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ADEQUATE SOLUTIONS OF JERK OSCILLATORS CONTAINING VELOCITY TIMES ACCELERATION-SQUARED: HAQUE'S APPROACH WITH MICKENS' ITERATION METHOD

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Abstract

Haque's Approach with Mickens' Iteration Method is used to find the exact analytic solution of the nonlinear equation involving velocity times acceleration squared. A truncated Fourier series is used in different rhythms with different repetition steps. Our results are very close to the exact results and our results are comparatively closer to the exact results than others. Our solution method is obtained around the second-order angular frequency using Newton's method. For some third-order (jerk) differential equations with cubic nonlinearities and nonlinear second-order differential equations; Mickens' iteration method is used to determine the exact analytical approximate periodic solution. A numerical experiment of general differential equations with third-order, one-dimensional, autonomous, quadratic, and cubic nonlinearity has uncovered several algebraically simple equations involving the shaking of time-dependent acceleration that contain chaotic solutions.

Keywords: Jerk equation; Truncated Fourier series; Newton's method; Angular frequency; Haque's Approach with Mickens' Iteration Method; Autonomous; Chaotic solutions.

I. Introduction

Mathematics is an integral part of our life. We find its application in many parts of our everyday life. In many areas of our real life, differential equations serve as mathematical tools that enable us to solve many difficult problems. With the help of differential equations, we can derive many important solution formulas in many areas of our life like physical, mental, economic, and medical principles. Hence the solution to a particular problem lies in solving the associated differential equations. Differential equations are usually linear or non-linear, independent or dependent.

Most phenomena in the world can be described by linear and non-linear equations. A well-known example of linearity is the small oscillation method, which is essentially non-linear. Methods for solving linear differential equations are relatively simple and highly advanced. Methods for solving linear differential equations are relatively simple and highly advanced. In contrast, we rarely have the general character of non-linear equations. However, there are some prominent methods for solving non-linear equations. These are the perturbation method, the harmonic balance method; the iteration method, etc. These methods have seen widespread use of the perturbation method where the term non-linear is short. These methods have the advantage of being used in the case of weak non-linear differential equations as they have good results. On the contrary, strong non-linear equations use harmonic balance methods and iteration methods. Between these two types of methods, the harmonic balance method has been developed by Mickens and further developed by Wu, Gottlieb, Alam, etc. We have to make some assumptions about perturbation, harmonic balance, and iteration methods but there are some differences between them. In the asymptomatic series of dependent variables in the perturbation method, in the harmonic balance method, we have to assume certain levels such as third harmonic, fifth harmonic, etc. which we have to hold till the end without any option of change but in the iteration method, it is better for each step maybe. So it is an advantage for us that we do not see this kind of limitation in the case of the iteration method as a method of perturbation and harmonic balance.

The term 'jerk', first introduced by Schot in 1978, is a third-order derivative of displacement. Most efforts in dynamical systems are concerned with second-order differential equations. By third-order nonlinear differential, some dynamic systems can be explained. This type of equation is called the non-linear jerk equation.

The general form of the jerk equation is

$$\dddot{x} = J(x, \dot{x}, \ddot{x}) \tag{1}$$

where $J(x, \dot{x}, \ddot{x})$ denotes the jerk functions.

