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n-KERNELS OF SKELETAL CONGRUENCES ON A DISTRIBUTIVE NEARLATTICE

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Abstract

In this paper, the author studied the skeletal congruences θ^* of a distributive nearlattice S, where $*$ represents the pseudocomplement. Then the author described $\theta(I)^*$, where $\theta(I)$ is the smallest congruence of S containing n-ideal I as a class and showed that I^+ is the n-kernel of $\theta(I)^*$.

In this paper, the author established the following fundamental results:

When n is an upper element of a distributive nearlattice S, the author has shown that the n-kernels of the skeletal congruences are precisely those n-ideals which are the intersection of relative annihilator ideals and dual relative annihilator ideals whose endpoints are of the form $x \vee n$ and $x \wedge n$ respectively.

For a central element n of a distributive nearlattice S, the author proved that $P_n(S)$ is disjunctive if and only if the n-kernel of each skeletal congruence is an annihilator n-ideal.

Finally, the author discussed that $P_n(S)$ is semi-Boolean if and only if the map $\theta \to Ker_n\theta$ is a lattice isomorphism of $SC(S)$ onto $K_nSC(S)$ whose inverse is the map $I \to \theta(I)$ where I is an n-ideal and n is a central element of S.

Keywords: n-Kernels of skeletal congruence, Pseudo complement, Annihilator n-ideal, Disjunctive nearlattice, Semi-Boolean algebra.

I. Introduction

In this paper, the author will be concerned with a distributive nearlattice S with a fixed element n. Skeletal congruences on distributive lattices have been studied by Cornish in [VIII]. Also, skeletal congruences on distributive nearlattices have been studied extensively by Akhter [VI]. Cornish [VIII] studied the Kernels of Skeletal congruences on the distributive lattice. On the other hand Latif in [III] has generalized the results of [VIII] for n-ideals in lattices.

In this paper, the author extended and generalized those results for nearlattices. A nearlattice S is a meet semilattice with the property that any two elements possessing a common upper bound, have a supremum. Nearlattice S is distributive if for all $x, y, z \in S$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ provided $y \vee z$ exists. For detailed literature on nearlattices and their congruences and ideals, we refer the reader to [VII], [IX], [I] and [II]. Here $C(S)$ denotes the lattice of congruences of S. For any $\theta \in C(S)$, θ^* denotes the pseudocomplement of θ. So by its definition, $\theta \cap \emptyset = \omega$ iff $\emptyset \leq \theta^*$, $\emptyset \in C(S)$. The existence of θ^* is guaranteed by the fact that $C(S)$ is a distributive algebraic lattice. A non-empty subset I of a nearlattice S is ideal if it is hereditary and closed under existent finite suprema. We denote the set of all ideals of S by $I(S)$. For a distributive nearlattice S with 0, $I(S)$ is pseudo-complimented. The pseudocomplement I^* of an ideal, I is the annihilator ideal

$I^* = \{x \in S : x \wedge i = 0 \; for \; all \; i \in I\}$.

The skeleton $SC(S) = \{\theta \in C(S) : \theta = \theta^{**}$.

The kernel of congruence θ is $ker\theta = \{x \in S : x \equiv 0(\theta)\}$. Of course $ker\theta(I) = I$.

We also denote $KSC(S) = \{ker\theta : \theta \in SC(S)\}$

For a fixed element $n \in S$, a convex subnearlattice of S containing n is called an n-ideal. Since the lattice of n-ideals $I_n(S)$ of a distributive nearlattice S is a distributive algebraic lattice, so $I_n(S)$ is pseudocomplemented

An element s of a nearlattice S is called standard if for all $t, x\ y \in S$,

$$t \wedge [(x \wedge y) \vee (x \wedge s)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge s).$$

The element s is called neutral if
(i) s is standard and

(ii) for all $x, y, z \in S$, $s \wedge [(x \wedge y) \vee (x \wedge z)] = (s \wedge x \wedge y) \vee (s \wedge x \wedge z)$.

An element n of a nearlattice S is called medial if $m(x, n, y) = (x \wedge y) \vee (x \wedge n) \vee (y \wedge n)$ exists in S for all $x, y \in S$. An element n of a nearlattice S is called an upper element if $x \vee n$ exists for all $x \in S$. An element n of a nearlattice S is called a central element of S if it is neutral, upper, and complemented in each interval containing it.

