JOURNAL OF MECHANICS OF CONTINUA AND MATHEMATICAL SCIENCES
www.journalimcms.org

ISSN (Online) : 2454 -7190 Vol.-19, No.-3, March (2024) pp 51-64 ISSN (Print) 0973-8975

OSCILLATORY BEHAVIOR OF SOLUTIONS OF FRACTIONAL MATRIX DIFFERENTIAL EQUATIONS

N. Sasikala [1], V Sadhasivam [2]

[1] Research Scholar, Post Graduate and Research Department of Mathematics, Thiruvalluvar Government Arts College, Rasipuram - 637 401, Namakkal, Tamil Nadu, India.

[2]Associate Professor, Post Graduate and Research Department of Mathematics, Thiruvalluvar Government Arts College, Rasipuram - 637 401, Namakkal, Tamil Nadu, India.

[1]krishivyattish@gmail.com; [2]ovsadha@gmail.com

Corresponding Author: **V. Sadhasivam**

https://doi.org/10.26782/jmcms.2024.03.00004

(Received: January 08, 2023; Revised: February 11, 2024; Accepted: March 10, 2024)

Abstract

In this article, new oscillation criteria for the second-order self-adjoint Matrix differential equations by using the Riccati technique are obtained. A suitable example is given to illustrate the significance and effectiveness of the result.

Keywords: Matrix Differential equations, oscillation, selfadjoint, damping.

I. Introduction

A Matrix differential equation is a mathematical equation in which the unknown is a matrix of functions appearing in the equation together with its derivative. In the last few years, the interest in Matrix differential equations has grown more and more in the field of mathematics. The oscillation theory of matrix differential equations was first studied by H.C. Howard in 1965 [VII]

$$Y''(x) + P(x)Y(x) = 0 \tag{1}$$

In 1991, Coles [V] introduced the self-adjoint in the second derivative.

$$(P(t)Y')' + Q(t)Y = 0 \tag{2}$$

Later, the oscillatory behavior of matrix differential equations was discussed by a few authors in the literature (see references [VIII, XVII, XXII]). Motivated by Philos, even more authors [XXV, XXVI, XXVII] acquired sufficient conditions for the oscillation of Matrix differential equations with damping.

$$(P(t)X'(t))' + R(t)X'(t) + Q(t)X(t) = 0 \tag{3}$$

There has been a significant number of publications on the oscillation theory of matrix linear differential equations [XII, XIV, XVIII] and extended their results in non-linear cases as in references [VI, XVII, XXII]. In 2005, Wan-Tong Li [XIII] studied the matrix differential equation by introducing the forcing term

$$[r(t)Y'(t)]' + p(t)Y'(t) + Q(t)G\big(Y'(t)\big)F\big(Y(t)\big) = e(t)I_n \tag{4}$$

Basci [I], has used $H(t,\tau)k(\tau)$ as opposed to $H(t,\tau)$ and established some new Kamenev-type oscillation criteria for the second order matrix differential system with damping. Yan Cong Xu [XXIV], obtained some new results with the use of the non-negative linear function and the generalized Riccatti technique. Further, Kumari and Umamaheswaram [IX] have also obtained the oscillation condition for a linear Hamiltonian matrix system. There are numerous applications of matrix differential equations both in mathematics and other field of Engineering. One of the physics-related applications is the study of electrical circuits, quantum mechanics, and optics. Many authors have expressed interest in studying the applications of matrices as in example [II, XX].

There are several definitions for fractional derivatives and integrals like Riemann Liouville, Caputo, Atangana Hadamard, etc. Riemann Liouville and Caputo derivatives are two common types of fractional derivative formulations that are frequently used to examine the properties of fractional order. A pivotal extension of calculus is the Riemann Liouville fractional derivative. This fractional derivative property has an extra few holdings when compared to a normal classical derivative. There is a great deal of analysis that leads to the conclusion of the Riemann Liouville derivative which is a more accurate option for fractional modeling. Numerous analyses lead to the conclusion that Riemann Liouvville is a more accurate option for fractional order derivative and analysis. Many authors used Riemann Liouville in their fractional model and contributed a notable amount of literature on oscillation theory [III, IV, X, XI, XVI]. In [XV], Nandhakumaran et al. studied the oscillatory theory in the classical case, and V. Sadhasivam et al. [XXI] in fractional partial differential equations.

As far as the authors are aware, there hasn't been any research on Riemann Liouville fractional derivative with matrix differential equations in the form of

$$\frac{d}{dt}\big(R(t)D_+^{\alpha}Y(t)\big) + Q(t)D_+^{\alpha}Y(t) + P(t)F\left(\int_0^t (t-s)^{-\alpha}Y(s)ds\right)G\big(D_+^{\alpha}Y(t)\big) = 0,$$
$$t \ge t_0 > 0 \tag{5}$$

Here R, Y, P, Q, F, G satisfy the condition

$(C_1)R(t), Q(t)$ and $P(t)$ are $n \times n$ matrices with real-valued continuous functions on the interval $[t_0,\infty), R(t)$ is symmetric, positive definite matrix and $Q(t)$ is symmetric for $t_0 \ge 0$.