Some researchers have presented analytical solutions to these oscillators. Among these researchers, Gottlieb [I] used the minimum order harmonic balance method to determine inferential periodic solutions and corresponding angular frequencies. Obtained by Gottlieb the solution and angular frequencies were not sufficiently accurate. The harmonic balance method, which demands analytical solutions, makes it highly challenging to build higher-order approximations to a collection of challenging nonlinear algebraic equations. Using an advanced harmonic balance approach and a residual harmonic balance method, respectively, Wu et al. [XX] and Leung et al. [XIII] solved the nonlinear jerk equations and their higher-order approximation to produce correct results for larger oscillation amplitudes. Ma et al. [XIV] employed the homotopy perturbation approach, and Hu [XI] used the parameter perturbation method to explain the high-order most feasible solution of the non-linear jerk equation. Their results were more accurate than those from the low-order harmonic balance method. Currently, it is discovered that Ramos [XVII-XIX] pioneered the order reduction method, a two-level iterative method, and a Volterra integral formulation, respectively, to solve the nonlinear jerk equations. He discovered that the third reduction method [XVII] appears to be extremely accurate for the first and second differential equations, and the second reduction approach merely assists in finding the correct solution for the beginning velocity close to the integral, but the absolute result or more absolute result is the fourth reduction method available by. All these are available in the parameter perturbation method. A modified method of Mickens for a nonlinear jerk equation was presented by Hu et al. [XII], and Newton's method of approximate angular frequency helped to obtain the second order. With cubic nonlinearities, a residual harmonic balance method has been introduced by Leung et al. [XIII] for describing the boundary cycles of the parity. Recently, Haque et al. [II, III, IV, V, VI, VII, VIII, IX, X] introduced a new approach of direct and extended iteration method [XV] to solve some important nonlinear oscillators including the nonlinear jerk oscillator containing acceleration and velocity of displacement time.

To get a periodic solution for jerk oscillators, on the oscillator's mathematical shape, there are unquestionably some restrictions. It is only necessary to take into account the third-order nonlinear functions introduced by Gottlieb [I] in the following order:

$$\dot{x}\ddot{x}^2 \tag{2}$$

The generic jerk function, which only has third-order nonlinearity and includes time- and space-reversal variation, can be represented as follows [I]:

$$\dddot{x} = \alpha x\dot{x}\ddot{x} - \beta\dot{x}\ddot{x}^2 - \gamma\dot{x} - \delta x^2\dot{x} - \varepsilon\dot{x}^3 \tag{3}$$

Where the derivative is shown by an over-dot and the parameters $\alpha,\ \beta,\ \gamma,\ \delta$ and ε the actual constant are given. The relevant prerequisites are:

$$x(0) = 0,\ \dot{x}(0) = \mathrm{A} \text{ and } \ddot{x}(0) = 0. \tag{4}$$

To fulfill the periodic requirement, these three beginning conditions in Eq. (4) must be met. In this case, at least one of $\alpha,\ \beta,\ \gamma,\ \delta$ and ε must not be zero.

In this article, we present Haque's Approach with Mickens' Iteration Method to ascertain the nonlinear jerk equation's approximate solution where the jerk function is the velocity times the acceleration squared.

II. The Method

Think about a nonlinear oscillator that is described by

$$\ddot{x} + f(\dot{x}, \ddot{x}) = 0, \quad x(0) = 0,\ \dot{x}(0) = \mathrm{A} \tag{5}$$

We select the frequency Ω of this system, where over dots signify differentiation with respect to time and t over dash denotes integration concerning time. Hence, by including $\Omega^2 x$ in both directions of Eq. (5), we get

$$\ddot{x} + \Omega^2 x = \Omega^2 x - f(\dot{x}, \ddot{x}) \equiv G(\dot{x}, \ddot{x}) \tag{6}$$

As a result of [XII], we define the iteration strategy as

$$\ddot{x}_{k+1} + {\Omega_k}^2 x_{k+1} = G(\dot{x}_k, \ddot{x}_k); k = 0,1,2 \tag{7}$$

in addition to

$$x_0(t) = A\cos(\Omega_0 \, \mathrm{t}) \tag{8}$$

Herein satisfies the conditions

$$x_{k+1}(0) = A, \dot{x}_{k+1}(0) \tag{9}$$

The restriction that secular terms [XV] should not appear in the solution determines Ω_k at each stage of the iteration. This process provides the following solutions in order: $x_0(\mathrm{t}), x_1(\mathrm{t}), \ldots\ldots.$ The approach can be applied to any level of approximation, but the solution is limited to a lower order, typically the second [XVI] due to increased algebraic complexity.

III. Solution procedure

We've thought about the function in this case. $\dot{x}\ddot{x}^2$ containing velocity times acceleration squared i.e. jerk function.

