94.2359

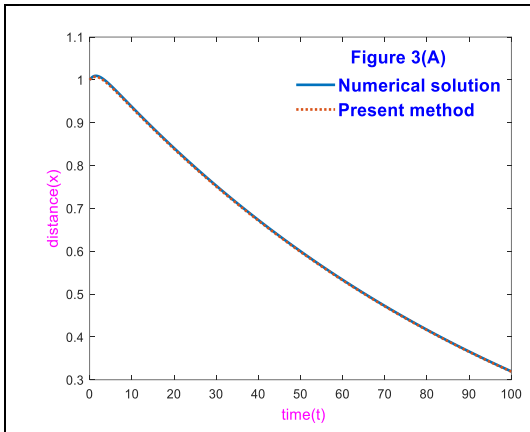

Fig. 3(A). The comparison of the line of the numerical solution, which is represented by the solid line, and the present solution, which is represented by the dotted line when $\varepsilon = 1.4$

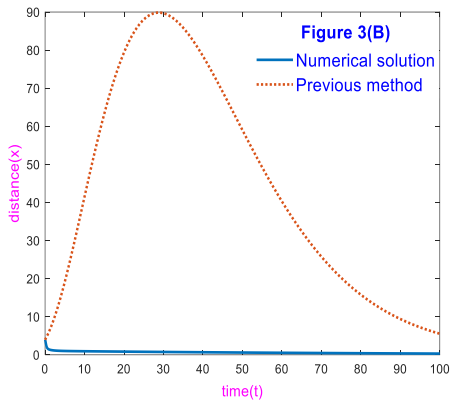

Fig. 3(B). The comparison of the line of the numerical solution, which is represented by the solid line, and the previous solution which is represented by the dotted line when $\varepsilon = 1.4$

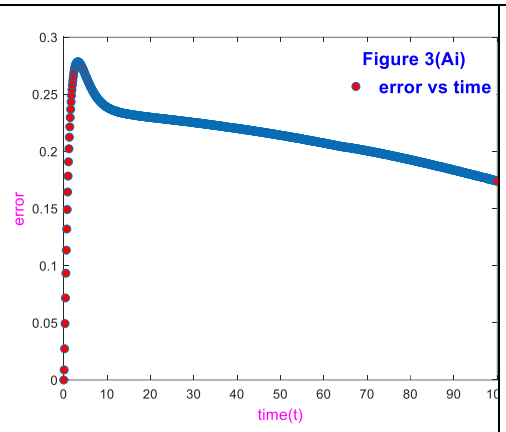

Fig. 3(Ai). The comparison of the line of error vs. time of the present method when $\varepsilon = 1.4$

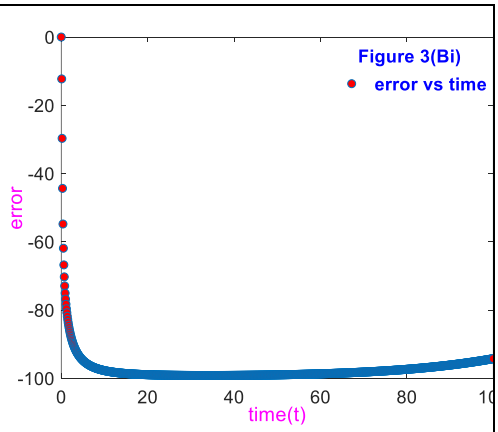

Fig. 3(Bi). The comparison of the line of error vs. time of the previous method when $\varepsilon = 1.4$

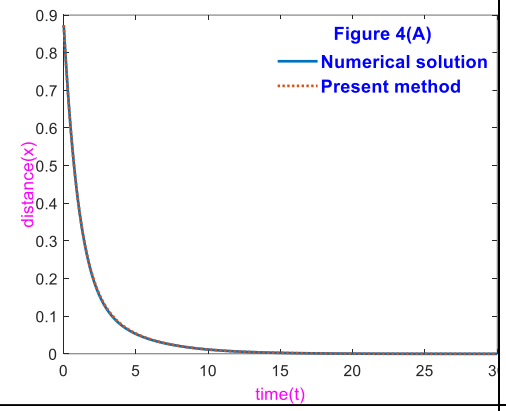

Fig. 4(A). The comparison of the line of the numerical solution, which is represented by the solid line, and the present solution, of third order which is represented by the dotted line when $\varepsilon = 1$

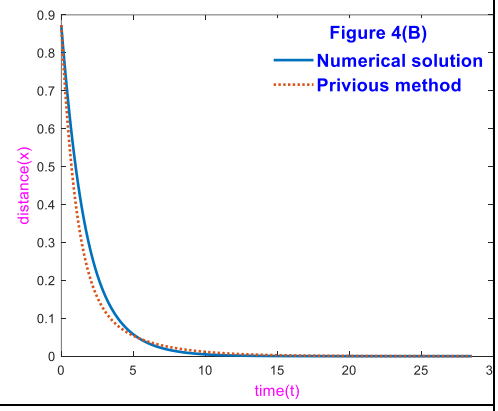

Fig. 4(B). The comparison of the line of the numerical solution, which is represented by the solid line, and the previous solution, of third order which is represented by the dotted line when $\varepsilon = 1$