The following assumptions are

(C_2) $F: M_n \to M_n$ and M_n is the vector space of all $n \times n$ real symmetric matrix, F is continuously differentiable in $R^{n^2}. Y(t)F(Y(t))$ is positive definite for $\det Y(t) \neq 0$,

$F'(K(t)) > \mu I_n$ where μ is the constant and I_n is the identity matrix. F^{-1} exist for $K \neq 0$ and $F(K(t))^{-1}$ is positive definite.

$(C_3) G \in C\left(R^{n^2}, R^{n^2}\right)$ and $G(D_+^\alpha Y(t))$ is a continuous and positive definite matrix such that $LI_n > 0$ where L is a positive constant at I_n is the identity matrix.

A solution $Y(t)$ of (5) is said to be non-trivial if $\det Y(t) \neq 0$ for at least one $t \in [t_0, \infty)$. A non-trivial solution $Y(t)$ of (5) is known as prepared or self-conjugate satisfies

$$(D_+^\alpha Y(t))^*(R(t)D_+^\alpha Y(t)) = (R(t)D_+^\alpha Y(t))^*(D_+^\alpha Y(t)) \text{ for } t \in [t_0, \infty),$$

where for any matrix A, the transpose A is denoted by A^*.

A non-trivial prepared solution $Y(t)$ of (5) is said to be oscillatory if for every $t_0 \geq 0$ It is possible to find a $t_1 \geq t_0$ such that $\det Y(t) = 0$, otherwise it is nonoscillatory.

II. Preliminaries

This section proceeded by stating the Riemann Liouville definitions for differentiation and integration. Additionally, a lemma is introduced which connects Riemann Liouville's derivative and that of classical.

Definition 2.1. The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ with respect to ' t ' of a function $Y(t)$ is given by

$$D_+^\alpha Y(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} Y(s) ds \tag{6}$$

provided that the right hand is pointwise defined on $[t_0, \infty)$, Γ is the gamma function.

Definition 2.2. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $Y: [t_0, \infty) \to R^{n^2}$ on the half-axis $[t_0, \infty)$ is given by

$$I_+^\alpha Y(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Y(s) ds \quad \text{for } t \geq t_0 \tag{7}$$

Definition 2.3. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $Y: [t_0, \infty) \to R^{n^2}$ on the half-axis $[t_0, \infty)$ is given by

$$D_+^\alpha Y(t) := \frac{d^{\lceil \alpha \rceil}}{dt^{\lceil \alpha \rceil}} \left[I_+^{\lceil \alpha \rceil} Y(t) \right] \quad \text{for } t \geq t_0 \tag{8}$$

provided that the right-hand side is pointwise defined on $[t_0, \infty)$ where $\lceil \alpha \rceil$ is the ceiling function of α.

Lemma 2.4. Let $Y(t) = \left[y_{ij}\right]_{n \times n}$ $i, j = 1, 2, \ldots, n$ be the solution of (5) and $K(t) = \int_0^t (t-s)^{-\alpha} Y(s) ds$ where $K(t) = \left[k_{ij}\right]_{n \times n}$, then $K'(t) = \Gamma(1-\alpha) D_+^\alpha Y(t)$ for $\alpha \in (0,1]$ and $t \geq 0$.

Proof : Consider $K(t)=\int_0^t (t-s)^{-\alpha}Y(s)ds$

Taking ij^{th} component we get,

$$k_{ij}(t)=\int_0^t (t-s)^{-\alpha}y_{ij}(s)ds \; i,j=1,2,\dots,n$$

Taking differentiation on both sides

$$\frac{d}{dt}\left(k_{ij}(t)\right)=\frac{d}{dt}\int_0^t (t-s)^{-\alpha}y_{ij}(s)ds$$

$$=\Gamma(1-\alpha)\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}y_{ij}(s)ds$$

$$=\Gamma(1-\alpha)\left[\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\int_0^t (t-s)^{\lceil\alpha\rceil-\alpha-1}y_{ij}(s)ds\right]$$

$$k'_{ij}(t)=\Gamma(1-\alpha)D_{+}^{\alpha}y_{ij}(t)$$

which implies

$$\left(k'_{ij}(t)\right)_{n\times n}=\Gamma(1-\alpha)D_{+}^{\alpha}\left[y_{ij}(t)\right]_{n\times n}$$

Therefore $K'(t)=\Gamma(1-\alpha)D_{+}^{\alpha}Y(t)$.

III. Oscillation with Monotonicity

This section deals with the monotonicity of F, hence the following assumption is derived

(C_4) F is monotone, satisfying the condition $F(K(t))^{-1}\geq \mu I_n>0$, here μ is a constant and I_n is the identity matrix.

Theorem 3.1. Let $\rho=\rho(t,s)$ belong to a class X, indicated by $\rho\in X$ if $\rho\in C(D,[0,\infty])$ where $D=\{(t_0,s):-\infty<s\leq t<\infty\}$ satisfying

(i) $\rho(t,t)=0$ and $\rho(t,s)>0$ on D

(ii) ρ has a partial derivative $\frac{\partial\rho}{\partial t}$ and $\frac{\partial\rho}{\partial s}$ on D such that

$$\frac{\partial\rho}{\partial t}=h_1(t,s)\sqrt{\rho(t,s)} \tag{9}$$

and

$$\frac{\partial\rho}{\partial s}=-h_2(t,s)\sqrt{b(t,s)} \tag{10}$$

where $h_1,h_2\in L_{loc}(D,R)$

Assume that condition $(C_1)-(C_4)$ holds.

