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GENERAL ANALYTICAL EXPRESSIONS FOR DEFLECTION AND SLOPE OF EULER-BERNOULLI BEAM UNDER DIFFERENT TYPES OF LOADS AND SUPPORTS

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Abstract

In this research paper, we solve the Euler-Bernoulli beam (EBB) differential equations by taking the general boundary conditions. Instead of finding a solution for the EBB model for a particular load and its particular boundary conditions, we derive the general analytical solution with general boundary conditions by using techniques of integration. The proposed general analytical solutions are neither load specific nor dependent on specific boundary conditions but can be used for any load and any boundary condition without having to integrate again and again. We have taken a general polynomial load function with general boundary conditions, and get the general analytical solution for the deflection and slope parameters of EBB. We find the direct solution for uniform distributed load and linearly varying load for a fixed beam.

Keywords. Euler Bernoulli Beam, General analytical solution, Deflection, Slope.

I. Introduction

Many engineers, scientists, and researchers study beam theory and its behavior when a load is applied to a beam [X], [XIII]. In 1750 circa, the Euler-Bernoulli beam (EBB) was described. Euler–Bernoulli beam is a type of beam where there is no shear deformation and the angle of cross-section remains constant concerning the neutral axis when a load is applied [V], [VIII]. The general analytical solution was attempted to find the solution for an elastic beam subjected to uniformly distributed loads and linearly varying loads in [V] and an infinite EBB in [IV, XIV]. Zamorska proposed a power series method to solve a first-order matrix differential equation that arises from the vibration problem of the EBB, which can be represented as a fourth-order differential equation with various coefficients [XV]. For the curved EBB, the authors in [XI], [XVI] used Eringen's two-phase local/nonlocal model, and by using Volterra integral equations and the Laplace transformation method, they were able to precisely solve for several boundary conditions. For both uniform and variable loads, the author derived analytical solutions to the boundary value problem for the Timoshenko beam (TB) model and thus explained the rotation and displacement parameters along with the structure of the beam [VII].

The EBB model is a special case of the TB model [XII]. The Timoshenko beam is thick the angle of the cross-section change after the deflection but the EBB is thin [II]. TB has both shear deformations as well as bending effects [VI]. In order to solve the mathematical model of the TB problem, some studies developed finite difference [VI] and finite element [IX] schemes for the Timoshenko beam model that was presented and implemented. The study in [VI] yielded successful results that are similar to the other methods in the literature for various parameters and step sizes.

From the literature review [I], [III], the numerical and other methods were used in the past extensively, but exact analytical methods were used in rare cases. Numerical methods are time-consuming and take a lot of time to obtain a fair approximation of the exact solution. In this research paper, we obtain the general analytical solution of the EBB model by considering a general polynomial load function and applying general boundary conditions by using integration techniques. This general solution leads to the deflection and slope of the EBB after getting some simplification. For comparison and validation, we use the previously obtained general analytical solution for the elastic beam model subjected to uniformly distributed load and linearly varying load and compare our results with those reported in the literature.

II. Mathematical Model of the EBB

The EBB model was described with the standard form of differential equations describing the relationship between shear force (Q), bending moment (M), slope (θ), and deflection (w) with the applied load (f) as in equations (1)-(4).

$$-\frac{dQ}{dx} = f(x) \tag{1}$$

$$-\frac{dM}{dx} = Q(x) \tag{2}$$

$$EI\frac{d\theta}{dx} = M(x) \tag{3}$$

$$\frac{dw}{dx} = \theta(x) \tag{4}$$

Here EI is called bending stiffness or flexural rigidity where E is Young's modulus (in Nm^{-2}) and the area of the second moment I (in m^4). By combining the equations (1)-(4), an EBB model may be described using a fourth-order differential equation relating the deflection of the beam with applied load $f(x)$ as;

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = f(x)$$

With the boundary conditions: $w(0) = A, w'(0) = B, w(L) = C, w'(L) = D$

III. Derivation of General Analytical Solution of EBB Model

Let the load function be defined generally as: $f(x) = \sum_{i=0}^{n} a_i x^i$, and substituting it in (1) gives:

$$-\frac{dQ}{dx} = \sum_{i=0}^{n} a_i x^i$$

Integrating both sides concerning x gives:

$$-Q = \sum_{i=0}^{n+1} \frac{a_i x^{i+1}}{i+1} + c_1$$

$$\Longrightarrow Q(x) = -\sum_{i=0}^{n+1} \frac{a_i x^{i+1}}{i+1} + c_1$$

Using the expression of Q in (2), and integrating again concerning x leads to M, which is:

$$M(x) = \sum_{i=0}^{n+2} \frac{a_i x^{i+2}}{(i+1)(i+2)} + c_1\, x + c_2$$

Substituting expression of $M(x)$ in (3) gives:

$$\frac{d\theta}{dx} = \frac{1}{EI}\left\{\sum_{i=0}^{n+2} \frac{a_i x^{i+2}}{(i+1)(i+2)} + c_1\, x + c_2\right\}$$