Let's have a look at the nonlinear jerk oscillator.

$$\dddot{x} + \dot{x} = -\in \dot{x}\ddot{x}^2 . \tag{10}$$

Indicating the space variable $y(\mathrm{t})$ by the relation $\dot{x} = y$ then equation (10) becomes

$$\ddot{\mathrm{y}} + \mathrm{y} = -\in \mathrm{y}\dot{\mathrm{y}}^2 . \tag{11}$$

Apparently, Eq. (11) can be expressed as

$$\ddot{\mathrm{y}} + \Omega^2 y = \Omega^2 y - (y + \in y\dot{y}^2) . \tag{12}$$

Now the iteration scheme is according to

$$\ddot{\mathrm{y}}_{k+1} + \Omega^2{}_k y_{k+1} = \Omega^2{}_k y_k - (y_k + \in y_k \dot{y}_k^2) . \tag{13}$$

Equation (8) is rewritten as

$$\mathrm{y}_0 = y_0(\mathrm{t}) = A\cos\mathrm{x} . \tag{14}$$

where $x = \Omega\mathrm{t}$, for k=0 the equation (13) becomes

$$\ddot{y}_1 + \Omega_0{}^2 y_1 = \Omega_0{}^2 y_0 - (y_0 + \varepsilon y_0 \dot{y}_0{}^2) . \tag{15}$$

Substituting the right-hand side of equation (15) into the elementary function equation (14) and expanding it into a cosine series, we get

$$\ddot{y}_1 + \Omega_0{}^2 y_1 = a_{11}\cos x + a_{13}\cos 3x, \tag{16}$$

where

$$a_{11} = \frac{1}{4}(-4A + 4A\Omega_0{}^2 - A^3\Omega_0{}^2) \tag{17}$$

We must eliminate $\cos x$ from the right-hand side of equation (16) to avoid secular terms in the solution.

$$\Omega_0{}^2 = \frac{4}{4-A^2} \tag{18}$$

To resolve Eq. (16) and meet the prerequisite of $y_1(0) = A$, we need,

$$y_1 = (A + A^3/32)cosx - A^3/32cos3x \tag{19}$$

The related Ω_1 must be calculated, and this is the first approximation of equation (11). From the solution, the value of Ω_1 will be determined.

$$\ddot{y}_2 + \Omega_1{}^2 y_2 = \Omega_1{}^2 y_1 - (y_1 + \varepsilon y_1 \dot{y}_1{}^2) \tag{20}$$

Substituting y_1 the right-hand side of equation (20) from equation (19) and then trigonometrically expanding we obtain

$$\ddot{y}_2 + \Omega_1{}^2 y_2 = a_{21}\cos x + a_{23}\cos 3x + a_{25}\cos 5x + a_{27}\cos 7x + a_{29}\cos 9x \tag{21}$$

where

$$\mathrm{a}21 = \frac{1}{131072}(-131072A - 4096A^3 + 131072Au^2 - 28672A^3u^2 + 2048A^5u^2 - 352A^7u^2 - 14A^9u^2)$$

$$\mathrm{a}23 = \frac{1}{131072}(4096A^3 + 28672A^3u^2 + 5120A^5u^2 + 224A^7u^2 + 12A^9u^2)$$

$$a25 = (-7168A^5u^2 - 352A^7u^2 - 4A^9u^2)/131072$$

$$a27 = (480A^7u^2 + 15A^9u^2)/131072$$

$$\mathrm{a}29 = -\frac{9A^9u^2}{131072} \tag{22}$$

We must eliminate $\cos x$ from the right-hand side of equation (21) to avoid secular terms in the solution

$$u^2 = \frac{2048(-32 - A^2)}{-65536 + 14336A^2 - 1024A^4 + 176A^6 + 7A^8} \tag{23}$$

