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CONSTRUCTION OF A SPLINE FUNCTION WITH MIXED NODE VALUES

Rama Nand Mishra[1], Akhilesh Kumar Mishra[2], Kulbhushan Singh[3]

[1]Lecturer, Department of Mathematical and Statistical Science, Shri Ramswaroop Memorial University, Barabanki, U.P., India

[2]Associate Professor, Department of Mathematical and Statistical Science, Shri Ramswaroop Memorial University, Barabanki, U.P., India

[3]Professor, Department of Mathematical and Statistical Science, Shri Ramswaroop Memorial University, Barabanki, U.P., India

Email: [1]ramanandmishra.ids@srmu.ac.in, [2]akhileshmishra.maths@srmu.ac.in, [3]kulbhushan.maths@srmu.ac.in

Corresponding Author: **Kulbhushan Singh**

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Abstract

The present paper deals with the lacunary interpolation problem called the mixed values problem or (0, 3; 0, 2) problem for which known data points are function values at all the points, third derivatives at even knots, and second derivatives at odd knots of the unit interval I = [0,1]. For this problem, we obtained an interpolating function. The paper is divided into two parts, where we have shown that the spline function exists and is convergent.

Keywords: Lacunary interpolation, spline functions, Taylor expansion, modulus of continuity, error bounds, convergence of function.

I. Introduction

In Mathematics, a spline is a special function defined by piecewise polynomials. In interpolating problems, spline Interpolation is often preferred to polynomial interpolation because it yields similar results, even when using low polynomials. In the computer science subfield of computer-aided design and computer graphics, the term spline more frequently refers to a piecewise polynomial curve. Splines are popular curves because of the simplicity of their construction, their ease and accuracy of evaluation, and their capacity to approximate complex shapes through curve fitting and interactive curve design.

Spline functions are useful for the representation of parametric curves in both interpolatory and B-spline-like forms. Using given continuity conditions and interpolatory data some special types of spline are constructed. These special spline functions are used to construct, design, and control the shape of the curves. Different parameters in the description of splines can be used for various applications including design in CAD/CAM, font design, image outline capture, multi-resolution, description of motion paths for moving objects such as robots, data visualization, reverse engineering, curve or surface editing, object recognition, and many other engineering fields.

In this paper we have discussed (0, 2 ; 0, 3) lacunary interpolation problem where the function value is prescribed at every node of the partition of the unit interval [0, 1] whereas third and second derivatives are prescribed alternately at even and odd nodes. To solve this problem we construct a quantic spline function.

For more related work one is referred to [II], [V], [XII] [XIII]. Let us denote by $S^3_{n,5}$, the class of quintic splines s(x) on the unit interval [0, 1] such that

(i)s(x) $\in C^3$ [0, 1]

(ii)s(x) $\in \pi_5$ on each [v/n, (v +1)/n] , $0 \le v \le n-1$.

We shall prove the following theorems.

II. Theorem 1

For every odd integer n and even node, we take second derivatives at even nodes and third derivative at odd nodes $f_0, f_2, \dots, f_n$; $f_0'', f_2'', f_4'' \dots, f_{n-2}''$; $f_1''', f_3''', \dots, f_{n-1}'''$; f_0''' ; f_n' there exists a unique splines(x) $\in S_{n,5}^{(3)}$ such that

$$s(k/n) = f_k; \qquad k = 0,1,\dots,n, \tag{1.1}$$

$$s''(2k/n) = f''_{2k}; \qquad k = 0,1,\dots,(n-1)/2, \tag{1.2}$$

$$s'''((2k+1)/n) = f'''_{2k+1}; \quad k = 0,1,\dots,(n-1)/2. \tag{1.3}$$

$$s'(0) = f'_0, \; s'(1) = f'_n \tag{1.4}$$

III. Theorem 2

Let $f \in C^4$ [0,1] and n be an odd integer. Then for the unique quintic spline $S_n(x)$ satisfying conditions of Theorem 1 with

$f_k = f(k/n)$, k = 0,1,…,n,

$f''_{2k} = f''(2k/n)$, k = 0,1,…,(n–1)/2,

$f'''_{2k+1} = f'''(2k+1/n)$, k = 0,1,...,(n–1)/2,

$f'_0 = f'(0)$ and $f'_n = f'(1)$;

We have

$$\|S_n^r(x) - f^r(x)\|_\infty \le K_v n^{r-3} \omega_4\left(\frac{1}{n}\right) + 2n^{r-4}\|f^4\|_\infty \tag{2.1}$$

Here K_v are different constants depending on k and ω_4(.)denotes the modulus of continuity of $f^{(4)}$.

