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LEVEL SEPARATION OF FUZZY PAIRWISE REGULAR BITOPOLOGICAL SPACES

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Abstract

This paper introduced four notions of Fuzzy pairwise regular (in short FP-R) bitopological spaces and established some relation among them. Also, prove that all of these definitions satisfy the "good extension" property. Further, prove that all of these notions are hereditary. Finally, observe that all concepts are preserved under one-one, onto, and continuous mapping.

Keywords: Fuzzy bitopological space, Regular space, FP-Continuous, FP – Open, FP – Close Map.

I. Introduction

The fundamental concept of fuzzy set introduced by L. A. Zadeh [XXX] provided a natural foundation for building new branches. In 1968 Chang [V] introduced the concept of fuzzy topological spaces and thereafter in 1980 Hutton and Reilly [XI] defined regular space in fuzzy topological spaces after then many fuzzy topologists have contributed various forms of separation axioms to the theory of fuzzy bitopological spaces. In 1987, Kandil et al [XII, XIV] introduced the concept of fuzzy bitopological spaces, and since then many concepts in classical topology have been extended to fuzzy bitopological spaces. The concepts of fuzzy pairwise Regular bitopological spaces were first introduced by Kandil and El-Shafee [XIII]. Later, Safiya [I,II], Nouh [XXIII], Mukherjee [XXII], Lee [XVIII], Ramadan and Abbas [XXVI], Yue [XXIX], Kumar [XVII], prova [XXIV, XXV], Amin [III], Mahbub [XXI] and Hossain [X] also introduced other types of Regular fuzzy bitopological spaces. Here add four more notions to this list; our notions are a good extension of their topological counterparts P-R in a bitopological space. Our definition α - FP – R(j) , j = i, ii, iii, iv. This seems to be most approximate in view of the fact that FP – R fuzzy bitopological spaces in this sense are precisely the R – objects in the category of 'fuzzy bitopological spaces and fuzzy pairwise continuous maps'. On comparison, it turns out that α - FP – R(j) , j = i , ii, iii, iv. is the weakest among all. We have proved that FP – R(j) is hereditary.

II. Notation and preliminaries

Through this paper, X will be a nonempty set, I = [0 , 1], I_0 = (0 , 1], I_1 = [0 , 1), I_{01} = (0 , 1) and FP(resp P) stands for fuzzy pairwise (resp pairwise). The class of all fuzzy sets on a universe X will be denoted by I^X and fuzzy sets on X will be denoted by u, v, w, etc. Crisp subset of X will be denoted by capital letters A, B, C, etc

Definition : (Chang [V]) **:-** Let I = [0 , 1], X be a nonempty set, and I^X be the collection of all mappings from X into I, i.e. the class of all fuzzy sets in X . A fuzzy topology on X is defined as a family t of members of I^X, satisfying the following conditions.

(1) $1 , 0 \in t$,
(2) If $u_i \in t$ for each $i \in \Lambda$, then $\cup_{i \in \Lambda} u_i \in t$.
(3) If $u_1 , u_2 \in t$ then $u_1 \cap u_2 \in t$.

The pair (X , t) is called a fuzzy topological space (fts, in short), and members of t are called t- open (or simply open) fuzzy sets. A fuzzy set v is called a t- closed (or closed) fuzzy set if $1 – v \in t$.

Definition : (Lowen [XIX]) **:** A fuzzy topology on a nonempty set X is a collection t of fuzzy subsets of X such that

(1) all constant fuzzy subsets of X belong to t .
(2) t is closed under the formation of the fuzzy union of an arbitrary collection of members of t.
(3) t is closed under the formation of the intersection of a finite collection of members of t.

Definition : (Kelly [XV]) **:** Let X be any nonempty set and S and T be any two general topologies on X then the triple (X ¸ S , T) is called bitopological space.

Definition : (Kandil [XIII])**:-** Let X be a nonempty set s and t be two fuzzy topologies on X then the triple (X , s , t) is called the fuzzy bitopological space.

Definition : (Lowen [XIX] and Weiss [XXVII]) **:** Let (X, T) be an ordinary topological space. The set of all lower semicontinuous functions from (X , T) into the closed unit interval equipped with the usual topology constitves a fuzzy topology associated with (X , T) and is denoted as (X , $\omega(T)$).