Integrating throughout concerning x gives:

$$\theta(x) = \frac{1}{EI}\left\{\sum_{i=0}^{n+3} \frac{a_i x^{i+3}}{(i+1)(i+2)(i+3)} + c_1\left(\frac{x^2}{2}\right) + c_2 x + c_3\right\} \quad (5)$$

Substitute expression of θ(x) in (4) gives:

$$\frac{dw}{dx} = \frac{1}{EI}\left\{\sum_{i=0}^{n+3} \frac{a_i x^{i+3}}{(i+1)(i+2)(i+3)} + c_1\left(\frac{x^2}{2}\right) + c_2 x + c_3\right\}$$

Integrating throughout concerning x gives:

$$w(x) = \frac{1}{EI}\left\{\sum_{i=0}^{n+4} \frac{a_i x^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + c_1\left(\frac{x^3}{6}\right) + c_2\left(\frac{x^2}{2}\right) + c_3 x + c_4\right\} \quad (6)$$

Equations (5) and (6) give general solution in terms of arbitrary constants of integration$c_1 - c_4$, which can be obtained using the boundary conditions (BCs) as worked now.

Applying BC $w(0) = A\,, w'(0) = B$ in equations (5) and (6), respectively, we get the value of c_3 and c_4.

As we know that:

$$\frac{dw}{dx} = \frac{1}{EI}\left\{\sum_{i=0}^{n+3} \frac{a_i x^{i+3}}{(i+1)(i+2)(i+3)} + c_1\left(\frac{x^2}{2}\right) + c_2 x + c_3\right\}$$

Applying BC: $w'(0) = B$, we get:

$$B = \frac{1}{EI}\{0 + 0 + 0 + c_3\}$$

$$\Longrightarrow c_3 = EIB \tag{7}$$

$$Since\ w(x) = \frac{1}{EI}\left\{\sum_{i=0}^{n+4} \frac{a_i x^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + c_1\left(\frac{x^3}{6}\right) + c_2\left(\frac{x^2}{2}\right) + c_3 x + c_4\right\}$$

Apply BC: $w(0) = A$, we get:

$$A = \frac{1}{EI}\{0 + 0 + 0 + c_4\}$$

$$\Longrightarrow c_4 = EIA \tag{8}$$

Similarly applying $w(L) = C, w'(L) = D$ in equations (5) and (6), we get other constants.

As, $w(x) = \frac{1}{EI}\left\{\sum_{i=0}^{n+4} \frac{a_i x^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + c_1\left(\frac{x^3}{6}\right) + c_2\left(\frac{x^2}{2}\right) + c_3 x + c_4\right\}$

Applying BC: $w(L) = C$ in equation (6), we have:

$$C = \frac{1}{EI}\left\{\sum_{i=0}^{n+4} \frac{a_i L^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + c_1\left(\frac{L^3}{6}\right) + c_2\left(\frac{L^2}{2}\right) + c_3 L + c_4\right\}$$

Here, we put the values of $c_3\ and\ c_4$ to get:

$$EIC = \sum_{i=0}^{n+4} \frac{a_i L^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + c_1\left(\frac{L^3}{6}\right) + c_2\left(\frac{L^2}{2}\right) + LEIB + EIA$$

Simplifying further as:

$$-c_1\left(\frac{L^3}{6}\right) - c_2\left(\frac{L^2}{2}\right) = \sum_{i=0}^{n+4} \frac{a_i L^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + LEIB + EIA - EIC$$

$$\frac{-c_1 L^3 - 3c_2 L^2}{6} = \sum_{i=0}^{n+4} \frac{a_i L^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + EI(A + LB - C)$$

$$-c_1 L^3 - 3c_2 L^2 = 6\left\{\sum_{i=0}^{n+4} \frac{a_i L^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + EI(A + LB - C)\right\}$$

Finally, we have:

$$c_1 L + 3c_2 = -\frac{6}{L^2}\left\{\sum_{i=0}^{n+4} \frac{a_i L^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + EI(A + LB - C)\right\} \tag{9}$$