Then solving the equation (11) and satisfying the initial condition $y_2(0) = A$, we have

$$y_2 = (A + \mathrm{a}23\frac{1}{8u^2} + \mathrm{a}25\frac{1}{24u^2} + \mathrm{a}27\frac{1}{48u^2} + \mathrm{a}29\frac{1}{80u^2})\cos x$$
$$+\mathrm{a}23\frac{1}{-8u^2}\cos 3x + a251/(-24u^2)cos5x \qquad (24)$$
$$+a271/(-48u^2)cos7x + a291/(-80u^2)cos9x$$

The related Ω_2 must be calculated, and this is the second approximation of equation (11). From the solution, the value of Ω_2 will be determined.

$$\ddot{y}_3 + \Omega_2{}^2 y_3 = \Omega_2{}^2 y_2 - (y_2 + epy_2\dot{y}_2{}^2) \qquad (25)$$

Substituting y_2 from equation (24) into the right-hand side of equation (25) and then expanding in a trigonometric reduction we obtain,

$$\ddot{y}_3 + \Omega_2{}^2 y_3 = \sum_{i=1}^{14} a_{3(2i-1)} \cos(2i-1)x \qquad (26)$$

where

$$a_{31} = -A - A^3/32 - (5A^5)/3072 - (53A^7)/196608 + (13A^9)/7864320$$
$$+Av^2 - \frac{7A^3v^2}{32} + \frac{53A^5v^2}{3072} + \frac{245A^7v^2}{196608} - \frac{523A^9v^2}{7864320} - \frac{18061A^{11}v^2}{188743680}$$
$$-\frac{14959A^{13}v^2}{2013265920} - \frac{69847A^{15}v^2}{85899345920} - \frac{221477A^{17}v^2}{3092376453120}$$
$$-\frac{506173A^{19}v^2}{158329674399744} - \frac{434041A^{21}v^2}{5629499534213120} - \frac{78229A^{23}v^2}{90071992547409920}$$
$$-\frac{1741511A^{25}v^2}{25940733853654056960} + \frac{332591A^{27}v^2}{864691128455135232000}$$

$$a_{33} = \frac{A^3}{32} + \frac{A^5}{256} + \frac{5A^7}{16384} - \frac{A^9}{524288} + \frac{7A^3v^2}{32} + \frac{9A^5v^2}{256} - \frac{9A^7v^2}{16384}$$
$$+\frac{925A^9v^2}{1572864} + \frac{5797A^{11}v^2}{62914560} + \frac{33415A^{13}v^2}{2415919104} + \frac{3271207A^{15}v^2}{2319282339840}$$
$$+\frac{3074959A^{17}v^2}{24739011624960} + \frac{14296031A^{19}v^2}{1979120929996800} + \frac{147282323A^{21}v^2}{759982437118771200}$$
$$+\frac{11992073A^{23}v^2}{4053239664633446400} + \frac{14502773A^{25}v^2}{129703669268270284800}$$
$$-\frac{7698191A^{27}v^2}{1556444031219243417600 0}$$

$$\cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots \qquad (27)$$

$$a_{327} = \frac{59049A^{27}v^2}{4611686018427387904000}$$

We must eliminate $\cos x$ from the right-hand side of equation (26) to avoid secular terms in the solution

$$\begin{aligned}
v^2 = -(&329853488332800(7864320 + 245760A^2 + 12800A^4 \\
&+ 2120A^6 - 13A^8)) / (-2594073385365405696000 \\
&+ 5674535530486824960000A^2 - 447545212969943040000A^4 \\
&- 3232564185661440000A^6 + 172513374398054400A^8 \\
&+ 248228493865779200A^{10} + 19274524734259200A^{12} \\
&+ 2109308770713600A^{14} + 185788373401600A^{16} + 8293138432000A^{18} \\
&+ 200006092800A^{20} + 2252995200A^{22} + 174151100A^{24} - 997773A^{26})
\end{aligned} \tag{28}$$