Definition : (Mukherjee [XXII]) **:** Let (X , T_1 , T_2) be a bitopological space and $\omega(T_i)$ (i = 1, 2) be the set of all completely lower semi-continuous functions defined from X into the closed unit interval I=[0 , 1], then $\omega(T_i)$ is a topology in X. The triple (X , $\omega(T_1)$, $\omega(T_2)$) is called a completely induced bifuzzy topological(CIBFT) space.

Definition : (Kandil [XIV]) **:** A fuzzy bitopological space (X , s , t) is called p-bitopology generated iff there exist two ordinary topologies S and T on X such that $\omega(S)= s$ and $\omega(T)= t$.

Definition : (Hutton[XI]) : A fuzzy space (X , t) is called a fuzzy regular space if and only if each fuzzy open subset δ of X is a union of fuzzy open subsets δ_n' s of X such that Cl $\delta_a \subseteq \delta$ for each a.

Definition : (Abu Safiya [II]) **:** A bfts (X , t_1 , t_2) is said to be P-regular if for every fuzzy point p in X and each t_i-closed set μ such that $p \in \mu^c$ then there exist a t_i-open fuzzy set γ and a t_j-open fuzzy set v such that $p \in \gamma, \mu \subseteq v$ and $\gamma \subseteq v^c, i \neq j,\ i,j = 1, 2$.

Definition : (Hossain [X]) **:** Let (X , t) be fuzzy topological space and $\alpha \in I_1$.

(a) (X , t) is an α - FR(i) space $\Leftrightarrow$ $\forall$ w $\in t^c$, $\forall$ x $\in$ X , with w(x) < 1, $\exists$ u, v $\in t$ such that u(x) = 1 , v(y) = 1 , y $\in w^{-1}\{1\}$ and $u \cap v \leq \alpha$.

(b) (X , t) is an α -FR(ii) space $\Leftrightarrow$ $\forall$ w $\in t^c$, $\forall$ x $\in$ X , with w(x) < 1, $\exists$ u, v $\in t$ such that u(x) > α , v(y) = 1 , y $\in w^{-1}\{1\}$ and $u \cap v \leq \alpha$.

(c) (X , t) is an α - FR(iii) space $\Leftrightarrow$ $\forall$ w $\in t^c$, $\forall$ x $\in$ X , with w(x) = 0 , $\exists$ u, v $\in t$ such that u(x) = 1 , v(y) = 1 , y $\in w^{-1}\{1\}$ and $u \cap v \leq \alpha$.

(d) (X , t) is an α -FR(iv) space $\Leftrightarrow$ $\forall$ w $\in t^c$, $\forall$ x $\in$ X , with w(x) = 0 , $\exists$ u, v $\in t$ such that u(x) > α , v(y) = 1 , y $\in w^{-1}\{1\}$, $u \cap v \leq \alpha$.

Definition : (Mukherjee [XXII]) : A bitopological space (X , T_1 , T_2) is called a P-almost regular space if and only if for each T_i- regular open subsets A of X is a union of T_i –regular open subsets A_j's of X such that Cl $A_j \subseteq A$ for each j.

Definition : (Mukherjee [XXII]) : A CIBFT space (X , $t(T_1)$, $t(T_2)$) is called a P-fuzzy regular space if and only if for each $t(T_i)$ –open subset δ of X is a union of $t(T_i)$ –open subset δ_n' s of X such that Cl $\delta_a \subseteq \delta$ for each a.

Definition : (Kandil [XIII]) **:** A fuzzy bitopological (X, t_1, t_2) is called fuzzy pairwise regular if $x_r \tilde{q} u$, u is t_i closed fuzzy set, there exist $\mu \in N(x_r , t_i)$ and $\gamma \in N(u\ , t_j)$ such that $\mu \tilde{q} \gamma$.

Further, let $A \subseteq X$ and $s_A = \{ u/A : u \in s\}$, $t_A = \{ v/A : v \in t \}$ denoted the subspace topology on A induced by s_A, t_A. Then (A, s_A, t_A) is called a subspace of (X, s, t) with the underlying set A.

A fuzzy bitopological property P is called hereditary if each subspace of a fbts with property P also has property P.