(a) Suppose Y be the non-negative solution of (5) on $[c,b)$. To any $h \in C'([t_0,\infty),(0,\infty))$. Let

$$U(t) = v(t)R(t)[D_+^\alpha Y(t)[F(K(t)]^{-1} + h(t)I_n], t \geq t_0 \text{ on } [c,b) \quad (11)$$

Next, for any $\rho \in X$

$$\int_c^b \rho(b,s)\Phi(s)ds \leq \rho(b,c)U(c) + \frac{1}{4\mu\Gamma(1-\alpha)}\int_c^b v(s)R(s)\left[h_2(b,s)I_n + \sqrt{\rho(b,s)}\right. \\ \left.(Q(s)R^{-1}(s) + 2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n)\right]^2 ds \quad (12)$$

(b) In the same way, suppose Y be the nonnegative solution of (5) on $(a,c]$. If any $h \in C'([t_0,\infty),(0,\infty))$ specify as said above. To any $\rho \in X$

$$\int_a^b \rho(s,a)\Phi(s)dx \leq -\rho(c,a)U(c) + \frac{1}{4\mu\Gamma(1-\alpha)}\int_a^c v(s)R(s)\left[h_1(s,a)I_n - \sqrt{\rho(s,a)}\right. \\ \left.(Q(s)R^{-1}(s) + 2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n)\right]^2 ds \quad (13)$$

Defining v and Φ as

$$v(t) = \exp\left[-2\mu\int_{t_0}^t h(s)ds\right]$$

$$\Phi(t) = v(t)[LP(t) - h(t)Q(t) - \mu\Gamma(1-\alpha)h^2(t)R(t) - (h(t)R(t))']$$

Proof : By substituting the value of $K(t) = \int_0^t (t-s)^{-\alpha}Y(s)ds$. (1) becomes

$$\frac{d}{dt}(R(t)D_+^\alpha Y(t)) + Q(t)D_+^\alpha Y(t) + P(t)F\big(K(t)\big)G\big(D_+^\alpha Y(t)\big) = 0 \quad (14)$$

From the equation (11) and (14) for $s \in [c,b)$,

$$U'(t) = -2\mu v(t)h(t)[R(t)(D_+^\alpha Y(t)[F(K(t))]^{-1} + h(t)I)] \\ +v(t)[(R(t)D_+^\alpha Y(t)F^{-1}K(t))' + (h(t)R(t))']$$

By (C_1), (C_2) and (C_3)

$$U'(t) \leq -2\mu v(t)h(t)R(t)D_+^\alpha Y(t)[F(K(t))]^{-1} - 2\mu v(t)h^2(t)R(t) \\ -v(t)Q(t)D_+^\alpha Y(t)[F(K(t))]^{-1} - Lv(t)P(t) \\ -v(t)R(t)D_+^\alpha Y(t)[F(K(t))]^{-1}\mu[F(K(t))]^{-1}K'(t) + v(t)(h(t)R(t))'U'(t) \\ \leq \frac{-\mu\Gamma(1-\alpha)U^2(t)R^{-1}(t)}{v(t)} - U(t)[Q(t)R^{-1}(t) + 2\mu h(t)I_n \\ -\Gamma(1-\alpha)2\mu h(t)I_n] - v(t)[LP(t) - h(t)Q(t) - \mu\Gamma(1-\alpha)h^2(t)R(t) - (h(t)R(t))']U'(t) \\ \leq \frac{-\mu\Gamma(1-\alpha)U^2(t)R^{-1}(t)}{v(t)} - U(t)[Q(t)R^{-1}(t) + 2\mu h(t)I_n - 2\mu\Gamma(1-\alpha)h(t)I_n] - \Phi(t)$$

Then

$$\Phi(t) \leq -U'(t) - \frac{\mu\Gamma(1-\alpha)U^2(t)R^{-1}(t)}{v(t)} - U(t)[Q(t)R^{-1}(t) + 2\mu h(t)I_n \\ -2\mu\Gamma(1-\alpha)h(t)I_n] \quad (15)$$

Multiplying (15) by $\rho(t,s)$ and make use of (14) and (10) and integrate it with respect to s from c to t

$$\int_c^t \rho(t,s)\Phi(s)ds \le -\int_c^t \rho(t,s)U'(s)ds - \int_c^t \frac{\mu\Gamma(1-\alpha)\rho(t,s)\phi^2(s)R^{-1}(s)}{v(s)}ds$$
$$-\int_c^t \rho(t,s)U(s)[Q(s)R^{-1}(s)+2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n]ds$$

$$\int_c^t \rho(t,s)\Phi(s)ds \le -\rho(t,c)U(c) - \int_c^t h_2(t,s)\sqrt{\rho(t,s)}U(s)ds$$
$$-\int_c^t \frac{\mu\Gamma(1-\alpha)\rho(t,s)U^2(s)R^{-1}(s)}{v(s)}ds$$
$$-\int_c^t \rho(t,s)U(s)[Q(s)R^{-1}(s)+2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n]ds$$

$$\int_c^t \rho(t,s)\Phi(s)ds \le b(t,c)U(c) + \frac{1}{4\mu\Gamma(1-\alpha)}\int_c^t v(s)R(s)[h_2(t,s)I_n$$
$$+\sqrt{\rho(t,s)}(Q(s)R^{-1}(s)+2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n)]^2 ds$$