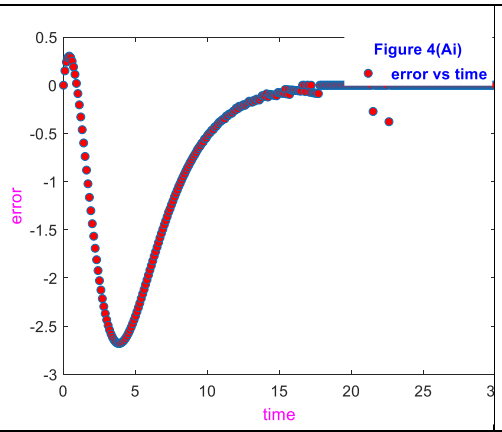

Fig. 4(Ai). The comparison of the line of error vs. time of the present method of third order when $\varepsilon = 1$

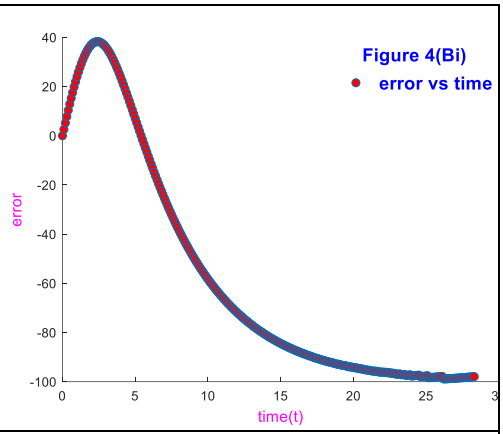

Fig. 4(Bi): The comparison of the line of error vs. time of the previous method of third order when $\varepsilon = 1$

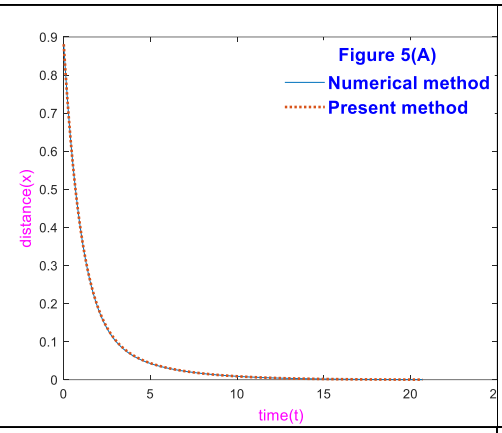

Fig. 5(A). The comparison of the line of the numerical solution, which is represented by the solid line, and the present solution, of third order which is represented by the dotted line when $\varepsilon = 1.4$

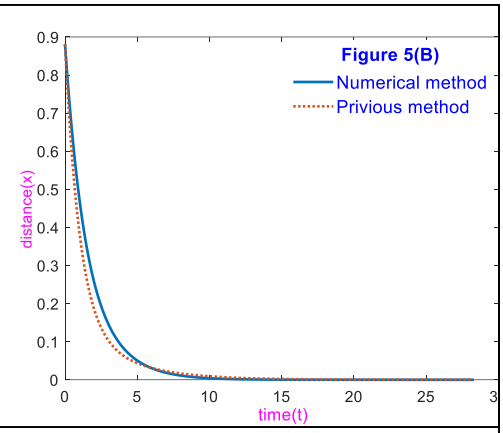

Fig. 5(B). The comparison of the line of the numerical solution, which is represented by the solid line, and the previous solution, of third order which is represented by the dotted line when $\varepsilon = 1.4$

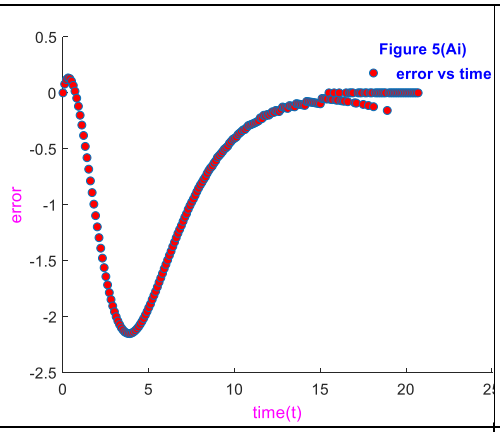

Fig. 5(Ai). The comparison of the line of error vs. time of the present method when $\varepsilon = 1.5$