Definition (Chang [V]) **:-** A mapping $f : (X, s) \rightarrow (Y, t)$ from a fuzzy topological space (X, s) into another fuzzy topology (Y, t) is said to be

(i) Continuous if and only if for every $v \in t, \Rightarrow f^{-1}(v) \in s$.

(ii) Open if and only if for each open fuzzy set u in (X, s), $\Rightarrow f(u)$ is open in (Y, t).

(iii) Closed if and only if for each closed fuzzy set u in (X, s), $\Rightarrow f(u)$ is closed in (Y, t).

(iv) The function f is called fuzzy homeomorphism if and only if f is bijective and both f and f^{-1} are fuzzy continuous.

Definition : (Mukherjee [XXII]) **:** A mapping $f : (X, s, t) \rightarrow (X, s_i, t_i)$ from a fuzzy bitopological (X, s, t) into another fuzzy bitopological space (X, s_i, t_i) is said to be

(i) FP – continuous if and only if $f : (X, s) \rightarrow (X, s_i)$ and $f : (X, t) \rightarrow (X, t_i)$ are both continuous.

(ii) FP – open if and only if $f : (X, s) \rightarrow (X, s_i)$ and $f : (X, t) \rightarrow (X, t_i)$ are both open.

(iii) FP – closed if and only if $f : (X, s) \rightarrow (X, s_i)$ and $f : (X, t) \rightarrow (X, t_i)$ are both closed.

(iv) FP – homeorphism if and only if f is bijective and both f and f^{-1} are fuzzy continuous.

III. Definition and properties of Regular Spaces

Definition 3.1. A fuzzy bitopological space (X, s, t) is called

(a) α - FP – R(i) space if $\forall\ w \in s^c$, $\forall\ x \in X$, with $w(x) < 1$, $\exists\ u \in s$, $v \in t$ such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$.

(b) α - FP – R(ii) space if $\forall\ w \in s^c$, $\forall\ x \in X$, with $w(x) < 1$, $\exists\ u \in s$, $v \in t$ such that $u(x) > \alpha$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$.

(c) α - FP – R(iii) space if $\forall\ w \in s^c$, $\forall\ x \in X$, with $w(x) = 0$, $\exists\ u \in s$, $v \in t$ such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$.

(d) α - FP – R(iv) space if $\forall\ w \in s^c$, $\forall\ x \in X$, with $w(x) = 0$, $\exists\ u \in s$, $v \in t$ such that $u(x) > \alpha$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$.

Theorem 3.2. Let (X , s , t) be a fuzzy bitopological space, then we have the following implication

$$\begin{tikzcd}
 & \alpha\text{ - FP- R(ii)} \arrow[dr] & \\
\alpha\text{ - FP – R(i)} \arrow[ur] \arrow[dr] & & \alpha\text{ - FP – R(iv).} \\
 & \alpha\text{ - FP – R(iii)} \arrow[ur] &
\end{tikzcd}$$

Proof:- First suppose that (X, s, t) be a α - FP–R(i). We shall prove that (X, s, t) is α - FP – R(ii). Let $w\in s^c$, $x \in X$, with w(x) < 1. Since (X , s , t) is α - FP –R(i) , for $\alpha\in I_1$ then $\exists$ $u\in s$, $v\in t$ such that u(x) =1, v(y) =1, $y\in w^{-1}\{1\}$ and $u\cap v \leq \alpha$. Now we see that $u(x) > \alpha$, v(y) =1, $y\in w^{-1}\{1\}$ and $u\cap v \leq \alpha$. Hence it is clear that the fuzzy bitopological space (X, s, t) is α -FP -R(ii).

Further, one can easily verify that

(X , s , t) is α - FP – R(i) $\Rightarrow$ (X, s, t) is α - FP – R(iii).

(X , s , t) is α - FP – R(ii) $\Rightarrow$ (X, s, t) is α - FP – R(iv).

and (X , s , t) is α - FP – R(iii) $\Rightarrow$ (X, s, t) is α - FP – R(iv).

None of the implications are reversible can see these through the following counter-examples.

Example (a)**.** Let X ={ x, y } and s be the fuzzy topology on X generated by $\{u\}\cup\{\text{Constants}\}$, where u(x) =0.9, u(y) =0, again let t be the fuzzy topology on X generated by $\{v\}\cup\{\text{Constants}\}$, where v(x) =0.4, v(y) = 0. For w =1-u and α =0.6, we see that the fuzzy bitopological space (X, s, t) is α - FP –R(ii), but (X, s, t) is not α - FP – R(i).