Taking $t \to b^-$in the inequality above

$$\int_c^b \rho(b,s)\Phi(s)ds \le \rho(b,c)U(c) + \frac{1}{4\mu\Gamma(1-\alpha)}\int_c^b v(s)R(s)[h_2(b,s)I_n$$
$$+\sqrt{\rho(b,s)}(Q(s)R^{-1}(s)+2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n]^2 ds$$

In the subsequent section multiply (15) by $\rho(s,t)$, make use of (9), (10) & integrate it with respect to s from c to t

$$\int_t^c \rho(s,t)\Phi(s)ds \le -\int_t^c \rho(s,t)U'(s)ds - \int_t^c \frac{\rho(s,t)\mu\Gamma(1-\alpha)U^2(s)R^{-1}(s)}{v(s)}ds$$
$$-\int_t^c \rho(s,t)U(s)[Q(s)R^{-1}(s)+2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n]ds$$

$$\int_t^c \rho(s,t)\Phi(s)ds \le -\rho(c,t)U(c) + \int_t^c h_1(s,t)\sqrt{\rho(s,t)}U(s)ds$$
$$-\int_t^c \frac{\mu\Gamma(1-\alpha)\rho(s,t)U^2(s)R^{-1}(s)}{v(s)}ds$$
$$-\int_t^\rho \rho(s,t)U(s)[Q(s)R^{-1}(s)+2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n]ds$$

$$\int_t^c \rho(s,t)\Phi(s)ds \le -\rho(c,t)U(c) + \frac{1}{4\mu\Gamma(1-\alpha)}\int_t^c v(s)R(s)[h_1(s,t)$$
$$-\sqrt{\rho(s,t)}(Q(s)R^{-1}(s)+2\mu h(s)I_n - 2\mu h(s)I_n)\Gamma(1-\alpha)]^2 ds$$

Taking $t \to a^{+}$in the inequality above

$$\int_a^c \rho(s,a)\Phi(s)ds \le -\rho(c,a)U(c) + \frac{1}{4\mu\Gamma(1-\alpha)}\int_a^c v(s)R(s)[h_1(s,a)I_n$$
$$-\sqrt{\rho(s,a)}(Q(s)R^{-1}(s) + 2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n]^2 ds$$

Hence the theorem is proved.

Theorem 3.2. Assume $(\mathrm{C}_1) - (\mathrm{C}_4)$ satisfies. Suppose that for some $c \in (a,b) \exists \rho \in X, h \in C'([t_0,\infty),(0,\infty))$ the next inequality is satisfied

$$\frac{1}{\rho(b,c)}\int_c^b \rho(b,s)\Phi(s)ds + \frac{1}{\rho(c,a)}\int_a^c \rho(s,a)\Phi(s)ds > \frac{1}{4\mu\Gamma(1-\alpha)}\int_c^b v(s)R(s)[h_2(b,s)I_n$$
$$+\sqrt{\rho(b,s)}(Q(s)R^{-1}(s) + 2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n]^2 ds$$
$$+\frac{1}{4\mu\Gamma(1-\alpha)}\int_a^c v(s)R(s)\big[h_1(s,a)I_n - \sqrt{\rho(s,a)}[Q(s)R^{-1}(s) + 2\mu h(s)I_n$$
$$-2\mu\Gamma(1-\alpha)h(s)I_n]^2 ds \tag{16}$$

where U, v, h_1, h_2, Φ and have been already defined, then each solution of (5) contains at least one zero in (a,b).

Proof: If not, suppose that without loss of generality, assume that (5) admit a solution $Y(t)$ such that $Y(t) > 0$ for $t \in (a,b)$. Take $c \in (a,b)$ then using the Theorem 3.1 we conclude (12) & (13) holds. By dividing (12) & (13) by $\rho(b,c)$ and $\rho(c,a)$ and their addition,

$$\frac{1}{\rho(b,c)}\int_c^b \rho(b,s)\Phi(s)ds + \frac{1}{\rho(c,a)}\int_a^c \rho(s,a)\Phi(s)ds$$
$$\le \frac{1}{4\mu\Gamma(1-\alpha)\rho(b,c)}\int_c^b v(s)R(s)[h_2(b,s)I_n$$
$$+\sqrt{\rho(b,s)}(Q(s)R^{-1}(s) + 2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n)^2\big]ds$$
$$+\frac{1}{4\mu\Gamma(1-\alpha)\rho(c,a)}\int_a^c v(s)R(s)[h_1(s,a)I_n$$
$$-\sqrt{\rho(s,a)}[(Q(s)R^{-1}(s) + 2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n)^2\big]ds$$

which leads to the contradiction to (16).

Theorem 3.3. Assume that $(\mathrm{C}_1) - (\mathrm{C}_4)$ holds. If for every $T \ge t_0$, symbol $\rho \in X, h \in C'([t_0,\infty),(0,\infty))$ and $a,b,c \in \mathbb{R}$ so that $T \le a < c < b$ and equation (16) satisfies, then every solution of (5) is oscillatory.