If n is a medial element, then for any n-ideal I of a distributive nearlattice S, we define $I^+ = \{x \in S : m(x, n, i) = n \; for \; all \; i \in I\}$. Obviously, I^+ is an n-ideal and $I \cap I^+ = \{n\}$. We call I^+ the annihilator n-ideal of I which is the pseudocomplement of I in $I_n(S)$.

We define the n-kernel of a congruence θ by

$$Ker_n\theta = \{x \in S: x \equiv n\theta\} \text{ which is an } n\text{-ideal.}$$

$\theta \in C(S)$ is called dense if $\theta^* = \omega$, while an n-ideal I is called dense if $I^+ = \{n\}$. A non-empty subset T of a nearlattice S is called join-dense if each $y \in S$ is the join of its predecessors in T, while T is called meet-dense if each $y \in S$ is the meet of its successors in T. A distributive nearlattice S with 0 is called disjunctive if $0 \leq a < b$ implies the existence of $x \in S$ such that $x \wedge a = 0$ and $0 < x \leq b$. A nearlattice S with 0 is called semi-Boolean if it is distributive and the interval $[0, x]$ is complemented for each $x \in S$.

An n-ideal generated by a single element a is called principal n-ideal and denoted by $< a >_n$. The set of principal n-ideals is denoted by $P_n(S)$. When $n \in S$ is standard and medial then for any $a \in S$

$$< a >_n = \{y \in S: y = (y \wedge a) \vee (y \wedge n) \vee (a \wedge n) .$$

When n is an upper element, then $< a >_n$ is the closed interval $[a \wedge n, a \vee n]$.

In this paper, we generalize several results of [5] on n-kernels of skeletal congruences in a distributive nearlattice.

II. Main results

To obtain the main results of this paper we need the following theorems.

The following theorems are due to [V]. These will be needed for further development of this paper.

Theorem 2.1. In a distributive nearlattice S the mapping $\mathrm{I} \to \theta(\mathrm{I})$ is an embedding from $\mathrm{I_n(S)}$ to $\mathrm{C(S)}$ where $\mathrm{I_n(S)}$ is the lattice of n-ideals of S and $\mathrm{C(S)}$ is the lattice of congruences of S.

Theorem 2.2. For a distributive nearlattice S with 0, the following conditions hold:

(i) For $a \leq b$ $(a, b \in S), x \equiv y(\theta(a, b)')$ if and only if $x \wedge b) \vee a = (y \wedge b) \vee a$ where $\theta(a, b)'$ is the complement of $\theta(a, b)$.

(ii) For any $\theta \in C(S)$, $x \equiv y(\theta^*)$ $(x, y \in S)$ if and only if for each $a, b \in S$ with $a \leq b$ and $a \equiv b(\theta)$; $(x \wedge b) \vee a = (y \wedge b) \vee a$.

(iii) For any $\theta \in C(S)$, $x \equiv y(\theta^*)$ if and only if $\theta(0, x) \cap \theta = \theta(0, y) \cap \theta$ if and only if $\psi_x \cap \theta = \psi_y \cap \theta$.

Theorem 2.3. Let S be a distributive nearlattice with an upper element n. Then for any $\theta \in \mathrm{C(S)}$, $\mathrm{x} \equiv \mathrm{y}(\theta^*)$ if and only if $\theta(\mathrm{n}, \mathrm{x}) \cap \theta = \theta(\mathrm{n}, \mathrm{y}) \cap \theta$.

Recall that the n-kernel of a congruence θ is given by $\mathrm{Ker}_n\theta = \{x \in S: x \equiv n\theta\}$, which is also an n-ideal.

Theorem 2.4. If S is a distributive nearlattice and n ∈ S is an upper element, then the following conditions hold :

(i) For any n-ideal I, $x \equiv y(\theta(I)^*)$ $(x, y \in S)$ if and only if $< x >_n \cap I =< y >_n \cap I$ i.e, if and only if $m(x, n, i) = m(y, n, i)$ for all $i \in I$.

(ii) For an n-ideal I, both $\theta(I^+)$ and $\theta(I)^*$ have I^+ as their n-kernel.

(iii) The n-kernels of the skeletal congruences are precisely those n-ideals that are the intersection of relative annihilator ideals and dual relative annihilator ideals whose endpoints are of the form $x \vee n$ and $x \wedge n$ respectively.

(iv) Each principal n-ideal in a distributive nearlattice is the intersection of relative annihilator ideals and dual relative annihilator ideals whose endpoints are of the form $x \vee n$ and $x \wedge n$ respectively.