Example (b)**.** Let X ={x, y} and s be the fuzzy topology on X generated by $\{u\}\cup\{\text{Constants}\}$, where u(x) =0.2, u(y) =0.3, again let t be the fuzzy topology on X generated by $\{v\}\cup\{\text{Constants}\}$, where v(x) =0.4, v(y) = 0.2. For w = 1-u and α =0.6, we see that the fuzzy bitopological space (X, s, t) is α - FP –R(iii), and α - FP – R(iv) but it is not α - FP – R(i) and α -FP – R(ii). As there does not exist $u\in s$, $v\in t$ such that $u(x)>\alpha$, v(y) =1, $y\in w^{-1}\{1\}$ and $u\cap v\leq\alpha$.

Example (c)**.** Let X = { x, y } and s be the fuzzy topology on X generated by$\{u, w\}\cup\{\text{Constants}\}$, where u(x) =0.9, u(y) =0, w(x)=1, w(y)=0 again let t be the fuzzy topology on X generated by $\{v\}\cup\{\text{Constants}\}$, where v(x) =0.4, v(y) = 1. For p = 1- w and α =0.7, we see that the fuzzy bitopological space (X, s, t) is α - FP –R(iv), but it is not α - FP – R(iii).

Theorem 3.3. Let $0 \le \alpha \le \beta \le 1$ and (X, s, t) be a fuzzy bitopological space then

(a) (X , s , t) is α - FP - R(i) $\Rightarrow$ (X , s , t) is β - FP -R(i).
(b) (X , s , t) is α - FP – R(iii) $\Rightarrow$ (X , s , t) is β - FP – R(iii).

Proof: - First suppose that (X, s, t) is α - FP – R(i). We shall prove that (X, s, t) is β - FP–R(i). Let $w \in s^c$ and $x \in X$ with $w(x) < 1$. Since (X , s , t) is α - FP - R(i), for $\alpha \in I_1$, then $\exists$ $u \in s$, $v \in t$ such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \le \alpha$, this implies that $u \cap v \le \beta$ as $\alpha \le \beta$. It is now clear that the fuzzy bitopological space (X, s, t) is β-FP-R(i).

Next, suppose that (X, s, t) is α - FP – R(iii). We shall prove that (X, s, t) is β - FP – R(iii). Let $w \in s^c$ and $x \in X$ with $w(x) = 0$. Since (X, s, t) is α - FP - R (iii), for $\alpha \in I_1$, then $\exists$ $u \in s$, $v \in t$ such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \le \alpha$. Since $0 \le \alpha \le \beta < 1$, then $u \cap v \le \beta$. Now it can be written as $\forall$ $w \in s^c$, $x \in X$ with $w(x) = 0$, $\exists$ $u \in s$, $v \in t$ such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \le \beta$. Hence it is clear that the fuzzy bitopological space (X, s, t) is β - FP –R(iii)

Example. Let X ={ x, y } and s be the fuzzy topology on X generated by $\{u\} \cup \{\text{Constants}\}$, where $u(x) = 1$, $u(y) = 0.5$, again let t be the fuzzy topology on X generated by $\{v\} \cup \{\text{Constants}\}$, where $v(x) = 0.6$, $v(y) = 1$. For $w = 1 - u$ and $\alpha = 0.4$, $\beta = 0.7$, we see that the fuzzy bitopological space (X, s, t) is β - FP–R(i) and is also β - FP – R(iii). But the fuzzy bitopological space (X, s, t) is neither α - FP – R(i) nor α - FP – R(iii).

Good extension and subspaces

Now we discuss goodness and hereditary properties of FP –R(j) concepts, where (j = i, ii, iii, iv, v.)