Similarly, apply BC $w'(L) = D$ in equation (5), i.e. through the derivative:

$$\frac{dw}{dx} = \frac{1}{EI}\left\{\sum_{i=0}^{n+3} \frac{a_i x^{i+3}}{(i+1)(i+2)(i+3)} + c_1\left(\frac{x^2}{2}\right) + c_2 x + c_3\right\}$$

We now have:

$$D = \frac{1}{EI}\left\{\sum_{i=0}^{n+3}\frac{a_i L^{i+3}}{(i+1)(i+2)(i+3)} + c_1\left(\frac{L^2}{2}\right) + c_2 L + c_3\right\}$$

Here, we put the value of $c_3 = EIB$

$$EID = \sum_{i=0}^{n+3}\frac{a_i L^{i+3}}{(i+1)(i+2)(i+3)} + c_1\left(\frac{L^2}{2}\right) + c_2 L + EIB$$

Again, simplifying we have:

$$-c_1\left(\frac{L^2}{2}\right) - c_2 L = \sum_{i=0}^{n+3}\frac{a_i L^{i+3}}{(i+1)(i+2)(i+3)} + EIB - EID$$

$$\frac{-c_1 L^2 - 2c_2 L}{2} = \sum_{i=0}^{n+3}\frac{a_i L^{i+3}}{(i+1)(i+2)(i+3)} + EIB - EID$$

$$c_1 L + 2c_2 = -\frac{2}{L}\left\{\sum_{i=0}^{n+3}\frac{a_i L^{i+3}}{(i+1)(i+2)(i+3)} + EIB - EID\right\} \tag{10}$$

Subtracting equation (10) from equation (9), we get:

$$c_2 = -\frac{6}{L^2}\left\{\sum_{i=0}^{n+4}\frac{a_i L^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + EI(A + LB - C)\right\} + \frac{2}{L}\left\{\sum_{i=0}^{n+3}\frac{a_i L^{i+3}}{(i+1)(i+2)(i+3)} + EI(B - D)\right\} \tag{11}$$

Putting value of c_2 in equation (8), we get:

$$c_1 L + 2\left[-\frac{6}{L^2}\left\{\sum_{i=0}^{n+4}\frac{a_i L^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + EI(A + LB - C)\right\} + \frac{2}{L}\left\{\sum_{i=0}^{n+3}\frac{a_i L^{i+3}}{(i+1)(i+2)(i+3)} + EIB - EID\right\}\right] - \frac{2}{L}\left\{\sum_{i=0}^{n+3}\frac{a_i L^{i+3}}{(i+1)(i+2)(i+3)} + EIB - EID\right\}$$

Simplifying, we finally have:

$$c_1 = \frac{12}{L^3}\left\{\sum_{i=0}^{n+4}\frac{a_i L^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + EI(A + LB - C)\right\} - \frac{6}{L^2}\left\{\sum_{i=0}^{n+3}\frac{a_i L^{i+3}}{(i+1)(i+2)(i+3)} + EI(B - D)\right\} \tag{12}$$

Therefore, we get the general analytical solution for the deflection in equation (6) and slope in equation (5) of the EBB model for general load function and general boundary conditions with the unknown constants determined in (7),(8), (11), (12).

The main results are summarized for brevity, now:

The proposed general analytical solution of an EBB model can be written as:

Slope: $\theta(x) = \frac{1}{EI}\left\{\sum_{i=0}^{n+3}\frac{a_i x^{i+3}}{(i+1)(i+2)(i+3)} + c_1\left(\frac{x^2}{2}\right) + c_2 x + c_3\right\}$

Deflection: $w(x) = \frac{1}{EI}\left\{\sum_{i=0}^{n+4} \frac{a_i x^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + c_1\left(\frac{x^3}{6}\right) + c_2\left(\frac{x^2}{2}\right) + c_3 x + c_4\right\}$

Where:

$$c_1 = \frac{12}{L^3}\left\{\sum_{i=0}^{n+4} \frac{a_i L^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + EI(A + LB - C)\right\} - \frac{6}{L^2}\left\{\sum_{i=0}^{n+3} \frac{a_i L^{i+3}}{(i+1)(i+2)(i+3)} + EI(B - D)\right\}$$

$$c_2 = -\frac{6}{L^2}\left\{\sum_{i=0}^{n+4} \frac{a_i L^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + EI(A + LB - C)\right\} + \frac{2}{L}\left\{\sum_{i=0}^{n+3} \frac{a_i L^{i+3}}{(i+1)(i+2)(i+3)} + EI(B - D)\right\}$$

$$c_3 = EIB$$

$$c_4 = EIA$$

The proposed equations can be used directly to obtain the parameters of deflection $w(x)$ and $\theta(x)$ for an EBB model under any type of load function and boundary conditions. The uniform distributed load or varying loads (Polynomial-like behaviour loads) can be solved by using the proposed formulas to obtain the parameters $w(x)$ and $\theta(x)$, which is the main and novel contribution of this study.