IV. Preliminaries

It can be verified that if P(x) is a quantic on [0,1] then

$P(x) = P(0)\, A_0(x) +.\, P(1)\, A_1(x) + P_0'(0)\, A_2(x) +$

$P'(1)\, A_3(x) + P''(0)\, A_4(x) + P'''(1) A_5(x).$ (3.1)

Where

$A_0(x) = 3/2\, x^5 – 5/2x^4 +1,$

$A_1(x) = 1/3\, (8x^5 – 25x^4 + 20\, x^3),$

$A_2(x) = 1/3\, (-5x^5+16\, x^4–14\, x^3+3x),$

$A_3(x) = – x^5+ 3x^4 - 2x^3,$

$A_4(x) = 1/6\, (-2x^5+7\, x^4–8\, x^3-3x^2),$

$A_5(x) = 1/18\, (x^5-2\, x^4+x^3).$

A quintic Q(x) on [1, 2] can be expressed as

$Q(x) = Q(2)\, A_0(2 – x) + Q(1)\, A_1(2 – x)$

$+ Q'(2)\, A_2(2 – x) + Q''(1)\, A_3(2 – x) +$

$+ Q'''(2)\, A_4(2 – x) + Q''(1)\, A_5(2– x).$ (3.2)

For later reference, we note that:

$A_0'(0) = 0,$	$A_0''(0) = 0,$	$A_0'''(0) = 0,$	$A_0^4(0) = -60,$	$A_0^5(0) = 180,$
$A_0'(1) = 5/2,$	$A_0''(1) = 0,$	$A_0'''(1) = 30,$	$A_0^4(1) = 60,$	$A_0^5(1) = 180,$
$A_1'(0) = 0,$	$A_1''(0) = 0,$	$A_1'''(0) = 0,$	$A_1^4(0) = -200,$	$A_1^5(0) = 320,$
$A_1'(1) = 0,$	$A_1''(1) = -20/3$	$A_1'''(1) = 0,$	$A_1^4(1) = 120,$	$A_1^5(1) = 320,$
$A_2'(0) = 1,$	$A_2''(0) = 0,$	$A_2'''(0) = -28,$	$A_2^4(0) = 128,$	$A_2^5(0) = -200,$
$A_2'(1) = 0,$	$A_2''(1) = 8/3,$	$A_2'''(1) = 0,$	$A_2^4(1) = -72,$	$A_2^5(1) = -200,$
$(A_3'(0) = 0,$	$A_3''(0) = 0,$	$A_3'''(0) = -12,$	$A_3^4(0) = 72,$	$A_3^5(0) = -120,$
$A_3'(1) = 1,$	$A_3''(1) = 4,$	$A_3'''(1) = 0,$	$A_3^4(1) = -48,$	$A_3^5(1) = -120,$
$A_4'(0) = 0,$	$A_4''(0) = -1,$	$A_4'''(0) = -8,$	$A_4^4(0) = 28,$	$A_4^5(0) = -40,$
$A_4'(1) = -2,$	$A_4''(1) = -5/3,$	$A_4'''(1) = 0,$	$A_4^4(1) = -12,$	$A_4^5(1) = -40,$
$A_5'(0) = 0,$	$A_5''(0) = 0,$	$A_5'''(0) = 1/3,$	$A_5^4(0) = -8/3,$	$A_5^5(0) = 20/3,$
$A_5'(1) = 0,$	$A_5''(1) = 1/9,$	$A_5'''(1) = 1,$	$A_5^4(1) = 4,$	$A_5^5(1) = 20/3,$

Equation (3.3)

A quintic P(x) in [0,1] can be expressed in the following form.

