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AN EXTENDED STUDY TO DETERMINE THE BEST LOSS FUNCTIONS FOR ESTIMATING THE EXPONENTIAL DISTRIBUTION PARAMETER UNDER JEFFERY AND GAMMA PRIORS

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Abstract

In this research, we compared the Bayesian estimators when estimating the scale parameter for the exponential distribution by using different loss functions under Jeffrey and Gamma priors, as most of the available symmetric and asymmetric loss functions were used, also the balanced and unbalanced loss functions. The simulation results proved the advantage of balanced loss functions with the Gamma prior, and the effectiveness of the balanced loss functions when using Jeffrey prior especially if the value of the weighted coefficient is equal to 0.5, so it is possible to use initial estimators as maximum likelihood estimator to compensate for the lack of prior information around the parameter to be estimated, also the advantage of the balanced general entropy loss function and the balanced weighted square error loss function under Jeffrey prior when the value of the scale parameter for the exponential distribution is less than 1, the preference of the balanced weighted square error loss function and the balanced K loss function if the value of the scale parameter for the exponential distribution is equal to 1, and the preference for the AL-Sayyad balanced loss function and the balanced AL-Bayyati loss function if the value of the scale parameter for the exponential distribution is greater or equal to 2.

Keywords: Bayes Method, Unbalanced Loss Functions, Balanced Loss Functions, Exponential Distribution.

I. Introduction

The estimation of the scale parameter for the exponential distribution has received a very large number of previous studies, and researchers in this field have used comparisons in terms of informational and non-informational prior distributions, as well as their use of different loss functions in those comparisons. In this research we try to achieve three main aims: The aim goal is collecting most of the loss functions and their Bayesian estimates under the Jeffrey and Gamma priors, the second aim is to use the balanced loss functions to compensate for the lack of initial information about the parameter to be estimated, the third aim is to find the best loss function to estimate the scale parameter for the exponential distribution by comparing the estimators of those loss functions.

II. Bayes Method [1], [5]

Let $y_1, y_2, \dots, y_n$ be a random sample from Exponential Distribution (ED), then the probability density function and the reliability function are respectively given as:

$$f(y,\mu) = \mu\, e^{-\mu y} \quad ,\mu\,,y > 0 \tag{1}$$

$$R(y,\mu) = e^{-\mu y}\,,\mu\,,y > 0 \tag{2}$$

μ is a scale parameter.

The likelihood function can be found as follows:

$$L(y_1, y, \dots, y_n|\mu) = \prod_{i=1}^{n} f(y_i,\mu) = \prod_{i=1}^{n} \mu\, e^{-\mu y_i} = \mu^n\, e^{-\mu \sum_{i=1}^{n} y_i} = \mu^n\, e^{-\mu\, Y} \tag{3}$$

Bayesian posterior for μ can be obtained based on the following formula:

$$P(\mu|y) = \frac{L(y_i|\mu)\,\pi(\mu)}{\int_{\forall\mu} L(y_i|\mu)\,\pi(\mu)d\mu} \tag{4}$$

where:

L(y|μ): likelihood function.

π(μ): prior distribution.

P(μ|y): posterior distribution.

Now, the Bayes estimator ($\hat{\mu}_B$) for (μ) under any loss function $L(\hat{\mu},\mu)$ is just an expectation of the risk function, thus ($\hat{\mu}_B$) is the value that makes the risk function as minimum as possible, that is

$$\hat{\mu}_B = Risk(\hat{\mu},\mu) = E\{L(\hat{\mu},\mu)\} = \int_{\forall\mu} L(\hat{\mu},\mu)P(\mu|y)d\mu \tag{5}$$

assuming that the unknown parameter μ has respectively the following Jeffery and Gamma distribution priors:

$$\pi_2(\mu) \propto \frac{1}{\mu} \tag{6}$$

$$\pi_2(\mu) \propto \mu^{\theta-1}e^{-\beta\mu} \qquad ,\mu,\theta,\beta > 0 \tag{7}$$

III. Bayesian Estimation under unbalanced loss functions: [2], [3], [4], [5], [7], [8], [10], [12]

We obtained Bayesian estimators for μ under different loss functions as in table (1) below, those functions are: squared error loss function (SLF), weighted squared error loss function (WLF), modified squared error loss function (MLF), K-loss function (KLF), precautionary loss function (PLF), General entropy loss function (GLF), AL-Bayyati loss function (ALBLF) and AL-Sayyad loss function (ASLF).