Proof: Consider a sequence $\{T_i\} \subset [t_0,\infty)$ so that $T_i \to \infty$ for i tends to ∞. By our assumption for $i \in \mathbb{N}$, the symbol a_i, b_i, c_i such that $T \le a_i < c_i < b_i$ and equation (16) holds. By the previous theorem, each solution $Y(t)$ has to contain at least one zero in (a_i, b_i). Letting T_i tends to ∞, we conclude that each solution of (5) is oscillatory.

Theorem 3.4. Suppose that condition $(C_1) - (C_4)$ satisfies. Assume that, $\exists \rho \in X, h \in C'([t_0, \infty), (0, \infty))$ such that if any $t_1 \geq t_0$

$$\limsup_{t\to\infty} \int_{t_1}^{t} \Big\{\Big(\rho(s,t_1)\Phi(s) - \frac{1}{4\mu\Gamma(1-\alpha)} v(s)R(s)\left[h_1(s,t)I_n - \sqrt{\rho(s,t_1)}(R^{-1}(s)Q(s) + 2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n)\right]^2\Big\} ds > 0 \quad (17)$$

and

$$\limsup_{t\to\infty} \int_{t_1}^{t} \Big\{\Big(\rho(t,s)\Phi(s) - \frac{1}{4\mu\Gamma(1-\alpha)} v(s)R(s)\big[h_2(t,s)I_n + \sqrt{\rho(t,s)}(R^{-1}(s)Q(s) + 2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n)\big]^2\Big\} ds > 0 \quad (18)$$

here $v(t)$ and $\Phi(t)$ were previously defined in Theorem 3.1. Then each solution of (5) is oscillatory.

Proof : For any $T \geq t_0$, sub $a = T$. In (17), let $t_1 = a$ then $\exists c > a$ like that

$$\int_{a}^{c} \Big\{\Big(\rho(s,a)\Phi(s) - \frac{1}{4\mu\Gamma(1-\alpha)} v(s)R(s)\big[h_1(s,a)I_n - \sqrt{\rho(s,a)}(R^{-1}(s)Q(s) + 2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n)\big]^2\Big\} ds > 0 \quad (19)$$

In (18), let $t_1 = c$. Then there exists $b > c$ such that

$$\int_{c}^{b} \Big\{\Big(\rho(b,s)\Phi(s) - \frac{1}{4\mu\Gamma(1-\alpha)} v(s)R(s)\big[h_2(b,s)I_n + \sqrt{\rho(b,s)}(R^{-1}(s)Q(s) + 2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n)\big]^2\Big\} ds > 0 \quad (20)$$

Divide the (19) by $\rho(c,a)$ and (20) by $\rho(b,c)$ and by adding

$$\frac{1}{\rho(c,a)} \int_{a}^{c} \rho(s,a)\Phi(s)ds + \frac{1}{\rho(b,c)} \int_{c}^{b} \rho(b,s)\Phi(s)ds > \frac{1}{4\mu\Gamma(1-\alpha)\rho(c,a)} \int_{a}^{c} v(s)R(s)[h_1(s,a)I_n - \sqrt{\rho(s,a)}(Q(s)R^{-1}(s) + 2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n)]^2 ds + \frac{1}{4\mu\Gamma(1-\alpha)\rho(b,c)} \int_{c}^{b} v(s)R(s)\big[h_2(b,s)I_n + \sqrt{\rho(b,s)}(Q(s)R^{-1}(s) + 2\mu h(s)I_n - 2\mu\Gamma(1-\alpha)h(s)I_n)]^2 ds$$

By following Theorem 3.3, then each solution of (5) is oscillatory.

IV. Oscillation without monotonicity

This section deals with the non-monotonicity of F, In the following assumption.

(C_5) F is non-monotone, satisfying the condition $F(K(t))K^{-1}(t) \geq \gamma I_n > 0$ where γ is a constant and I_n is the identity matrix.

Theorem 4.1. Assume $(C_1)-(C_3)$ and (C_5) holds.

(i) Let Y be a solution of (5) such that $Y(t)>0$ in $[c,b)$. To any $h \in C'([t_0,\infty),(0,\infty))$. Take

$$\chi(t) = v(t)R(t)[D_+^{\alpha}Y(t)K^{-1}(t)+h(t)I_n],\ t \geq 0 \tag{21}$$

on $[c,b)$. Then each $\rho \in X$,

$$\int_c^b \rho(b,s)\Phi(s)ds \leq \rho(b,c)\chi(c) + \frac{1}{4\Gamma(1-\alpha)}\int_c^b v(s)R(s)\big[h_2(b,s)I_n + \sqrt{\rho(b,s)} \\ (Q(s)R^{-1}(s)+2h(s)I_n - 2\Gamma(1-\alpha)h(s)I_n)]^2 ds \tag{22}$$

(ii) Let Y be a solution of (5) such that $Y(t)>0$ on $(a,c]$ and χ be mentioned as above. That each $\rho \in X$,

$$\int_a^c \rho(s,a)\Phi(s)ds \leq -\rho(c,a)\chi(c) + \frac{1}{4\Gamma(1-\alpha)}\int_a^c v(s)R(s)\big[h_1(s,a)I_n - \sqrt{b(s,a)} \\ (Q(s)R^{-1}(s)+2h(s)I_n - 2\Gamma(1-\alpha)h(s)I_n)]^2 ds \tag{23}$$

where $v(t) = \exp\left[-2\int_{t_0}^{t} h(s)ds\right]$ and

$\Phi(t) = v(t)[L\gamma P(t) - Q(t)v(t) - \Gamma(1-\alpha)h^2(t)R(t) - (h(t)R(t))']$.