$$P(x) = P(0)\,B_0(x) + P(1)\,B_1(x) + P'(0)\,B_2(x) + P'(1)\,B_3(x) +$$

$$+ P''(0)\,B_4(x) + P'''(1)\,B_5(x) \qquad (3.4)$$

Where

$B_0(x) = 1/3(-8x^5 + 25x^4 - 20x^3)$,

$B_1(x)$ = 1/3(8x5 - 25x4 + 20x3),

$B_2(x) = 1/3(-5x^5 + 16x^4 - 14x^3 + 3x)$,

$B_3(x) = (-x^5 + 3x^4 - 2x^3)$,

$B_4(x) = 1/2\,(2x^5 - 3x^4 + 2x^2)$,

$B_5(x) = 1/6(-7x^5 + 6x^4 + x^3)$,

Also, a quintic Q (x) in [1, 2] can be written as

$$Q(x) = Q(2)B_0(2-x) + Q(1)B_1(2-x) - Q'(2)B_2(2-x) - Q'(1)B_3(2-x) -$$

$$Q'''(2)\,B_4(2-x) + Q''(1)\,B_5(2-x) \qquad (3.5)$$

For later reference we have

$B_0'(0)=0$	$B_0''(0)=0$	$B_0'''(0)=-120/3$	$B_0^{(4)}(0)=200$
$B_0'(1)=0$	$B_0''(1)=20/3$	$B_0'''(1)=0$	$B_0^{(4)}(1)=-120$
$B_1'(0)=0$	$B_1''(0)=0$	$B_1'''(0)=40$	$B_1^{(4)}(0)=-200$
$B_1'(1)=0$	$B_1''(1)=-20/3$	$B_1'''(1)=0$	$B_1^{(4)}(1)=120$
$B_2'(0)=1$	$B_2''(0)=0$	$B_2'''(0)=-28$	$B_2^{(4)}(0)=128$
$B_2'(1)=0$	$B_2''(1)=8/3$	$B_2'''(1)=0$	$B_2^{(4)}(1)=-72$
$B_3'(0)=0$	$B_3''(0)=0$	$B_3'''(0)=-12$	$B_3^{(4)}(0)=72$
$B_3'(1)=1$	$B_3''(1)=4$	$B_3'''(1)=0$	$B_3^{(4)}(1)=-48$
$B_4'(0)=0$	$B_4''(0)=2$	$B_4'''(0)=0$	$B_4^{(4)}(0)=-36$
$B_4'(1)=1$	$B_4''(1)=4$	$B_4'''(1)=24$	$B_4^{(4)}(1)=84$
$B_5'(0)=0$	$B_5''(0)=0$	$B_5'''(0)=1$	$B_5^{(4)}(1)=24$
$B_5'(1)=-4/3$	$B_5''(1)=-31/3$	$B_5'''(1)=-45$	$B_5^{(4)}(1)=-232$

(3.6)

Using equation (3.4) and (3.6) we have

$$P'''(0) = -120/3P(0) + 40P(1) - 28P'(0) - 12P'(1) + 0.P'''(0) + 1.P''(1) \quad (3.7)$$

$$P''(1) = 20/3P(0) - 20/3P(1) + 8/3P'(0) + 4P'(1) + 4.P'''(0) - 31/3P''(1) \quad (3.8)$$

$$P^4(0) = 200P(0) - 200P(1) + 128P'(0) + 72P'(1) - 36P'''(0) + 24P''(1) \quad (3.9)$$

$$P^4(1) = -120P(0) + 120P(1) - 72P'(0) - 48P'(1) + 84P'''(0) - 232P''(1) \quad (3.10)$$

Similarly from equation (3.5) and (3.6), we get

$$Q'''(2) = -120/3Q(2) + 4Q(1) - 28Q(2) - 12Q(1) + 0.Q(2) + 1.Q(1) \quad (3.11)$$

$$Q''(2) = -\frac{20}{3}Q(2) - \frac{20}{3}Q(1) + \frac{8}{3}Q(2) + 4Q(1) + 4Q(2) - 31/3Q(1) \quad (3.12)$$

$$Q^4(2) = 200Q(2) - 200Q(1) + 128Q(2) + 72Q(1) - 36Q(2) + 24Q(1) \quad (3.13)$$

$$Q^4(1) = 120Q(2) + 120Q(1) - 72Q(2) - 48Q(1) + 84Q(2) - 232Q(1) \quad (3.14)$$

Proof of Theorem 1

For a given s (x) $\in S_{n,5}^{(3)}$ set h = 1/n and

$M_v = s^{(4)}(vh+)$, $\quad v = 0, 1, \ldots, n-1$,

$N_v = s^{(4)}(vh-)$, $\quad v = 0, 1, \ldots, n$.