Then solving equation (26) and satisfying the initial condition

$$y_3(0) = A, \text{ we have } y_3 = \sum_{j=1}^{14} b_{2j-1} \cos(2i-1)x \tag{29}$$

where

$$\begin{aligned}
\mathrm{b}_1 = -(&A(147031371570260622171527184384000000 \\
&+ 918946072314128888572044902400000A^2 + 622203069796024768303988736000000A^4 \\
&+ 142089643342842194684805120000000A^6 + 8525378600570531681088307200000A^8 \\
&+ 1889116149170228957282304000000A^{10} + 209106609219019074856550400000A^{12} \\
&+ 21929326360803287865753600000A^{14} + 1873509959345723827814400000A^{16} \\
&+ 127757090566156786335744000A^{18} + 4080485188096653000704000A^{20} \\
&+ 183763095151974547456000A^{22} + 12809151604503085056000A^{24} \\
&+ 234644755833438617600A^{26} + 3078874214586188800A^{28} \\
&+ 10619115588909600A^{30} - 83143004270290A^{32} + 48008990497A^{34})) / \\
&(18696005703005551932211200000(-7864320 - 245760A^2 - 12800A^4 \\
&- 2120A^6 + 13A^8))
\end{aligned}$$

$$\begin{aligned}
b_3 = (&A^3(30600934808995300984750080000 + 47813960639055157788672000000A^2 \\
&+ 10957365979783473659904000A^4 + 88717309779496874803200000A^6 \\
&+ 15571249754829018365952000A^8 + 17820574840519026278400000A^{10} \\
&+ 18647203515248463052800 0A^{12} + 16064292042074750976000A^{14} \\
&+ 1160397156319546572800A^{16} + 44743332907188224000A^{18} \\
&+ 2049463634257510400A^{20} + 133924104018329600A^{22} \\
&+ 3083897589432320A^{24} + 33726136985600A^{26} \\
&+ 871227569600A^{28} - 8902960600A^{30} + 10276913A^{32})) / \\
&(124515522497539473408000(-7864320 - 245760A^2 \\
&- 12800A^4 - 2120A^6 + 13A^8))
\end{aligned}$$

$$\cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \tag{30}$$

$$b_{27} = -(59049A^2 7) / 335730742141513839411200 0$$

The related Ω_3 must be calculated, and this is the third approximation of equation (11). From the solution, the value of Ω_3 will be determined.

$$\ddot{y}_4 + {\Omega_3}^2 y_4 = {\Omega_3}^2 y_3 - (y_3 + epy_3{\dot{y}_3}^2) \tag{31}$$

Substituting y_3 from equation (29) into the right-hand side of equation (11) and then expanding trigonometrically we get,

$$y_4 = \sum_{k=1}^{41} c_{2k-1}\cos(2k-1)x \tag{32}$$

where

$$\begin{aligned}
&c1 = -(A(-279558165905478414954568193377437504715310821563020655840603 50\\
&8243395614608855643502099332346847325121161723877684803095916617961098 4\\
&448000000000000000000000 - 524171561072772028039815362582695321341207 79\\
&0430663729701131577956366777391604331566436248150338734602178232270659 0\\
&058048436586770595840000000000000000000000000A^2 - 682515053480171911510176 2\\
&5336288453299636431045659339804834840879735257472865147339379719811242 0\\
&2273465862399357539138083901805690880000000000000000000000000A^4 - \cdots\\
&+345049231097276724147428936012202497986312096630624393512188505347A^{138}))\\
&/(39091167742404878595814630503582283220265845135743791311360017527364 99\\
&07174623136490455040000000000000000(7864320 + 245760A^2 + 12800A^4 + 2120A^6 - 13A^8)^3\\
&(14703137157026062217152718438400000 + 918946072314128888572044902400000A^2\\
&+\cdots + 48008990497A^{34}))
\end{aligned}$$