IV. Results and discussion

To validate the proposed exact expressions and the method, we determine the deflection and slope of the EBB model under both uniformly distributed and linearly varying loads. The salient feature of the direct exact expression, however, is in the cases of varying loads, making it highly easy to use, quicker, and versatile.

Case 1. We determine the deflection and slope of an EBB which is fixed on both sides with uniformly distributed load. We use $f(x) = a, 0 \le x \le L$.

Using the proposed method, we outline that:

$$n = 0,\ a_0 = a,\ a_1 = a_2 = \cdots a_n = 0\ , L = L, A = B = 0 \textit{ and } C = D = 0$$

First we find the values of constants:

$$c_1 = 12\sum_{i=0}^{0+4} \frac{a_i L^{i+1}}{(i+1)(i+2)(i+3)(i+4)} - 6\sum_{i=0}^{0+3} \frac{a_i L^{i+1}}{(i+1)(i+2)(i+3)}$$

$$c_1 = \frac{12 a_0 L}{(0+1)(0+2)(0+3)(0+4)} - \frac{6 a_0 L}{(0+1)(0+2)(0+3)}$$

$$c_1 = \frac{12aL}{24} - \frac{6aL}{6} = \frac{aL}{2} - aL = -\frac{aL}{2}$$

$$c_1 = -\frac{aL}{2}$$

$$c_2 = 2\sum_{i=0}^{0+3} \frac{a_i L^{i+2}}{(i+1)(i+2)(i+3)} - 6\sum_{i=0}^{0+4} \frac{a_i L^{i+2}}{(i+1)(i+2)(i+3)(i+4)}$$

$$c_2 = \frac{2a_0L^2}{(0+1)(0+2)(0+3)} - \frac{6a_0L^2}{(0+1)(0+2)(0+3)(0+4)}$$

$$c_2 = \frac{2aL^2}{6} - \frac{6aL^2}{24} = aL^2\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$c_2 = \frac{aL^2}{12}$$

And, $c_3 = c_4 = 0$

Now we put the values of constants in equations (6) and (5) to get expressions of w(x) and θ(x) as:

$$w(x) = \frac{1}{EI}\left\{\sum_{i=0}^{0+4} \frac{a_i x^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + c_1\left(\frac{x^3}{6}\right) + c_2\left(\frac{x^2}{2}\right)\right\}$$

$$w(x) = \frac{1}{EI}\left\{\frac{a_0 x^{0+4}}{(0+1)(0+2)(0+3)(0+4)} + c_1\left(\frac{x^3}{6}\right) + c_2\left(\frac{x^2}{2}\right)\right\}$$

$$w(x) = \frac{1}{EI}\left\{\frac{ax^4}{24} + \left(-\frac{aL}{2}\right)\left(\frac{x^3}{6}\right) + \left(\frac{aL}{12}\right)\left(\frac{x^2}{2}\right)\right\}$$

$$w(x) = \frac{1}{EI}\left\{\frac{ax^4}{24} - \frac{aLx^3}{12} + \frac{aL^2x^2}{24}\right\}$$

$$w(x) = \frac{a}{24EI}\{x^4 - 2Lx^3 + L^2x^2\}$$

$$\theta(x) = \frac{1}{EI}\left\{\sum_{i=0}^{0+3} \frac{a_i x^{i+3}}{(i+1)(i+2)(i+3)} + c_1\left(\frac{x^2}{2}\right) + c_2 x\right\}$$

$$\theta = \frac{1}{EI}\left\{\frac{a_0 x^{0+3}}{(0+1)(0+2)(0+3)} + \left(-\frac{aL}{2}\right)\left(\frac{x^2}{2}\right) + \left(\frac{aL^2}{12}\right)x\right\}$$

$$\theta = \frac{1}{EI}\left\{\frac{ax^3}{6} - \frac{aLx^2}{4} + \frac{aL^2x}{12}\right\}$$

$$\theta(x) = \frac{a}{12EI}\{2x^3 - 3Lx^2 + L^2x\}$$

Our results for $w(x)$ and $\theta(x)$ match those found in books and literature for an elastic beam under uniformly distributed load (UDL).