Proof. (i) For any two n-ideals I and J of S, the author $\theta(I \cap J) = \theta(I) \cap \theta(J)$. Also, since n is upper so $\theta(n, x) = \theta(n \wedge x, n \vee x) = \theta(< x >_n)$. Then by Theorem 2.3, $x \equiv y(\theta(I)^*)$ if and only if $\theta(n, x) \cap \theta(I) = \theta(n, y) \cap \theta(I)$ if and only if $\theta(< x >_n) \cap \theta(I) = \theta(< y >_n) \cap \theta(I)$ if and only if $\theta(< x >_n \cap I) = \theta(< y >_n \cap I)$ if and only if $< x >_n \cap I =< y >_n \cap I$ by theorem 2.1, if and only if $m(x, n\ i) = m(y, n, i)$ for all $i \in I$.

Hence (i) holds.

(ii) If $x \in Ker_n(\theta(I)^*)$, then $x \equiv n(\theta(I)^*)$. Then by above (i), $< x >_n \cap I =< n >_n \cap I$ if and only if $m(x, n, i) = m(n, n, i) = n$ for all $i \in I$ and so $x \in I^+$. Thus (ii) holds.

(iii) Let $a, b \in S$ with $a \leq b$. Since $\theta(a, b)^* = \theta(a, b)'$, so by Theorem 2.2, $x \in Ker_n(\theta(a, b)^*)$ if and only if $(x \wedge b) \vee a = (n \wedge b) \vee a$ (Since $a \leq b$, $(x \wedge b) \vee a$ and $(n \wedge b) \vee a$ exist by the upper bound property of S).

Now, we shall show that $(x \wedge b) \vee a = (n \wedge b) \vee a$ is equivalent to $x \in< b \vee n, a \vee n > \cap < a \wedge n, b \wedge n >_d$. Since $(x \wedge b) \vee a = (n \wedge b) \vee a$ implies $x \wedge b \leq a \vee n$, then the author $x \wedge (b \vee n) = (x \wedge b) \vee (x \wedge n) \leq a \vee n$, and so $x \in< b \vee n, a \vee n >$. Again from $(x \wedge b) \vee a = (n \wedge b) \vee a$, the author $b \wedge n \leq (x \wedge b) \vee a$. So $(b \wedge n) \leq (x \wedge b \wedge n) \vee (a \wedge n) \leq x \vee (a \wedge n)$, which implies that $x \in < a \wedge n, b \wedge n >_d$.

Hence $x \in< b \vee n, a \vee n > \cap < a \wedge n, b \wedge n >_d$.

Conversely, let $x \in< b \vee n, a \vee n > \cap < a \wedge n, b \wedge n >_d$. .

Then $x \in < b \vee n, a \vee n >$ and $\in < a \wedge n, b \wedge n >_d$.

So $x \wedge (b \vee n) \leq a \vee n$ and $x \vee (a \wedge n) \geq b \wedge n$.

Now, $x \wedge (b \vee n)$ $(x \wedge b) \vee a = (n \wedge b) \vee a \leq a \vee n$ implies

$$x \wedge b = x \wedge b \wedge (b \vee n) \leq (a \vee n) \wedge b$$

$= (a \wedge b) \vee (b \wedge n) = a \vee (b \wedge n)$ and so $(x \wedge b) \vee a \leq (b \wedge n) \vee a$.

On the other hand, $b \wedge n \leq x \vee (a \wedge n)$ implies $b \wedge n \leq b \wedge \left(x \vee (a \wedge n)\right)$

$= (x \wedge b) \vee (a \wedge b \wedge n) = (x \wedge b) \vee (a \wedge n)$ and so $(n \wedge b) \vee a \leq (x \wedge b) \vee a$. Hence $(x \wedge b) \vee a = (n \wedge b) \vee a$.

Since for any $\theta \in C(S)$, $\theta^* = \cap \{\theta(a, b)^* : a \equiv b\theta\}$, hence the result follows.

(iv) Since each principal n-ideal

$$< a >_n = \ker_n \theta(< a >_n) = \ker_n \theta(a \wedge n, a \vee n)$$

and since $\theta(a \wedge n, a \vee n)$ is skeletal so by (iii) the result follows.

A non-empty subset T of a nearlattice S is called $large$ if $x \wedge t = y \wedge t$ for all $t \in T$, $x, y \in S$ implies $x = y$ while recall that T is join-dense if each $z \in S$ is the join of its predecessors in T.