Theorem 4.2. Suppose, $(C_1)-(C_3)\&(C_5)$ holds. Assume that for some $c \in (a,b)\exists P \in X$, $h \in C'([t_0,\infty),(0,\infty)]$ the next inequality is satisfied

$$\frac{1}{\rho(c,a)}\int_a^{\rho} b(s,a)\Phi(s)ds + \frac{1}{\rho(b,c)}\int_c^b \rho(b,s)\Phi(s)ds \\ > \frac{1}{4\Gamma(1-\alpha)}\frac{1}{\rho(c,a)}\int_a^c v(s)R(s)\big[h_1(s,a)I_n - \sqrt{\rho(s,a)}(Q(s)R^{-1}(s)+2h(s)I_n \\ -2\Gamma(1-\alpha)h(s)I_n]^2 ds + \frac{1}{4\Gamma(1-\alpha)}\frac{1}{\rho(b,c)}\int_{\rho}^b v(s)R(s)[h_2(b,s)I_n \\ +\sqrt{\rho(b,s)}[Q(s)R^{-1}(s)+2h(s)I_n - 2\Gamma(1-\alpha)h(s)I_n)\big]^2 ds \tag{24}$$

where U, v, h_1, h_2, Φ has been already defined, then each solution of (5) contains at least one zero in (a,b).

Theorem 4.3. Suppose that condition $(C_1)-(C_3)$ and (C_5) satisfies. If for every $T \geq t_0, \exists\ b \in X, h \in C'([t_0,\infty),(0,\infty))$ and $a,b,c, \in \mathbb{R}$ so that $T \leq a < c < b$ and (24) satisfies, then each solution of (5) is oscillatory.

Theorem 4.4. Assume that condition $(C_1),(C_2)$ and $(C_3),(C_5)$ holds. Suppose that there exist $\rho \in X, h \in C'([t_0,\infty),(0,\infty))$ such that for any $t_1 \geq t_0$.

$$\limsup_{t\to\infty}\int_{t_1}^{t}\Big\{\Big(\rho(s,t_1)\Phi(s) - \frac{1}{4\Gamma(1-\alpha)}v(s)R(s)\Big[h_1(s,t_1) - \sqrt{\rho(s,t_1)}(Q(s)R^{-1}(s) \\ + 2h(s)I_n - 2\Gamma(1-\alpha)I_n)]^2\Big\}ds > 0$$

and

$$\lim_{t\to\infty}\sup\int_{t_1}^{t}\left\{\left(\rho(t,s)\Phi(s)-\frac{1}{4\Gamma(1-\alpha)}v(s)R(s)\left[h_2(b,s)I_n-\sqrt{\rho(b,s)}Q(s)(R^{-1}(s)\right.\right.\right.$$
$$\left.+2h(s)I_n-2\Gamma(1-\alpha)h(s)I_n\right]^2\}ds>0$$

then each solution of (6) is oscillatory.

Corollary 4.5. Assume C_1-C_3 and (C_5) holds. If for every $T\geq t_0$ and $\lambda>1$ then $\exists$a function $b\in X, h\in C'([T_0,\infty),(0,\infty))$ such that

$$\lim_{t\to\infty}\frac{1}{t^{\lambda-1}}\int_{t_1}^{t}(s-t_1)^{\lambda}\left\{\Phi(s)-\frac{1}{4\Gamma(1-\alpha)}v(s)R(s)\right.$$
$$\left.\left(\frac{\lambda}{(s-t_1)}-(Q(s)R^{-1}(s)+2h(s)I_n-2\Gamma(1-\alpha)h(s)I_n)\right)^2\right\}ds>0$$

and

$$\lim_{t\to\infty}\frac{1}{t^{\lambda-1}}\int_{t_1}^{t}(t-s)^{\lambda}\left\{\Phi(s)-\frac{1}{4\Gamma(1-\alpha)}v(s)R(s)\right.$$
$$\left.\left(\frac{\lambda}{(t-s)}-(Q(s)R^{-1}(s)+2h(s)I_n-2\Gamma(1-\alpha)h(s)I_n)\right)^2\right\}ds>0$$

Then every solution of equation (5) is oscillatory.

V. Example

This section deals with an example to show the validity of the new result.