Case 2. We also attempt to determine the exact expressions for the deflection and slope of an EBB if the beam is fixed on both sides with the triangular (linearly varying) load:

$$f(x) = 100x, 0 \le x \le L.$$

Using the proposed method, we outline that:

$n = 1,\ a_0 = 0,\ a_1 = 100, a_2 = a_3 = \cdots = a_n = 0\,, L = L,\ A = B = 0\ and\ C = D = 0$

First we find the values of constants:

$$c_1 = 12\sum_{i=0}^{1+4}\frac{a_i L^{i+1}}{(i+1)(i+2)(i+3)(i+4)} - 6\sum_{i=0}^{1+3}\frac{a_i L^{i+1}}{(i+1)(i+2)(i+3)}$$

$$c_1 = \frac{12a_1L^2}{(1+1)(1+2)(1+3)(1+4)} - \frac{6a_1L^2}{(1+1)(1+2)(1+3)}$$

$$c_1 = \frac{12a_1L^2}{120} - \frac{6a_1L^2}{24} = a_1L^2\left(\frac{1}{10} - \frac{1}{4}\right)$$

$$c_1 = -\frac{3a_1L^2}{20}$$

$$c_1 = -\frac{300L^2}{20}$$

$$c_1 = -15L^2$$

Now,

$$c_2 = 2\sum_{i=0}^{1+3}\frac{a_i L^{i+2}}{(i+1)(i+2)(i+3)} - 6\sum_{i=0}^{1+4}\frac{a_i L^{i+2}}{(i+1)(i+2)(i+3)(i+4)}$$

$$c_2 = \frac{2a_1L^3}{(1+1)(1+2)(1+3)} - \frac{6a_1L^3}{(1+1)(1+2)(1+3)(1+4)}$$

$$c_2 = \frac{2a_1L^3}{24} - \frac{6a_1L^3}{120} = \frac{a_1L^3}{12} - \frac{a_1L^3}{20} = \frac{a_1L^3}{30} = \frac{100L^3}{30}$$

$$c_2 = \frac{10L^3}{3}$$

Again, $c_3 = c_4 = 0$.

Finally, the deflection can be obtained to:

$$w(x) = \frac{1}{EI}\left\{\sum_{i=0}^{1+4}\frac{a_i x^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + c_1\left(\frac{x^3}{6}\right) + c_2\left(\frac{x^2}{2}\right)\right\}$$

$$w(x) = \frac{1}{EI}\left\{\frac{a_1 x^{1+4}}{(1+1)(1+2)(1+3)(1+4)} + c_1\left(\frac{x^3}{6}\right) + c_2\left(\frac{x^2}{2}\right)\right\}$$

$$w(x) = \frac{1}{EI}\left\{\frac{a_1 x^5}{120} + (-15L^2)\left(\frac{x^3}{6}\right) + \left(\frac{10L^3}{3}\right)\left(\frac{x^2}{2}\right)\right\}$$

$$w(x) = \frac{1}{EI}\left\{\frac{100x^5}{120} - 15\left(\frac{L^2x^3}{6}\right) + \left(\frac{10L^3}{3}\right)\left(\frac{x^2}{2}\right)\right\}$$

$$w(x) = \frac{1}{EI}\left\{\frac{5x^5}{6} - 15\left(\frac{L^2x^3}{6}\right) + \left(\frac{10L^3x^2}{6}\right)\right\}$$

$$w(x) = \frac{5}{6EI}(x^5 - 3L^2x^3 + 2L^3x^2)$$

The slope can be determined accordingly as:

$$\theta = \frac{1}{EI}\left\{\frac{100x^{1+3}}{(1+1)(1+2)(1+3)} + (-15L^2)\left(\frac{x^2}{2}\right) + \left(\frac{10L^3}{3}\right)x\right\}$$

$$\theta = \frac{1}{EI}\left\{\frac{100x^{1+3}}{24} - 15\left(\frac{L^2x^2}{2}\right) + \left(\frac{10L^3x}{3}\right)\right\}$$

$$\theta = \frac{1}{EI}\left\{\frac{25x^{1+3}}{6} - 15\left(\frac{L^2x^2}{2}\right) + \left(\frac{10L^3x}{3}\right)\right\}$$