Theorem 3.4. Let (X, s, t) be fuzzy bitopological space and $I_\alpha(s) = \{ u^{-1}(\alpha, 1] \ : \ u \in s \}$, $I_\alpha(t) = \{ v^{-1}(\alpha, 1] \ : v \in t \}$ then

(X, s, t) is 0 – FP – R(i) $\Rightarrow$ (X , $I_0(s)$, $I_0(t)$) is Pairwise Regular space

Proof: Let the fuzzy bitopological space (X, s, t) be a 0 - FP – R(i). We shall prove that (X, $I_0(s)$, $I_0(t)$) is Pairwise Regular space. Let V be a closed set in $I_0(s)$ and $x \in X$ such that $x \notin V$, then $V^c \in I_0(s)$ and $x \in V^c$. So by the definition of $I_0(s)$, there exists an $u \in s$ such that $V^c = u^{-1}(0, 1]$, i.e. $u(x) > 0$, since $u \in s$ then u^c is closed fuzzy set in s and $u^c(x) < 1$. Since (X, s, t) is 0 – FP – R(i) then $\exists$ $v \in s$, $w \in t$ such that $v(x)=1$, $w \ge 1_{(u^c)^{-1}\{1\}}$, $u \cap v = 0$.

(a) Since $v\in s$, $w\in t$ then $v^{-1}(0,1]\in I_0(s)$, $w^{-1}(0,1]\in I_0(t)$ and $x\in u^{-1}(0,1]$.

(b) Since $w\geq 1_{(u^c)^{-1}\{1\}}$ then $w^{-1}(0,1]\supseteq \left(1_{(u^c)^{-1}\{1\}}\right)^{-1}(0,1]$.

(c) And $u\cap v=0$, mean $(v\cap w)^{-1}(0,1]=v^{-1}(0,1]\cap w^{-1}(0,1]=\phi$.

Now, we have

$$\begin{aligned}\left(1_{(u^c)^{-1}\{1\}}\right)^{-1}(0,1] &= \{x : 1_{(u^c)^{-1}\{1\}}(x)\in(0,1]\}\\ &= \{x : 1_{(u^c)^{-1}\{1\}}(x)=1\}\\ &= \{x : x\in (u^c)^{-1}\{1\}\}\\ &= \{x : u^c(x)=1\}\\ &= \{x : u(x)=0\}\\ &= \{x : x\notin V^c\}\\ &= \{x : x\in V\}\\ &= V\end{aligned}$$

Put $W=v^{-1}(0,1]$ and $W^*=w^{-1}(0,1]$, then $x\in W$, $W^*\supseteq V$ and $W\cap W^*=\phi$. Hence it is clear that the bitopological space $(X, I_0(s), I_0(t))$ is pairwise Regular space.

Theorem 3.5. Let (X, t) be a fuzzy topological space $A\subseteq X$, and $t_A=\{u/A : u\in t\}$, then $1_{((u/A)^c)^{-1}\{1\}}(x) = (1_{(u^c)^{-1}\{1\}}/A)(x)$

Proof: Let w be a closed fuzzy set in t_A ie $w\in t_A{}^c$, then $u/A=w^c$, where $u\in t$.

Now we have

$$\begin{aligned}1_{((u/A)^c)^{-1}\{1\}}(x) &= \begin{cases}0 & \text{if} \quad x\notin ((u/A)^c)^{-1}\{1\}\\ 1 & \text{if} \quad x\in ((u/A)^c)^{-1}\{1\}\end{cases}\\ &= \begin{cases}0 & \text{if} \quad x\notin \{y : (u/A)^c(y)=1\}\\ 1 & \text{if} \quad x\in \{y : (u/A)^c(y)=1\}\end{cases}\\ &= \begin{cases}0 & \text{if} \quad (u/A)^c(x)<1\\ 1 & \text{if} \quad (u/A)^c(x)=1\end{cases}\\ &= \begin{cases}0 & \text{if} \quad w(x)<1\\ 1 & \text{if} \quad w(x)=1\end{cases}\end{aligned}$$

Again $\quad 1_{(u^c)^{-1}\{1\}}(x) = \begin{cases}0 & \text{if} \quad x\notin (u^c)^{-1}\{1\}\\ 1 & \text{if} \quad x\in (u^c)^{-1}\{1\}\end{cases}$

$$= \begin{cases} 0 & if \quad x \notin \{y : u^c(y) = 1\} \\ 1 & if \quad x \in \{y : u^c(y) = 1\} \end{cases}$$

$$= \begin{cases} 0 & if \quad u^c(x) < 1 \\ 1 & if \quad u^c(x) = 1 \end{cases}$$