Example 5.1. Consider the fractional differential equation

$$\frac{d}{dt}(R(t)D_+^{\alpha}Y(t))+Q(t)D_+^{\alpha}Y(t)+P(t)F\big(K(t)\big)G\big(D_+^{\alpha}Y(t)\big)=0 \qquad (25)$$

Here $R(t)=\begin{bmatrix}t^4 & 0\\ 0 & t^4\end{bmatrix}, Q(t)=\begin{bmatrix}-4t^3 & 0\\ 0 & -4t^3\end{bmatrix}, Y(t)=\begin{bmatrix}\cos t & 0\\ 0 & \cos t\end{bmatrix}, \alpha=\frac{1}{3}$

$$P(t)=\frac{\left(\frac{\sqrt{3}}{2}t^4\sin t+\frac{t^4}{\cos t}\right)\sqrt{3}\Gamma\left(\frac{1}{3}\right)I_2}{\pi(\sqrt{3}\cos t-\sin t)\left(5+2\cos^2 t-\sqrt{3}\sin^2 t\right)}$$

$$F'(K(t)) = K'(t) = \Gamma(1-\alpha)D_+^\alpha Y(t) = \frac{\pi}{\Gamma\left(\frac{1}{3}\right)\sqrt{3}}(\sqrt{3}\cos t - \sin t)I_2$$

$$F'(K(t)) > \left[\frac{\pi}{\Gamma\left(\frac{1}{3}\right)\sqrt{3}}(\sqrt{3}-1)\right]I_2 \geq I_2 = \mu I_2$$

$$G(D_+^\alpha Y(t)) = I_2 + \left(D_+^{1/3}Y(t)\right) > I_2 = LI_2$$

$$\Phi(s) = v(s)[LP(s) - h(s)Q(s) - \Gamma(1-\alpha)h^2(s)R(s) - (h(s)R(s))']$$

$$v(s) = \frac{1}{s^2}, h(s) = \frac{1}{s}, Q(s) = -4s^3 I_2, R(s) = s^4 I_2$$

$$R^{-1}(s) = \frac{1}{s^4}I_2, \lambda = 2, n = 2$$

$$\Phi(s) = \frac{1}{s^2}\left[\left(\frac{\left(\frac{\sqrt{3}}{2}s^4\sin s + \frac{s^3}{2}\cos s\right)\sqrt{3}\Gamma\frac{1}{3}I_2}{\pi(\sqrt{3}\cos s - \sin s)\left(5 + 2\cos^2 s - \sqrt{3}\sin 2s\right)}\right) + 4s^2 I_2\right.$$

$$\left.-\Gamma\left(\frac{2}{3}\right)s^2 I_2 - (s^3 I_2)'\right]$$

Consider

$$\limsup_{t\to\infty}\frac{1}{t^{\lambda-1}}\int_{t_1}^{t}(s-t_1)^\lambda\left[\Phi(s) - \frac{1}{4\Gamma(1-\alpha)}v(s)R(s)\left(\frac{\lambda}{(s-t_1)}\right.\right.$$

$$\left.\left.-(Q(s)R^{-1}(s) + 2h(s)I_n - 2\Gamma(1-\alpha)h(s)I_n)\right)^2\right]ds$$

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_1}^{t}(s-t_1)^2\left\{\left[\left(\frac{\left(\frac{\sqrt{3}}{2}s^4\sin s + \frac{s^4}{2}\cos s\right)\sqrt{3}\Gamma(1/3)I_2}{\pi(\sqrt{3}\cos s - \sin s)\left(5 + 2\cos^2 s - \sqrt{3}\sin 2s\right)}\right) + 4s^2 I_2\right.\right.$$

$$\left.\left.-\Gamma\left(\frac{2}{3}\right)s^2 I_2 - (s^3 I_2)'\right] - \frac{1}{4\Gamma\left(1-\frac{1}{3}\right)}s^2 I_2\left(\frac{2}{(s-t_1)} - \left(\frac{-4}{5} + \frac{2}{5}I_2 - 2\Gamma\left(1-\frac{1}{2}\right)\frac{1}{s}I_2\right)\right)^2\right\}$$

$$> \limsup_{t\to\infty}\frac{1}{t}\int_{t_1}^{t}(s-t_1)^2\frac{1}{4\Gamma\left(\frac{2}{3}\right)}s^2\left[\frac{4s-2t_1}{s(s-t_1)}\right]^2 I_2 ds$$

$$> \limsup_{t\to\infty}\frac{1}{4\Gamma\left(\frac{2}{3}\right)}\int_{t_1}^{t}(4s-2t_1)^2 I_2 ds = \infty$$

and

$$\limsup_{t\to\infty}\frac{1}{t^{\lambda-1}}\int_{t_1}^{t}(t-s)^{\lambda}\left[\Phi(s)-\frac{1}{4\Gamma(1-\alpha)}v(s)R(s)\left(\frac{\lambda}{t-s}-(Q(s)R^{-1}(s)\right.\right.$$
$$\left.\left.+2h(s)I_n-2\Gamma(1-\alpha)h(s)I_n)\right)^2\right]ds$$

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_1}^{t}(t-s)^2\left[\Phi(s)-\frac{1}{4\Gamma\left(\frac{2}{3}\right)}s^2I_2\left(\frac{2}{t-s}-\left(\frac{-4}{s}I_2+\frac{2}{s}I_2-2\Gamma\left(1-\frac{1}{3}\right)\frac{1}{s}I_2\right)\right)^2\right]ds$$
$$>\limsup_{t\to\infty}\frac{1}{4\Gamma\left(\frac{2}{3}\right)}\frac{1}{t}\int_{t_1}^{t}(4s-2t_1^2)I_2ds=\infty$$

Hence, all conditions of Corollary 4.5 are satisfied. Therefore $Y(t)=\cos tI_2$ is oscillatory solution of (25).