$$\theta = \frac{5}{6EI}\{5x^4 - 9L^2x^2 + 4L^3x\}$$

Further, the plots have been used to exhibit the deflection and slope profiles. For example, in case 2, with $L = 30ft$ and $EI = 161111\, units$, we can verify the deflection and slope as in particular forms:

$$w(x) = \frac{5}{6EI}(x^5 - 3L^2x^3 + 2L^3x^2)$$

$$w(x) = \frac{5}{6 \times 161111}(x^5 - 3(30)^2x^3 + 2(30)^3x^2)$$

$$w(x) = \frac{5}{966666}(x^5 - 2700x^3 + 5400x^2)$$

$$w(x) = \frac{5x^2}{966666}(x^3 - 2700x + 5400)$$

Similarly,

$$\theta = \frac{5}{6EI}\{5x^4 - 9L^2x^2 + 4L^3x\}$$

$$\theta = \frac{5}{6 \times 161111}\{5x^4 - 9(30)^2x^2 + 4(30)^3x\}$$

$$\theta = \frac{5x}{966666}\{5x^3 - 8100x + 108000\}$$

Figures 1 and 2, respectively, show the deflection and slope parameters for the case 2 load, i.e. a fixed EBB at both ends when subjected to linearly varying load and further $L = 30ft$ and $EI = 161111\,units$. As expected from the literature, the point of maximum deflection lies around $x = 16ft$ as in Figure 1. At this point, the slope vanishes as shown in Figure 2. The results confirm the accuracy and validity of the proposed exact generalizations of the deflection and slope parameters of an EBB regardless of any particular boundary conditions or types of load. These equations will surely prove to be a concrete aid for structural engineers and in theory of beams.

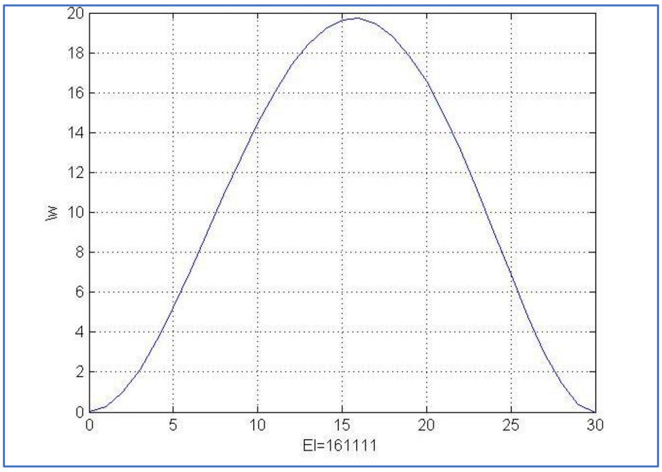

Fig. 1. Deflection of a particular EBB for linearly varying load

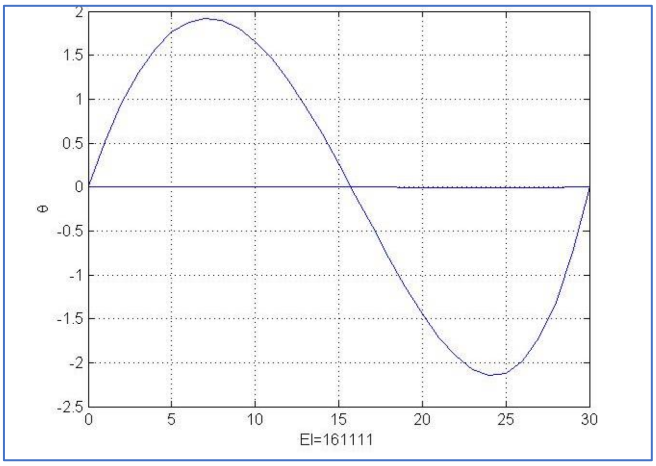

Fig. 2. Slope of a particular EBB for linearly varying load

V. Conclusion

In this study, the EBB model was focused and the exact analytical expressions for the slope and deflection parameters were targeted so that a solution without having to depend on a particular load function or boundary types can be attained. The proposed expressions were achieved using direct integration only once on the generalized load function in the form of a power series. The main aid the proposed equations provide is that the solution remains valid for any type of load and boundary conditions without having to solve the model each time. The proposed general analytical solution of the EBB model for different types of load and different types of boundary conditions was validated for a fixed beam with constant and linearly varying load. We have checked the general analytical solution on uniformly distributed and triangular (linearly varying load) with the boundary conditions. We have found the deflection and slope parameters that are matching with those in the literature.

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Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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