Now $(1_{(u^c)^{-1}\{1\}}/A)(x) = \begin{cases} 0 & if \quad (u^c/A)(x) < 1 \\ 1 & if \quad (u^c/A)(x) = 1 \end{cases}$

$$= \begin{cases} 0 & if \quad (u/A)^c(x) < 1 \\ 1 & if \quad (u/A)^c(x) = 1 \end{cases}$$

$$= \begin{cases} 0 & if \quad w(x) < 1 \\ 1 & if \quad w(x) = 1 \end{cases}$$

Hence it is clear that $1_{((u/A)^c)^{-1}\{1\}}(x) = (1_{(u^c)^{-1}\{1\}}/A)(x)$.

Theorem 3.6. Let (X, *s*, *t*) be a fuzzy bitopological space, $A \subseteq X$ and $s_A = \{ u/A : u \in s \}$, $t_A = \{ v/A : v \in t \}$ then

(a) (X, *s*, *t*) is α - FP – R(i) $\Rightarrow$ (A , s_A , t_A) is α - FP – R(i).
(b) (X, *s*, *t*) is α - FP – R(ii) $\Rightarrow$ (A , s_A , t_A) is α - FP – R(ii).
(c) (X, *s*, *t*) is α - FP – R(iii) $\Rightarrow$ (A , s_A , t_A) is α - FP – R(iii).
(d) (X, *s*, *t*) is α - FP – R(iv) $\Rightarrow$ (A , s_A , t_A) is α - FP – R(iv).

Proof: Suppose (X, *s*, *t*) be a fuzzy bitopological space and is α - FP – R(i). We shall prove that (X, s_A, t_A) is α - FP – R(i). Let w be a closed fuzzy set in s_A and $x^* \in A$ such that $w(x^*) < 1$. This implies that $w^c \in s_A$ and $w^c(x^*) > 0$. So there exists an $u \in s$ such that $u/A = w^c$ and clearly u^c is closed in *s* and $u^c(x^*) = (u/A)^c(x^*) = w(x^*) < 1$. Since (X , *s* , *t*) is α- FP–R(i), for $\alpha \in I_1$ then $\exists$ $v \in s$, $v^* \in t$ such that $v(x^*) = 1$, $v^* \geq 1_{(u^c)^{-1}\{1\}}$ and $v \cap v^* \leq \alpha$. Since $v \in s$ and $v^* \in t$, then $v/A \in s_A$, $v^*/A \in t_A$ and $v/A(x^*) = 1$, $v^*/A \geq (1_{(u^c)^{-1}\{1\}}/A)$ and $v/A \cap v^*/A = (v \cap v^*)/A \leq \alpha$. But $1_{((u/A)^c)^{-1}\{1\}} = 1_{(u^c)^{-1}\{1\}}/A = 1_{w^{-1}\{1\}}$, then $v^*/A \geq 1_{w^{-1}\{1\}}$. Hence it is clear that (X , s_A , t_A) is α - FP – R(i).

The proof of (b), (c), and (d) are similar.

Hence, we see that α -FP – R(i) , α - FP –R(ii), α - FP – R(iii), α - FP – R(iv) properties are hereditary.

Theorem 3.7. Let (X, S, T) be a bitopological space. Consider the following statements.

(1) (X, S, T) be a Pairwise Regular space.
(2) (X, ω(S) , ω(T)) be an α - FP- R(i) space .
(3) (X, ω(S) , ω(T)) be an α- FP- R(ii) space .
(4) (X, ω(S) , ω(T)) be an α- FP- R(iii) space .
(5) (X, ω(S) , ω(T)) be an α- FP- R(iv) space .