Since $S^{(4)}(x)$ is linear in each interval [vh, (v+1)h], it is completely determined by the 2n constants $\{M_v\}^{n-1}$ and $\{N_v\}_1^n$. Also if s(x) satisfies the requirements of theorem 1, it follows from equations (1.1)– (1.3) and (3.1) – (3.2) that for

$2vh \le x \le (2v+1)h$, $v = 0, 1, \ldots, (n-1)/2$, it must have the form v

$$S(x) = f_{2V} A_0\left(\frac{(2V+1)h-x}{h}\right) + f_{2V+1}A_0\left(\frac{(x-2Vh)}{h}\right) + h^2 f''_{2V} A_1\left(\frac{(x-2Vh)}{h}\right) + h^2 f''_{2V+1}A_2\left(\frac{(x-2Vh)}{h}\right) + h^4 M_{2V} A_3\left(\frac{(x-2Vh)}{h}\right) + h^4 N_{2V+1}A_4\left(\frac{(x-2Vh)}{h}\right) \quad (4.1)$$

$$S'(x) = f_0A_0(1) + f_1A'_0(0) + h^2 f''_0 A'_1(0) + h^2 f''_1 A'_2(0) + h^4 M_0 A'_3(0 + h^4 N_1 A'_4(0)$$

$$f'_0 = \frac{1}{120}\{(f_0 + f_1) + 40h^2 f''_0 + 60h^2 f''_1 + 16h^4 M_0 + 9h^4 N_1\}$$

$$\frac{h^4}{120}(16M_0 + 9N_1) = f'_0 - f_0 - f_1 + \frac{1}{3}h^2 f''_0 + \frac{1}{2}h^2 f''_1$$

For (2v +1) h ≤ 3)/2

$$S(x) = \quad f_{2V+1}A_0\left(\frac{(2V+2)h-x}{h}\right) + f_{2V+2}A_0\left(\frac{x-(2V+1)h}{h}\right) + h^2 f''_{2V+2}A_1\left(\frac{(2V+2)h-x}{h}\right) + h^2 f''_{2V+1}A_2\left(\frac{(2V+2)h)-x}{h}\right) + h^4 M_{2V+2}A_3\left(\frac{(2V-2)h-x}{h}\right) + h^4 N_{2V+1}A_4\left(\frac{(2V-2)h-x}{h}\right) \quad (4.2)$$

Since $S^4(0) = \ f_0^4$, therefore equation (1.1) implies

Putting x = (2v+1) h, v = 0,

$$S(x) = f_{2V}A_0(1) + f_{2V+1}A_0(0) + h^2 f''_{2V}A_1(0) + h^2 f''_{2V+1}A_2(0) + h^4 M_{2V}A_3(0) + h^4 N_{2V+1}A_4(0)$$

$$S^4(0) = f_0 A_0^4(1) + f_1 A_0^4(0) + h^2 f_0'' A_1^4(0) + h^2 f_1'' A_2^4(0) + h^4 M_0 A_4^4(0) + h^4 N_1 A_3^4(0)$$

$$S^4(0) = 60f_0 - 60f_1 - 200h^2 f_0'' + 128\, h^2 f_1'' + 28h^4 M_0 + 72h^4 N_1$$

$$f^4(0) = 60f_0 - 60f_1 - 200h^2 f_0'' + 128\, h^2 f_1'' + 28h^4 M_0 + 72h^4 N_1$$

$$28M_0 + 72N_1 = h^{-4}\left(f^4(0) - 60f_0 + 60f_1 + 200h^2 f_0'' - 128\, h^2 f_1''\right) \quad (4.3)$$