Table 1: different unbalanced loss functions (UBLF), Risk, and Bayes Estimators

L.F. formula	$E(\hat{\mu}, \mu)$	Bayes I	Bayes II
$S = (\mu - \hat{\mu})^2$	$E(\mu)$	$\hat{\mu}B1S = \frac{n}{\sum_n^{i=1} y_i}$	$\hat{\mu}B2S = \frac{n+\theta}{\beta + \sum_n^{i=1} y_i}$
$W \frac{(\mu - \hat{\mu})^2}{\mu}$	$[E(\mu^{-1})]^{-1}$	$\hat{\mu}B1W = \frac{n-1}{\sum_n^{i=1} y_i}$	$\hat{\mu}B2W = \frac{n+\theta-1}{\beta + \sum_n^{i=1} y_i}$
$M = \left(\frac{\mu - \hat{\mu}}{\mu}\right)^2$	$\frac{E(\mu^{-1})}{E(\mu^{-2})}$	$\hat{\mu}B1M = \frac{n-2}{\sum_n^{i=1} y_i}$	$\hat{\mu}B2M = \frac{n+\theta-2}{\beta + \sum_n^{i=1} y_i}$
$K = \frac{(\mu - \hat{\mu})^2}{\mu\,\hat{\mu}}$	$\sqrt{\frac{E(\mu)}{E(\mu^{-1})}}$	$\hat{\mu}B1K = \frac{\sqrt{n(n-1)}}{\sum_n^{i=1} y_i}$	$\hat{\mu}B2K = \frac{\sqrt{(n+\theta)(n+\theta-1)}}{\beta + \sum_n^{i=1} y_i}$
$P = \frac{(\mu - \hat{\mu})^2}{\hat{\mu}}$	$\sqrt{E(\mu^2)}$	$\hat{\mu}B1P = \frac{\sqrt{n(n+1)}}{\sum_n^{i=1} y_i}$	$\hat{\mu}B2P = \frac{\sqrt{(n+\theta)(n+\theta+1)}}{\beta + \sum_n^{i=1} y_i}$
$G = \left(\frac{\hat{\mu}}{\mu}\right)^r - r\log\left(\frac{\hat{\mu}}{\mu}\right) - 1$	$[E(\mu^{-r})]^{-\frac{1}{r}}$	$\hat{\mu}B1G = \frac{\left[\frac{\Gamma(n-r)}{\Gamma(n)}\right]^{-\frac{1}{r}}}{\sum_n^{i=1} y_i}$	$\hat{\mu}B2G = \frac{\left[\frac{\Gamma(n+\theta-r)}{\Gamma(n+\theta)}\right]^{-\frac{1}{r}}}{\beta + \sum_n^{i=1} y_i}$
$AB = \mu^b(\mu - \hat{\mu})^2$	$\frac{E(\mu^{b+1})}{E(\mu^b)}$	$\hat{\mu}B1AB = \frac{n+b}{\sum_n^{i=1} y_i}$	$\hat{\mu}B2AB = \frac{n+\theta+b}{\beta + \sum_n^{i=1} y_i}$
$AS = \mu^c(\mu^d - \hat{\mu}^d)^2$	$\left[\frac{E(\mu^{c+d})}{E(\mu^c)}\right]^{\frac{1}{d}}$	$\hat{\mu}B1AS = \frac{\left[\frac{\Gamma(n+c+d)}{\Gamma(n+c)}\right]^{\frac{1}{d}}}{\sum_n^{i=1} y_i}$	$\hat{\mu}B2AS = \frac{\left[\frac{\Gamma(n+\theta+c+d)}{\Gamma(n+\theta+c)}\right]^{\frac{1}{d}}}{\beta + \sum_n^{i=1} y_i}$

IV. Bayesian Estimation under balanced loss functions: [1], [6], [9], [11]

Zellner introduced Balanced Loss Functions (BLF) in (1994), Zellner's formula is defined as follows:

$$L_\omega(\hat{\mu}, \mu) = \omega\, L(\hat{\mu}, \mu_0) + (1 - \omega)\, L(\hat{\mu}, \mu) \quad (8)$$

where:

L_ω: Balanced loss function.

ω: weighted coefficient, where $w \in (0,1)$.

μ_0: any initial estimator for (μ) depends on the sample.

$L(\hat{\mu}, \mu)$: Unbalanced loss function.