Then the following implications are true;

$$\begin{tikzcd}
 & & (3) \arrow[dr] & & \\
(1) \arrow[r] & (2) \arrow[ur] \arrow[dr] & & (5) \arrow[r] & (1) \\
 & & (4) \arrow[ur] & &
\end{tikzcd}$$

Proof: Let the bitopological space (X, S, T) be a pairwise Regular. Suppose w be a closed fuzzy set in ω(S) and $x \in$ X such that $w(x) < 1$, then $w^c \in$ ω(S) and $w^c(x) > 0$. Now we have $(w^c)^{-1}(0 , 1] \in S$, $x \in (w^c)^{-1}(0 ,1]$. Also it is clear that $((w^c)^{-1}(0 , 1])^{-1} = w^{-1}\{1\}$ be a closed in S and $x \notin w^{-1}\{1\}$. Since (X, S, T) is pairwise Regular space, then $\exists\ V \in S$, $V^* \in T$ such that $x \in V$, $V^* \supseteq w^{-1}\{1\}$ and $V \cap V^* = \phi$. But by the definition of lower semi-continuous function $1_V \in$ ω(S), $1_{V^*} \in$ ω(T) and $1_V(x) = 1$, $1_{V^*} \supseteq 1_{w^{-1}\{1\}}$, $1_V \cap 1_{V^*} = 1_{V \cap V^*} = 0$. Put $u = 1_V$ and $v = 1_{V^*}$, then it is clear that $u(x) = 1$, $v \supseteq 1_{w^{-1}\{1\}}$ and $u \cap v \leq \alpha$. Hence it is clear that the fuzzy bitopological space (X, ω (S), ω (T)) is α - FP- R(i).

Further, it can easy to show that (2) ⇒(3), (3) ⇒ (5), (2) ⇒ (4), and (4) ⇒ (5).

We, therefore, prove that (5) ⇒ (1)

Suppose the fuzzy bitopological space (X, ω(S), ω(T)) be an α- FP- R(iv). We shall prove that the bitopological space (X, S, T) is Pairwise Regular. Let $x \in X$, V be a closed set in S, such that $x \notin$ V. This implies that $V^c \in S$ and $x \in V^c$. But from the definition of ω(S), $1_{V^c} \in$ ω(S) and $(1_{V^c})^C = 1_V$ closed in ω(S) and $1_V(x) = 0$. Since (X, ω (S), ω (T)) be an α- FP- R(iv), for $\alpha \in I_1$ then $\exists\ u \in$ ω(S), $v \in$ ω(T) such that $u(x) > \alpha$, $v \geq 1_{(1_V)^{-1}\{1\}} = 1_V$ and $u \cap v \leq \alpha$. Since $u \in$ ω(S) , $v \in$ ω(T), then $u^{-1}(\alpha , 1] \in S$, $v^{-1}(\alpha , 1] \in T$ and $x \in u^{-1}(\alpha , 1]$, since $v \geq 1_V$ then $v^{-1}(\alpha , 1] \supseteq (1_V)^{-1}(\alpha , 1] = V$ and $u \cap v \leq \alpha$ implies $(u \cap v)^{-1}(\alpha , 1] = u^{-1}(\alpha , 1] \cap v^{-1}(\alpha , 1] = \phi$. Now from the above, it is clear that the bitopological space (X, S, T) is Pairwise Regular.

Thus it is seen that α - FP – R(j) is a good extension of its topological counterpart. (j = i , ii , iii , iv)

Mappings in Regular Spaces

We observe here that α - PR –R(j), (j = i, ii, iii, iv, v) concepts are preserved under continuous, one-one, and open maps.

Theorem 3.8. Let (X, s, t) and (Y, s_1, t_1) be two fuzzy bitopological spaces and $f : X \rightarrow Y$ be a one-one onto and FP- open map, one-one, onto and FP – continuous map then,

(a) (X, s, t) is α - FP- R(i) $\Rightarrow$ (Y, s_1, t_1) is α - FP- R (i) .

(b) (X, s, t) is α-FP- R(ii) $\Rightarrow$ (Y, s_1, t_1) is α-FP- R(ii) .

(c) (X, s, t) is α-FP- R(iii) $\Rightarrow$ (Y, s_1, t_1) is α-FP- R(iii) .

(d) (X, s, t) is α-FP- R(iv) $\Rightarrow$ (Y, s_1, t_1) is α-FP- R(iv) .