Similarly using conditions, $S^4(0) = \ f_0^4$ we have from equation (4.2)

$$S^4(1) = f_{n-1}A_0^4(0) + f_n A_0^4(1) + h^2 f''_{n-1}A_1^4(1) + h^2 f_n'' A_2^4(1) + h^4 M_{n-1}A_4^4(1) + h^4 N_n A_3^4(1)$$

$$S^4(1) = -60f_{n-1} + 60f_n + 120h^2 f''_{n-1} - 72\, h^2 f_n'' - 12h^4 M_{n-1} - 48h^4 N_n A_3^4(1)$$

As taking, $S^4(n) = f^4(n)$

$$f^4(n) = -60f_{n-1} + 60f_n + 120h^2 f''_{n-1} - 72\, h^2 f_n'' - 12h^4 M_{n-1} - 48h^4 N_n$$

$$12M_{n-1} + 48N_n = -h^{-4}\left[f^4(n) + 60f_{n-1} - 60f_n - 120h^2 f''_{n-1} + 72\, h^2 f_n''\right] \quad (4.4)$$

Also using, S ((2v+1) /h)−) =S ((2v +1)/ h +)

$$S(x) = S((2v+1)/h)-) = \quad f_{2V}A_0(0) + f_{2V+1}A_0(1) + h^2 f''_{2V}A_1(1) + h^2 f''_{2V+1}A_2(1) + h^4 M_{2V}A_3(1) + h^4 N_{2V+1}A_4(1)$$

$$S^4(x) = S^4((2v+1)/h)-) = -60f_{2V} + 60f_{2V+1} + 120h^2 f''_{2V} - 72\, h^2 f''_{2V+1} + 48h^4 M_{2V} - 12h^4 N_{2V+1} \quad (4.5)$$

And

$$S(x) = S^4((2v\ +1)\ /h\ )+$$
$$= f_{2V}A_0(1) + f_{2V+1}A_0(0) + h^2 f''_{2V} A_1(0) + h^2 f''_{2V+1} A_2(0)$$
$$+h^4 M_{2V} A_3(0) + h^4 N_{2V+1} A_4(0)$$

$$S^4(x = f_{2V} A_0^4(1) + f_{2V+1} A_0^4(0) + h^2 f''_{2V+1} A_1^4(0) + h^2 f''_{2V+1} A_2^4(0)$$
$$+ h^4 M_{2V} A_3^4(0) + h^4 N_{2V+1} A_4^4(0)$$

$$S^4(x = 60 f_{2V} - 60 f_{2V+1} - 200 h^2 f''_{2V+1} + 128\, h^2 f''_{2V+1} + 72 h^4 M_{2V} + 28 h^4 N_{2V+1} A_4^4(0) \tag{4.6}$$

From (4.5) and (4.6),

$$3M_{2V} + N_{2V+1} = h^{-4}[-3 f_{2V} + 3 f_{2V+1} + 8\, h^2 f''_{2V+1} - 5\, h^2 f''_{2V+1} \tag{4.7}$$

Similarly from ' S((2*v* +2)/h–) = s ' ((2*v* +2)/h +)

And

S^4((2*v* +2)/h–) = S^4((2*v* +2)/h +),

we get

$$S^4(2v+2)/h-)S^4(2v+2)/h-)$$
$$= f_{2V+1} A_0(0) + f_{2V+2} A_0(1) + h^2 f''_{2V+2} A_1(1)$$
$$+ h^2 f''_{2V+1} A_2(1) + h^4 N_{2V+2} A_3(1) + h^4 M_{2V+1} A_4(1)$$

$$S^4(x) = f_{2V+1} A_0^4(0) + f_{2V+2} A_0^4(1) + h^2 f''_{2V+2} A_1^4(1) +$$
$$h^2 f''_{2V+1} A_2^4(1) + h^4 N_{2V+2} A_3^4(1) + h^4 M_{2V+1} A_4^4(1)$$

$$S^4(x) = \quad -60 f_{2V+1} + 60 f_{2V+2} + 120 h^2 f''_{2V+2} - 72\, h^2 f''_{2V+1} - 48 h^4 N_{2V+2} - 12 h^4 M_{2V+1} \tag{4.8}$$