$$\begin{aligned}
&c3 = -(A^3(9571279230532637861070005629806986668898311827912101222101785887\\
&744396879419737701861817478185923006614271995076231036928000000000000000\\
&0000 + 269192228358730439842593908338321500627650201600278468716127280928\\
&1116223368012286486361657397908456102639986151899791360000000000000000000\\
&0A^2 + 339605936239602059523642777880521337597382777481832790394188352096 8\\
&074865128626611423766442782083584666524982529595801600000000000000000000A^4\\
&+\cdots + 8540114790177794046842699923089956103353248135560442A^{134}\\
&-7759403988844385371820548627440371863267474381921A^{136}))/\\
&(428278651147427737741281943928376128993103411557911854394748760942767 50\\
&540800000000000(7864320 + 245760A^2 + 12800A^4 + 2120A^6 - 13A^8)^3\\
&(14703137157026062217152718438400000 + 918946072314128888572044902400000A^2\\
&+6222030697960247683039887360000A^4 + 142089643342842194684805120000 0A^6\\
&+85253786005705316810883072000 0A^8 + 18891161491702289572823040000 0A^{10}\\
&+2091066092190190748565504000 0A^{12} + \cdots + 10619115588909600A^{30} - 83143004270290A^{32}\\
&+48008990497A^{34}))
\end{aligned}$$

$$\cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \tag{33}$$

$$\begin{aligned}
&c81 = 1500946352969991 21A^{81}/(99297236533923907996295417102785\\
&02397071124064268722433465324352307200000000000)
\end{aligned}$$

Similarly, the method can be obtained by higher order approximation and respectively Eqs. (19), (24), (29), (32),…. represent the approximate frequency of the oscillator (11). However, due to the increasing algebraic complexity, most approximation techniques are applied to the second or third approximation.

IV. Results and Discussions

An iteration approach based on Mickens' [XV] iteration method is created to solve a class of nonlinear jerk equations. We demonstrate the viability of the modified iterative method in this section by contrasting it with results from earlier iterations and with precise results for nonlinear jerk equations. We looked at the percentage error (denoted by error (%)) by the definition to confirm the accuracy.

$$\text{Error} = \left| \frac{T_e - T_k}{T_e} \right| \times 100\%$$

Where the various approximate periods obtained by $T_0; k = 0, 1, 2, ...$ are illustrated the modified method and T_e represents the corresponding exact period of the oscillator.

Now we show the comparison of oscillator results. That is, the velocity of the displacement time is for the acceleration of time, and the jerk function contains the velocity. Nowadays, nonlinear jerk oscillators (provided by Eq. (11)) have approximate solutions, frequencies, and approximate periods that have been discovered by various methods without iteration by Gottlieb [I], Ma et al. [XIV], Ramos [XVII-XIX], and Leung & Guo [XIII]. To get a rough solution for the oscillator, we changed the iterative technique. The process is pretty easy to follow. It has been demonstrated that, in the majority of circumstances, our method produces results that are noticeably superior to those of other solutions, and that, on occasion, it is almost exact. The first, second, third, and fourth approximate frequencies $\Omega_0, \Omega_1, \Omega_2$ and Ω_3 have been calculated here and the corresponding periods are T_0, T_1, T_2 and T_3 .

Tables 1 and 2 contain all the data. To compare the estimated frequencies, Tables 1 and 2 also include the results of previous studies conducted independently by Gottlieb [I], Zheng et al. [XXI], and Haque [VII]. A graph is provided in Figure 1 where the comparative graph of our obtained result and the exact result is presented.

Table 1: The periods obtained by present's technique of $\ddot{x}+\dot{x}=-\in\dot{x}\ddot{x}^2$ with percentage errors:

A	T_{exact}	Modified T_0 Er(%)	Modified T_1 Er(%)	Modified T_2 Er(%)	Modified T_3 Er(%)	Modified T_4 Er(%)
0.1	6.275333	6.275326 1.05 e^{-4}	6.275334 1.23 e^{-5}	6.275334 1.25 e^{-5}	6.275334 1.25 e^{-5}	6.275334 1.25 e^{-5}
0.2	6.251809	6.251690 1.90 e^{-3}	6.251808 1.35 e^{-5}	6.251809 3.39 e^{-7}	6.275334 3.76 e^{-1}	6.251809 3.62 e^{-7}
0.5	6.088449	6.083668 7.85 e^{-2}	6.088246 3.34 e^{-3}	6.088451 3.24 e^{-5}	6.088449 5.63 e^{-6}	6.088449 1.39 e^{-6}
1	5.527200	5.441398 1.55	5.513543 2.47 e^{-1}	5.527511 5.62 e^{-3}	5.527659 8.30 e^{-3}	5.527239 7.10 e^{-4}
1.5	4.690247	4.155936 11.39	4.518740 3.66	4.683187 1.51 e^{-1}	4.715160 5.31 e^{-1}	4.698991 1.86 e^{-1}

T_0, T_1, T_2, T_3 and T_4 respectively denote initial, first, second, third and fourth modified approximate periods. Er(%) denotes percentage error.