A non-empty subset T of a nearlattice S is called *small* if for all $x, y \in S$ with $x \leq y$ and $y = x \vee (y \wedge t)$ for all $t \in T$ imply $x = y$ while recall that T is meet-dense if each $z \in S$ is the meet of its successors in T.

The following lemma is due to [II] and it will be needed for our next theorem.

Lemma 2.5. A convex superlattice J of a distributive nearlattice S is large if and only if it is join-dense in S.

Theorem 2.6. Let S be a distributive nearlattice with an upper element n. Then for any n-ideal I of S, θ(I) is dense in C(S) if and only if I is both meet and join-dense.

Proof. Let θ(I) be dense in C(S), that is $\theta(\mathrm{I})^* = \omega$. Suppose $x \wedge i = y \wedge i$ for all $i \in I$. Then $m(x, n\, i) = m(y, n, i)$ for all $i \in I$. Then by Theorem 2.4 (i), the author $x \equiv y\theta(I)^* = \omega$. Hence $x = y$. This implies I is large and so by Lemma 2.5, I is join-dense.

Again for $x, y \in S$ with $x \leq y$ let $y = x \vee (y \wedge i)$ for all $i \in I$. Since $n \in I$, so $y = x \vee (y \wedge n)$. This implies $x \vee n = y \vee n$; as n is upper.

Now $m(x, n, y \wedge i) = (x \vee n) \wedge \big(n \vee (y \wedge i)\big) \wedge (x \vee (y \wedge i))$

$= (y \vee n) \wedge \big(n \vee (y \wedge i)\big) \wedge y$

$= m(y, n, y \wedge i)$ for all $i \in I$.

Hence $(x \wedge n) \vee (x \wedge i) \vee (y \wedge n \wedge i) = (y \wedge i) \vee (y \wedge n) \vee (n \wedge y \wedge i)$

and so

$(x \wedge n) \vee (x \wedge i) \vee (n \wedge i) = (y \wedge i) \vee (y \wedge n) \vee (n \wedge i)$.

That is, $m(x, n, i) = m(y, n, i)$ for all $i \in I$. Hence by Theorem 2.4 (i), the author $x \equiv y\theta(I)^* = \omega$. This implies that $x = y$ and so I is meet-dense.

Conversely, let I be both meet join-dense and $x \equiv y\theta(I)^*$ with $x \leq y$. Then by Theorem 2.4 (i), the author $m(x, n, i) = m(y, n, i)$ for all$i \in I$.

Now, $(x \vee n) \wedge i = m(x, n, i) \wedge i = m(y, n, i) \wedge i = (y \vee n) \wedge i$ for all $i \in I$. So by Lemma 2.5, $x \vee n = y \vee n$.

Again $(x \wedge n) \vee ((y \wedge n) \wedge i) = \big(y \wedge (x \wedge n)\big) \vee (y \wedge (n \wedge i))$ as $x \leq y$

$= y \wedge [(x \wedge n) \vee (n \wedge i)]$

$= y \wedge m(x \wedge n, n, i)$

$= y \wedge m(y \wedge n, n, i)$

$= y \wedge [(y \wedge n) \vee (n \wedge i)]$

$= (y \wedge n) \vee (y \wedge n \wedge i)$

This implies $(x \wedge n) \vee \big((y \wedge n) \wedge i\big) = y \wedge n$ for all $i \in I$

Since I is meet-dense, so $x \wedge n = y \wedge n$.

Hence by the distributivity of S, $x = y$. That is, $\theta(I)^* = \omega$.

Therefore, $\theta(I)$ is dense in $C(S)$.

The following result is due to [IV] which will be needed for the next theorem of this paper.

Theorem 2.7. For a neutral element n of a nearlattice S, the following conditions are equivalent:

(i) n is central in S.

(ii) n is upper and the map $\phi: P_n(S) \to (n]^d \times [n)$ defined by $\phi(< a >_n) = (a \wedge n, a \vee n)$ is an isomorphism, where $(n]^d$ represents the dual of the lattice $(n]$.

Recall that a distributive nearlattice S with 0 is called disjunctive if $0 \leq a < b$ implies the existence of $x \in S$ such that $x \wedge a = 0$ and $0 < x \leq b$.

Theorem 2.8. Let S be a distributive nearlattice with a central element n. Then the following conditions are equivalent:

(i) $P_n(S)$ is disjunctive.

(ii) For each congruence ϕ, $\phi^* = \theta(Ker_n\phi)^*$.