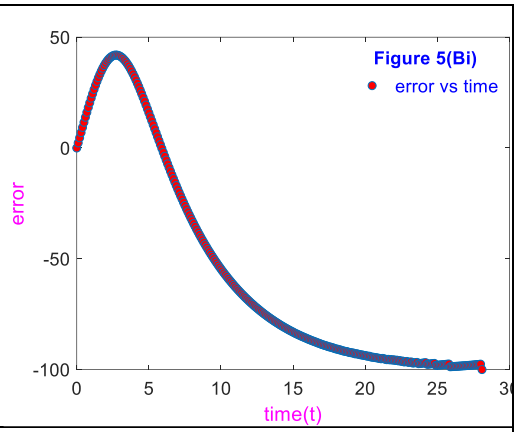

Fig. 5(Bi). The comparison of the line of error vs. time of the previous method of third order when $\varepsilon = 1.4$

IV. Conclusion

This study presents a novel approach to the problem of solving nonlinear, non-autonomous vibration systems with coefficients that are subject to progressive change. This technique is built on the foundations of the enlarged Krylov-Bolyubov-Mitropolsky (KBM) methodology as well as the harmonic balancing (HB) approach. Particularly in situations in which one eigenvalue is much bigger (far more than 250 times) than another, it is a tremendously successful method. When dealing with systems that are very nonlinear, the solutions that are generated via the use of this approach are quite similar to numerical solutions and provide improved accuracy.

In this article, we have shown two-dimensional graphs using the mathematical tool MATLAB to visually showcase the distinction between our solutions obtained by the perturbation technique and numerical solutions. These graphs clearly indicate that our technique is somewhat superior to numerical solutions, highlighting the practicality, dependability, and efficiency of our approach. Additionally, several graphs illustrate the variation in error over time. Despite this, it is essential to emphasize that this tactic does have several drawbacks. As long as the value of the parameter that expresses the rate of change is up to 1.4, it functions effectively. If this threshold is exceeded, mistakes occur, and these errors are serious.

V. Appendix A

Discussion of Shamsul's Unified Theory:

The author's selection of an approximate solution for equation (1)

$$x(t,\varepsilon) = a(t)e^{-\lambda t} + b(t)e^{-\mu t} + \varepsilon u_1(a,b,t) + \varepsilon^2 \ldots, \qquad \text{(A.1)}$$

where a and b satisfy the differential equations

$$\dot{a_1} = \varepsilon A_1(a,b,t) + \varepsilon^2 \ldots,$$

$$\dot{a_2} = \varepsilon B_1(a,b,t) + \varepsilon^2 \ldots, \qquad \text{(A.2)}$$

The equations

$$\left(\frac{\partial}{\partial t} - \lambda + \mu\right) A_1 e^{-\lambda t} + \left(\frac{\partial}{\partial t} - \lambda + \mu\right) B_1 e^{-\mu t} = -\left(3ab^2 e^{(\lambda+2\mu)t} + b^3 e^{-3\mu t}\right) \qquad \text{(A.3)}$$

When $\lambda \approx 3\mu$, (A3) separated into two following equations

$$\left(\frac{\partial}{\partial t} - \lambda + \mu\right) A_1 e^{-\lambda t} = -(b^3 e^{-3\mu t}) \qquad \text{(A.4)}$$

$$\left(\frac{\partial}{\partial t} - \lambda + \mu\right) B_1 e^{-\mu t} = -\left(3ab^2 e^{(\lambda+2\mu)t}\right) \qquad \text{(A.5)}$$

Therefore, A_1 and B_1 becomes

$$A_1 = b^3 e^{(\lambda-3\mu)t}/2\mu,$$

$$B_1 = 3ab^2 e^{(\lambda+\mu)t}/2\mu \qquad \text{(A.6)}$$

Thus B_1 does not contain a term, $\mu\rangle 0$. However, the above functions of A_1 and B_1 are valid if μ is small. The values of A_1 and B_1 from (A. 6) and then integrating with respect to t, we obtain,

$$\left(\frac{\partial}{\partial t} - \lambda + \mu\right)\left(\frac{\partial}{\partial t} - \lambda + \mu\right) u_1 = -\left(3a^2 b e^{(2\lambda+\mu)t} + a^3 e^{-3\lambda t}\right) \qquad \text{(A.7)}$$

The solution of (A.7) is

$$u_1 = -a^3 e^{-3\lambda t}/2\lambda(3\lambda - \mu) - 3a^2 b e^{-(2\lambda+\mu)}/2\lambda(\lambda + \mu) \qquad \text{(A.8)}$$

$$a = a_0 + b_0 \Big/ \left(1 + \varepsilon b_0^2 (e^{-2\mu t - 1})/\mu(3\mu - \lambda)\right)^{\frac{1}{2}}$$

$$b = b_0 \Big/ \left(1 + \varepsilon b_0^2 (e^{-2\mu t - 1})/\mu(3\mu - \lambda)\right)^{\frac{1}{2}}$$

Therefore, the first-order solution of (A. 1) is

$$x(t,\varepsilon) = a(t)e^{-\lambda t} + b(t)e^{-\mu t} + \varepsilon u_1 \qquad \text{(A.9)}$$

Where a and b are given by (A.6) and u_1 is given by (A.8).

VI. Acknowledgements

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Conflict of Interest:

The author declares that there is no conflict of interest regarding this paper.

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