Similarly

$$S^4\text{((2v+2)/h+)} = f_{2V+1} A_0^4(1) + f_{2V+2} A_0^4(0) + h^2 f''_{2V+2} A_1^4(0) +$$
$$h^2 f''_{2V+1} A_2^4(0) + h^4 N_{2V+2} A_3^4(0) + h^4 M_{2V+1} A_4^4(0)$$

$$S^4(x) = 60 f_{2V+1} - 60 f_{2V+2} - 200 h^2 f''_{2V+2} + 128\, h^2 f''_{2V+1} + 72 h^4 N_{2V+2} + 28 M_{2V+1} \tag{4.9}$$

From (4.8) and (4.9)

$$120h^4 N_{2v+2} + 40M_{2v+1} = 120f_{2v+1} - 120f_{2v+2} - 320h^2 f''_{2v+2} + 200\, h^2 f''_{2v+1}$$

$$h^4[3N_{2v+2} + M_{2v+1}] = 3f_{2v+1} - 3f_{2v+2} - 8h^2 f''_{2v+2} + 5\, h^2 f''_{2v+1}$$

$$\left[3N_{2v+2} + M_{2v+1}\right] = h^{-4}[3f_{2v+1} - 3f_{2v+2} - 8h^2 f''_{2v+2} + 5\, h^2 f''_{2v+1}]$$

From these equations (4.5), (4.7)……, we see that constants M_n *and* N_n all are zero. Which completes the proof of theorem 1.

V. Lemma 1

Let f $\in C^4$ [0,1], n any odd integer and h = 1/n. Then for the unique spline $S_n(x)$ Theorem (1),

We have

$|A_{2v}| = |S_n(2vh) - f'(2vh)|$ (5.1)

$= O(h^2/18\ \omega_4(h))$, $v = 0,1,\ldots,(n-1)/2$

and

$|A_{2v+1}| = |S_n'((2v+1)h) - f'(2v+1)h)|$ (5.2)

$= O(h^2\,\omega 3(h))$, $v = 0,1,\ldots,(n-1)/2$

Where O are the different constants depending on *v*.

VI. Lemma 2

Let f $\in C^4$ [0, 1], n any odd integer and h = 1/n. Then for $s_n(x) = S_n(f,x)$ of theorem1, we have

$|S^4((2v+1)h) - f^4_{2v+1}| = O(\omega_4(h))$,

$|M_{2v} - N_{2v+1}| = O(\frac{1}{h}\omega_4(h))$, (5.3)

where O are different constants depending on *v*.

VI. Proof of Theorem 2

For $2vh \le x \le (2v+1)h$, $v = 0, 1,\ldots,(n-1)/2$, we have from equation (1.5)

$$s^{(4)}(x) = s^{(4)}(2vh)A_0\frac{((2v\ +1)\mathrm{h} - \mathrm{x})}{h} + s^{(4)}((2v+1)h)A_0\frac{(\mathrm{x} - 2vh)}{h} + h^2 s^{(5)}(2vh)A_1\frac{(\mathrm{x} - 2vh)}{h}$$

$$s^{(4)}(x) = f_{2\nu}A_0^{(4)}(1) + f_{2\nu+1}A_0^{(4)}(0) + +h^2 f''_{2\nu} A_1^{(4)}(0) + h^2 f''_{2\nu+1}A_2^{(4)}(0) + h^4 M_{2\nu}A_3^{(4)}(0) + h^4 N_{2\nu+1}A_4^{(4)}(0)$$

$$= 60 f_{2\nu} - 60 f_{2\nu+1} - 200\, h^2 f''_{2\nu+1} + 128\ h^2 f''_{2\nu+1} + 72\, h^4 M_{2\nu} + +28\, h^4 N_{2\nu+1} \tag{6.1}$$

Now from equation (1.5) and (P.3) we have

$$s^{(5)}(2\nu h +) = -h^{-1}(M_{2v} - N_{2v+1}).$$

Since

$$A_0\frac{((2\nu+1)h - x)}{h} + A_0\frac{(x-2\nu h)}{h} = 1,$$

We have

$$s^{(4)}(x) - f^{(4)}(x) = (s'''(2vh+) - f'''(x))\quad A_0\frac{((2v+1)h - x)}{h} +$$

$$+ (s'''((2v+1)h-) - f'''(x))\quad A_0\frac{(x-2vh)}{h} -$$

$$- h(M_{2v} - N_{2v+1})A_1\frac{(x-2vh)}{h}.$$

$$= I_1 + I_2 + I_3, \quad \text{say.} \tag{6.2}$$

Here $|A_0| \le 1$, $|A_1| \le 1$.