Table 2: Comparison between the approximate periods produced by our method and other results already published and the exact periods T_e of $\ddot{x}+\dot{x}=-\in\dot{x}\ddot{x}^2$

A	T_{exact}	Modified T_4 Er(%)	T_G Er(%)	T_Z Er(%)	T_{HD} Er(%)
0.1	6.275333	6.275334 1.25 e^{-5}	6.2753264 1.18 e^{-4}	6.2753338 2.75 e^{-7}	6.275334 2.39 e^{-7}
0.2	6.251809	6.251809 3.62 e^{-7}	6.251690 1.90 e^{-3}	6.2518088 2.68 e^{-6}	6.251809 3.39 e^{-7}
0.5	6.088449	6.088449 1.39 e^{-6}	6.083668 7.85 e^{-2}	6.0884161 5.41 e^{-4}	6.088451 3.24 e^{-5}
1	5.527200	5.527239 7.10 e^{-4}	5.441398 1.55	5.525570 2.95 e^{-2}	5.527511 5.62 e^{-3}
1.5	4.690247	4.698991 1.86 e^{-1}	4.155936 11.39	4.672129 3.86 e^{-1}	4.683187 1.51 e^{-1}

T_4 denotes fourth modified approximate periods; $T_G, T_Z,$ and T_{HD} respectively denote approximate periods obtained by Gottlieb, Zheng et al., and Haque Er(%) denote percentage error.

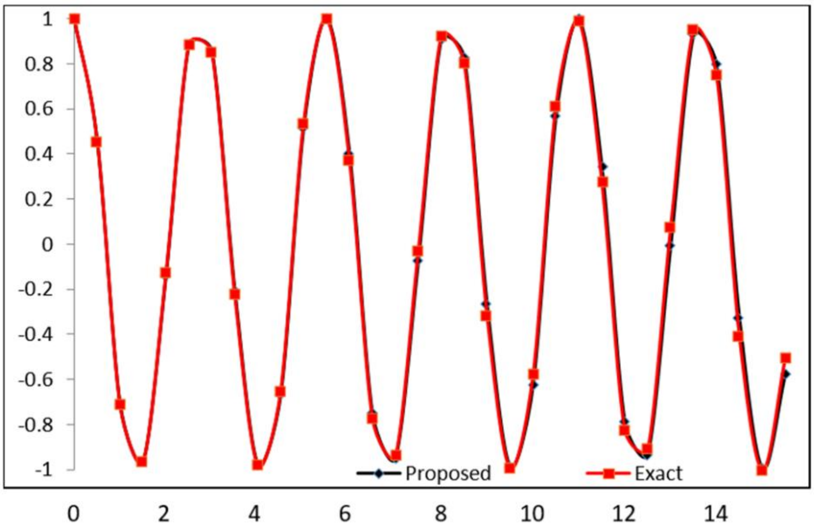

Fig. 1. The fourth-order approximate solutions for A=1 of $\ddot{x}+\dot{x}=-\in\dot{x}\ddot{x}^{2}$ compare with the corresponding numerical solution.

Conclusion:

For the nonlinear oscillator, we employed a very simple but effective strategy. Applying the suggested method, it has been shown that the nonlinear oscillators' first, second, third, and fourth approximate periods yield results that are quite similar to the exact result. The method is not only effective for this model but also effective for other nonlinear models. The current approach is also broadly useful in engineering and science and can be used as an example for several other applications in the search for periodic solutions of nonlinear oscillations.

Conflict of Interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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