Proof: Suppose the fuzzy bitopological space (X, s, t) is α - FP - R(i). We shall prove that (Y, s_1, t_1) is α-FP- R(i). Let $w \in s_1^c$ and $p \in Y$ such that $w(p) < 1$ then $f^{-1}(w) \in s^c$ as f is FP- continuous and $x \in X$ such that $f(x) = p$ as f is one-one and onto. Hence $f^{-1}(w)(x) = wf(x) = w(p) < 1$. Since (X, s, t) is α-FP-R(i), for $\alpha \in I_1$ then $\exists$ $u \in s$ and $v \in t$ such that $u(x) = 1$, $v(y) = 1$, $y \in (f^{-1}(w))^{-1}\{1\}$ and $u \cap v \leq \alpha$.
This implies that

$$f(u)(p) = \{ \sup u(x) \ : f(x) = p \}$$
$$= 1.$$

$$f(v)(y) = \{ \sup v(y) \} = 1 \text{ , as } f(f^{-1}(w)) \subseteq w \Rightarrow f(y) \in w^{-1}\{1\}$$

and $f(u \cap v) \leq \alpha$ as $u \cap v \leq \alpha \Rightarrow f(u) \cap f(v) \leq \alpha$.

Now it is clear that $\exists$ $f(u) \in s_1$, $f(v) \in t_1$ such that $f(u)(x)=1$, $f(v)(f(y))=1$, $f(y) \in w^{-1}\{1\}$ and $f(u) \cap f(v) \leq \alpha$. Hence it is clear that the fuzzy bitopological space (Y, s_1, t_1) is α-FP- R(i) .

Similarly (b), (c) and (d) can be proved.

Theorem 3.9. Let (X, s, t) and (Y, s_1, t_2) be two fuzzy bitopological spaces and $f : X \rightarrow Y$ be FP- continuous, FP -closed, and one-one map then,

(a) (Y, s_1, t_1) is α - FP- R(i) $\Rightarrow$ (X, s, t) is α -FP- R(i).

(b) (Y, s_1, t_1) is α-FP- R(ii) $\Rightarrow$ (X, s, t) is α-FP- R(ii).

(c) (Y, s_1, t_1) is α-FP- R(iii) $\Rightarrow$ (X, s, t) is α-FP- R(iii).

(d) (Y, s_1, t_1) is α-FP- R(iv) $\Rightarrow$ (X, s, t) is α-FP- (iv).

Proof: Suppose the fuzzy bitopological space (Y, s_1, t_1) is α-FP- R(i). We shall prove that the fuzzy bitopological space (X, s, t) is α-FP- R(i). Let w$\in s^c$ and x$\in$ X with w(x) < 1, then f(w)$\in s_1{}^c$ as f is FP-closed and we find p$\in$Y such that f(x) = p as f is one-one.

Now we have $f(w)(p) = \{ \sup w(x) \ : \ f(x) = p\} < 1$.

Since (Y , s_1 , t_1) is α-FP-R(i), for $\alpha \in I_1$, then $\exists$ u$\in s_1$, v$\in t_1$ such that u(f(x)) = 1, v(y) = 1, y $\in (f(w))^{-1}\{1\}$ and $u \cap v \leq \alpha$. This implies that $f^{-1}(u) \in s$, $f^{-1}(v) \in t$ as f is FP-continuous, and u$\in s_1$, v$\in t_1$. Now $f^{-1}(u)(x) = uf(x) = u(p) =1$ and $f^{-1}(v)(q) = vf(q) = v(y) =1$ as $f(q) = y$, y $\in (f(w))^{-1}\{1\}$ i.e. $f(p) \in (f(w))^{-1} \Rightarrow q \in w^{-1}\{1\}$ and $f^{-1}(u) \cap f^{-1}(v) = f^{-1}(u \cap v) \leq \alpha$, as $u \cap v \leq \alpha$. Now we observe that $\exists$ $f^{-1}(u) \in s$, $f^{-1}(v) \in t$ such that $f^{-1}(u)(x) =1$, $f^{-1}(v)(q)=1$, $q \in w^{-1}\{1\}$ and $f(u) \cap f(v) \leq \alpha$. Hence it is clear that the fuzzy bitopological space (X, s, t) is α-FP- R(i).

The proof of (b), (c), and (d) are similar.

IV. Conclusion

In this paper new four notions are more general other than Hutton [XI], Kandil [XIII], Mukherjee [XXII], and Abu Safiya [II] notions, also all of these notions are satisfied "good extension" properties, hence defined notions are appropriate. Further, it is clear that all notions are preserved under one-one, onto, and continuous mapping.

Conflict of Interest:

The author declares that there was no conflict of interest regarding this paper.

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