(iii) For each congruence ϕ, $Ker_n(\phi^*) = (Ker_n\phi)^+$.

(iv) For each congruence ϕ, $Ker_n(\phi^{**}) = (Ker_n\phi)^{++}$.

(v) The n-Kernel of each skeletal congruence is an annihilator n-ideal.

Proof. (i) $\Rightarrow$ (ii). Suppose (i) holds. Since $\theta(Ker_n\phi) \subseteq \phi$, so the author $\phi^* \subseteq \theta(Ker_n\phi)^*$. So it is sufficient to prove that $\phi \cap \theta(Ker_n\phi)^* = \omega$. Suppose $x \leq y$ and $x \equiv y\ (\phi \cap \theta(Ker_n\phi)^*)$ implies $x \equiv y\phi$ and $x \equiv y\theta(Ker_n\phi)^*$.

If $x < y$, then either $x \wedge n < y \wedge n\ or\ x \vee n < y \vee n$. Suppose $x \vee n < y \vee n$. Since $P_n(S)$ is disjunctive, so by Theorem 2.7, $[n)$ is also disjunctive. So there exists $n < a \leq y \vee n$ such that $a \wedge (x \vee n) = n$. Then $n = a \wedge (x \vee n) \equiv a \wedge (y \vee n = a(\phi)$ and so, $a \in Ker_n\phi$.

Since $x \equiv y\theta(Ker_n\phi)^*$ so $x \vee n \equiv y \vee n\theta(Ker_n\phi)^*$ and since $a \in Ker_n\phi$, so by Theorem 2.4,$m(x \vee n, n, a) = m(y \vee n, n\ a)$ that is,

$$\big((x \vee n) \wedge n\big) \vee \big(a \wedge (x \vee n)\big) \vee (n \wedge a) = \big((y \vee n) \wedge n\big) \vee \big(a \wedge (y \vee n)\big) \vee (n \wedge a)$$

and so $n \vee \big(a \wedge (x \vee n)\big) = n \vee a$. This implies $n = a$ which is a contradiction.

Therefore $x = y$ and so $\phi \cap \theta(Ker_n\phi)^* = \omega$. Thus $\theta(Ker_n\phi)^* \subseteq \phi^*$.

Hence $\phi^* = \theta(Ker_n\phi)^*$.

Since by Theorem 2.4 (ii), $\theta(I)^*$ and $\theta(I^+)$ have I^+ as their n-kernels,

so (ii) $\Rightarrow$ (iii) is obvious. (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) are also obvious. Finally, we need to prove that (v) $\Rightarrow$ (i).

Suppose (v) holds. Let $n \leq a < c$. Then by Theorem 2.4(iii), $< c, a >$ is the n-kernel of a skeletal congruence. Since (v) holds, so there is an annihilator n-ideal J such that $< c, a >= J = J^{++}$. Then $a \wedge c \leq a$ implies $a \in < c, a >= J = J^{++}$. Since $a < c$ implies $c \notin < c, a >= J = J^{++}$.

So there exists $e \in J^+$ such that $m(c, n, e) \neq n$. But $m(a, n, e) = n$ implies $(a \wedge e) \vee n = n$. That is, $a \wedge (e \vee n) = n$ and so $a \wedge ((e \vee n) \wedge c) = n$.

Also $m(c,n,e) \neq n$ implies $(e \vee n) \wedge c > n$ and so $n < (e \vee n) \wedge c \leq c$ with $a \wedge ((e \vee n) \wedge c) = n$. Thus $[n)$ is disjunctive. A dual proof of this gives that $(n]$ is dual disjunctive and so by Theorem 2.7, $P_n(S)$ is disjunctive.

Recall that a nearlattice S with 0 is semi-Boolean if it is distributive and the interval $[0, x]$ is complemented for each $x \in S$.

Theorem 2.9. Let S be a distributive nearlattice with a central element n. Then $P_n(S)$ is semi-Boolean if and only if the map $\theta \rightarrow Ker_n\theta$ is a lattice isomorphism of $SC(S)$ onto $K_nSC(S)$ whose inverse is the map $I \rightarrow \theta(I)$, where I is an n-ideal of S.

III. Conclusion

In this paper, we extend the concept of n-Kernels of skeletal congruences on a distributive nearlattice and establish several fundamental results on n-Kernels of skeletal congruences. We also give the notion of n-Kernels of skeletal congruences and prove some interesting results on n-Kernels of skeletal congruences in a distributive nearlattice.

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Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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