$$|I_1| = |(s'''(2\nu h +) - f'''(x)|$$

$$= |(s'''(2\nu h +) - f'''(2\nu h) + (x - 2\nu h)f^{(4)}(\alpha)|, \quad 2\nu h \le \alpha \le x$$

$$= |(s'''(2\nu h +) - f'''(2\nu h)| + h\Omega, \quad \text{where } \Omega = \|f^{(4)}\|_\infty$$

Or

$$|I_1| \le h\Omega. \tag{6.3}$$

$$|I_2| = |(s'''(2vh + 1)h -) - f'''(x)|$$

$$= |s'''(2vh+1) - f'''(2vh+1) + (x - (2\nu+1)h)f^{(4)}(\beta)|,$$

$(2\nu+1)h \le \beta \le x$.

Using Lemma 2, we have

$$|I_2| \le K_{1,v}\omega_4(h) + h\Omega \tag{6.4}$$

and

$$|I_3| = -h(M_{2v} - N_{2v+1})$$

$$|I_3| \le K_{2,v}\omega_4(h). \tag{6.5}$$

Thus from equations (2.2) – (2.5) we have the theorem for

$2vh \le x \le (2v+1)h$ and $r = 3$.

Further let $(2v+1)h \le x \le (2v+2)h$, $v = 0, 1, \ldots, (n-3)/2$.

From equation (1.6) we have

$$s'''(x) = s'''((2v+1)h)\, A_0 \frac{((2v+2)h - x)}{h} + s'''((2v+2)h)\, A_0 \frac{(x-(2v+1)h)}{h} +$$

$$+ h^2 s^{(5)}((2v+2)h)\, A_1 \frac{((2v+2)h - x)}{h}.$$

Using equations (1.5) and (1.3) we get

$$s'''(x) = s'''((2v+1)h)\, A_0 \frac{((2v+2)h - x}{h} + s'''((2v+2)h)\, A_0 \frac{(x-(2v+1)h)}{h} +$$

$$+ h\,(M_{2v+1} - N_{2v+2})\; A_1 \frac{((2v+2)h - x)}{h}.$$

Following similar arguments, we can prove the result for

$(2v+1)h \le x \le (2vh+2)h$ and $r = 3$.

Next for $r = 0, 1, 2$, using interpolatory condition we can write

$$|s''(x) - f''(x)| = \left| \int_x^{(2v+1)h} (s'''(t) - f'''(t))\, dt \right|$$

$$\le \int_x^{(2v+1)h} |s'''(t) - f'''(t)|\, dt$$

$$\le K_{1,v}\, h\, \omega_4(h).$$

Also we can write

$$|s'(x) - f'(x)| = \left| \int_\lambda^x (s''(t) - f''(t))\, dt \right|, \quad vh \le \lambda \le (v+1)h,$$

Therefore,

$$|s'(x) - f'(x)| \le h\, |s''(t) - f''(t)$$

$$\le K_{1,v}\, h^2 \omega_4(h).$$

Similarly,

$$|s(x) - f(x)| = \left| \int_{2vh}^x (s'(t) - f'(t))\, dt \right|$$

$$\le K_{1,v}\, h^3 \omega_4(h).$$

This proves Theorem 2 completely.

VII. Conclusion:

We have proved here the considered interpolation problem (0,3 ; 0,2) by showing the unique existence of it. Then we found the error bounds and showed that the derived spline function is convergent also. Based on the above problem one can construct different types of spline functions which can approximate a given function with given interpolatory and few known data points.

VIII. Scope and Significance of the result:

In the same way, we can use Spline functions for solving many lacunary interpolation problems for computer animation and design. Also, we noticed that the interpolation technique is useful for image processing.

Spline functions are useful in various fields like data smoothing, curve fitting, Computer-aided design, computer graphics, Numerical analysis, signal processing, data reconstruction, finite element analysis, and path planning. Collectively we can say that spline interpolation is a versatile and widely used technique with applications across various disciplines.

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Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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