V. Conclusions

In this article, the oscillation problem of the second-order self-adjoint matrix differential equations is obtained. Further, using the Riccati technique, new sufficient conditions for philos-type criteria for the matrix fractional equations are also established. To prove the result, the paper includes pertinent examples as and where needed. The obtained new result will be put forth to third order with impulse in the subsequent studies. The results obtained are fundamentally novel with enhanced and extended specific results.

Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

References

I. Basci,Y. Kamenev type oscillation criteria for second order matrix differential systems with damping, Hacettepe Journal of Mathematics and Statistics, volume 47(5) (2018), 1248-1267.

II. Bellmann, R. Introduction to Matrix Analysis, SIAM, Mc. Graw-Hill Book Company, New York, 1970.

III. Chatzarakis, G.E. Deepa,M. Nagajothi, N. Sadhasivam, V. On the oscillatory behavior of a class of mixed fractional order nonlinear differential equations, Pacific Journal of Applied Mathematics,(2018).

IV. Chatzarakis, G.E.Deepa, M.Nagajothi,N. Sadhasivam,V. Oscillatory properties of a class of mixed fractional differential equations., Applied Mathematics & Information Science, 14, No.1.109-117 (2020).

V. Coles, W.J. Oscillation for self adjoint second order matrix differential equations, Differential and integral equations, vol 4, no.1 (1991) pp 195-204.

VI. Cong, Xu. Yan. and Feng, M. Oscillation theorems for certain second order nonlinear Matrix differential systems, Journal of East China Normal University (Natural Science 2007) vol.2007, issue (5) pp 34-38

VII. Howard, H.C. Oscillation criteria for matrix differential equations., University of Wisconsin., Milwaukee, pp184-199.

VIII. Kreith, K. Oscillation criteria for nonlinear matrix differential equations, Proc. Amer. Maths.Soc, 26(1970) 270-272.

IX. Kumari, T.S. and Umamaheswaran,S. Oscillation criteria for linear matrix hamiltonian systems, J.Differential equations, vol 165(2000) pp 174-198.

X. Kumar, M.S.Deepa,M .Kavitha,J. Sadhasivam,V. Existence theory of fractional order three-dimensional differential system at resonance, Mathematical Modelling and Control, 2023;3(2):127-38.

XI. Kumar, M. S Ganesan, V. Asymptotic behavior of solutions of third-order neutral differential equations with discrete and distributed delay, Aims Mathematics, 5 (2020), pp.3851-3874.

XII. Liu,H. and Meng, F. Oscillation criteria for second order linear matrix differential systems with damping, Journal of computational and applied mathematics, (2009) 222- 229.

XIII. Tong Li, W. and Zhuang, Rong-kun Interval oscillation criteria for second order forced nonlinear matrix differential equations, Electronic Journal of Differential equations, vol (2005) no.69 pp1-6.

XIV. Meng, F. and man, cuiqin Oscillation results for linear second order matrix differential systems with damping, Applied Mathematics and computations, 187 (2007) 844-855.

XV. Nandakumaran, A.K. and.Panigrahi, S. Oscillation criteria for differential equations of second order, Mathematica slovoca, (59) (2009) no.4 pp 433-454.

XVI. Pan, Yuanyuan and Xu, Run. Some new oscillation criteria for a class of nonlinear fractional differential equations, Fractional Differential calculus, 6(1) (2016), 17-33.

XVII. Parhi, N. & Praharaj, N. Oscillation criteria for second order self adjoint matrix differential equations, Annales Polonici mathematics vol 72(1999) pp 1-14.

XVIII. Parhi,N.and Praharaj,P. Suffcient condition for oscillation of linear second order matrix differential systems., Rocky Mountain, Journal of Mathematics, Volume 32, Number 3, fall 2002.

XIX. Parhi, N.& Praharaj, N. Oscillation of nonlinear matrix differential equations, Math. Slovoca, 57 (2007) no.5 455-474.

XX. Pullman, N.J. Matrix theory and its applications, Marcel Dekker, 1976.

XXI. Sadhasivam,V. and Kavitha, J. Oscillation criteria for fractional partial differential equation with damping term, Applied Mathematics, 7 (2016), 272-291.

XXII. Wang, Qi-Ru Oscillation of self adjoint matrix differential systems, Applied Mathematics Letters 17 (2004) 1299-1305.

XXIII. Wenying shi, Shaoqin Gao and Wensheng Zhao, Oscillation results for nonlinear second order matrix differential systems with damping, International Journal of Pure and Applied Mathematics, vol 84. no.1 (2013) 1-12.

XXIV. Xu, Yan Cong and Zhu, De Ming. New oscillation criteria for second order linear matrix differential equations with damping, Acta mathematics Sinica English series (2008) vol.24, no.6, pp 925-936.

XXV. Yang, Qi Gui. Oscillation theorems of second order linear matrix differential systems with damping, Acta Mathematica Sinica English series, (2005), vol. 21, no.1, pp 17-30.

XXVI. Yang,Qui Gui. and cheng, Sui sun. Kamenev type oscillation criteria for second order matrix differential systems with damping , Annales polonici mathematics, 85(2005).

XXVII. Zheng, Z. Interval oscillation criteria of matrix differential systems with damping (2005).