$L(\hat{\mu}, \mu_0)$: Unbalanced loss function for the initial estimator.

The balanced loss function heavily depends on the weighted coefficient (ω) and the initial estimator (μ_0).

Lemma:

For estimating (μ) under a balanced loss function $L_{\omega\mu_0}$ and for a prior of, the Bayes estimator depends on $L(\hat{\mu}, \mu,)$ and $P^*(\mu|y)$, where:

$$P^*(\mu|y) = \omega_{\mu_0}(\mu) + (1-\omega)\, E\{P(\mu \mid y)\} \tag{9}$$

that means $P^*(\mu|y)$ is a mixture of (μ_0) and $P(\mu|y)$.

Now if (μ_0) represents Bayes estimators under non-informative prior $\pi_0(\mu) \propto 1$, then Bayes expected formulas under different balanced loss functions and the two priors Jeffry, Gamma $(j = 1,2)$ can be obtained based on the previous lemma as follows:

Balanced squared error loss function (BSLF):

$$\hat{\mu}_{BBjS} = E\{P_j^*(\mu)\} = \omega_{\mu_0}(\mu) + (1-\omega)\, E\{P^*(\mu|y)\} \tag{10}$$

Balanced weighted squared error loss function (BWLF):

$$\hat{\mu}_{BBjW} = [EP^*(\mu^{-1})]^{-1} = \{\omega_{\mu_0}(\mu)^{-1}\} + \left[(1-\omega)\, E\{P_j^*(\mu^{-1}|y)\}\right]^{-1} \tag{11}$$

Balanced modified squared error loss function (BMLF):

$$\hat{\mu}_{BBjM} = \frac{EP_j^*(\mu^{-1})}{EP_j^*(\mu^{-2})} = \frac{\{\omega_{\mu_0}(\mu)^{-1}\}+(1-\omega)\, E\{P_j^*(\mu^{-1}|y)\}}{\{\omega_{\mu_0}(\mu)^{-2}\}+(1-\omega)\, E\{P_j^*(\mu^{-2}|y)\}} \tag{12}$$

Balanced K-loss function (BKLF):

$$\hat{\mu}_{BBjK} = \sqrt{\frac{EP_j^*(\mu)}{EP_j^*(\mu^{-1})}} = \sqrt{\frac{\omega_{\mu_0}(\mu)+(1-\omega)\, E\{P_j^*(\mu|y)\}}{\{\omega_{\mu_0}(\mu)^{-1}\}+(1-\omega)\, E\{P_j^*(\mu^{-1}|y)\}}} \tag{13}$$

Balanced precautionary loss function (BPLF):

$$\hat{\mu}_{BBjP} = \sqrt{EP_j^*(\mu^2)} = \sqrt{\omega_{\mu_0}(\mu^2) + (1-\omega)\, E\{P_j^*(\mu^2|y)\}} \tag{14}$$

Balanced general entropy loss function (BGELF):

$$\hat{\mu}_{BGES_j} = [EP_j^*(\mu^{-r})]^{-\frac{1}{r}} = [\omega_{\mu_0}(\mu^{-r}) + (1-\omega)\, E\{P_j^*(\mu^{-r}|y)\}]^{-\frac{1}{r}} \tag{15}$$

Balanced AL-Bayyati loss function (BABLF):

$$\hat{\mu}_{BBjAB} = \frac{EP_j^*(\mu^{b+1})}{EP_j^*(\mu^{b})} = \frac{\{\omega_{\mu_0}(\mu)^{b+1}\}+(1-\omega)\, E\{P_j^*(\mu^{b+1}|y)\}}{\{\omega_{\mu_0}(\mu)^{b}\}+(1-\omega)\, E\{P_j^*(\mu^{b}|y)\}} \tag{16}$$

Balanced EL-Sayyad loss function (BASLF).

$$\hat{\mu}_{BBjAS} = \left[\frac{EP_j^*(\mu^{c+d})}{EP_j^*(\mu^{c})}\right]^{\frac{1}{d}} = \left[\frac{\{\omega_{\mu_0}(\mu^{c+d})\}+(1-\omega)\, E\{P_j^*(\mu^{c+d}|y)\}}{\{\omega_{\mu_0}(\mu^{c})\}+(1-\omega)\, E\{P_j^*(\mu^{c}|y)\}}\right]^{\frac{1}{d}} \tag{17}$$

Therefore, the Bayes estimators under the two priors and different BLFs are given in table (2) below:

Table 2: different balanced loss functions, Risk, and Bayes Estimators

BBj	Bayes I estimator	Bayes II estimator
$\hat{\mu}BBjS$	$\omega\left[\frac{n+1}{\sum_{n}^{i=1} y_i}\right]+\delta\left[\frac{n}{\sum_{n}^{i=1} y_i}\right]$	$\omega\left[\frac{n+1}{\sum_{n}^{i=1} y_i}\right]+\delta\left[\frac{n+\theta}{\beta+\sum_{n}^{i=1} y_i}\right]$
$\hat{\mu}BBjW$	$\omega\left[\frac{n}{\sum_{n}^{i=1} y_i}\right]+\delta\left[\frac{n-1}{\sum_{n}^{i=1} y_i}\right]$	$\omega\left[\frac{n}{\sum_{n}^{i=1} y_i}\right]+\delta\left[\frac{n+\theta-1}{\beta+\sum_{n}^{i=1} y_i}\right]$
$\hat{\mu}BBjM$	$\omega\left[\frac{n-1}{\sum_{n}^{i=1} y_i}\right]+\delta\left[\frac{n-2}{\sum_{n}^{i=1} y_i}\right]$	$\omega\left[\frac{n-1}{\sum_{n}^{i=1} y_i}\right]+\delta\left[\frac{n+\theta-2}{\beta+\sum_{n}^{i=1} y_i}\right]$
$\hat{\mu}BBjK$	$\omega\left[\frac{\sqrt{n(n+1)}}{\sum_{n}^{i=1} y_i}\right]+\delta\left[\frac{\sqrt{n(n-1)}}{\sum_{n}^{i=1} y_i}\right]$	$\omega\left[\frac{\sqrt{n(n+1)}}{\sum_{n}^{i=1} y_i}\right]+\delta\left[\frac{\sqrt{(n+\theta)(n+\theta-1)}}{\beta+\sum_{n}^{i=1} y_i}\right]$
$\hat{\mu}BBjP$	$\omega\left[\frac{\sqrt{(n+1)(n+2)}}{\sum_{n}^{i=1} y_i}\right]+\delta\left[\frac{\sqrt{n(n+1)}}{\sum_{n}^{i=1} y_i}\right]$	$\omega\left[\frac{\sqrt{(n+1)(n+2)}}{\sum_{n}^{i=1} y_i}\right]+\delta\left[\frac{\sqrt{(n+\theta)(n+\theta+1)}}{\beta+\sum_{n}^{i=1} y_i}\right]$
$\hat{\mu}BBjG$	$\omega\left[\frac{\left\{\frac{\Gamma(n-r)}{\Gamma(n)}\right\}^{-\frac{1}{r}}}{\sum_{n}^{i=1} y_i}\right]+\delta\left[\frac{\left\{\frac{\Gamma(n+r)}{\Gamma(n)}\right\}^{-\frac{1}{r}}}{\sum_{n}^{i=1} y_i}\right]$	$\omega\left[\frac{\left\{\frac{\Gamma(n-r)}{\Gamma(n)}\right\}^{-\frac{1}{r}}}{\sum_{n}^{i=1} y_i}\right]+\delta\left[\frac{\left\{\frac{\Gamma(n+\theta+r)}{\Gamma(n+\theta)}\right\}^{-\frac{1}{r}}}{\beta+\sum_{n}^{i=1} y_i}\right]$
$\hat{\mu}BBjAB$	$\omega\left[\frac{n+b+1}{\sum_{n}^{i=1} y_i}\right]+\delta\left[\frac{n+b}{\sum_{n}^{i=1} y_i}\right]$	$\omega\left[\frac{n+b+1}{\sum_{n}^{i=1} y_i}\right]+\delta\left[\frac{n+\theta+b}{\beta+\sum_{n}^{i=1} y_i}\right]$
$\hat{\mu}BBjAS$	$\omega\left[\frac{\left\{\frac{\Gamma(n+c+d+1)}{\Gamma(n+c+1)}\right\}^{\frac{1}{d}}}{\sum_{n}^{i=1} y_i}\right]+\delta\left[\frac{\left\{\frac{\Gamma(n+c+d)}{\Gamma(n+c)}\right\}^{\frac{1}{d}}}{\sum_{n}^{i=1} y_i}\right]$	$\omega\left[\frac{\left\{\frac{\Gamma(n+c+d+1)}{\Gamma(n+c+1)}\right\}^{\frac{1}{d}}}{\sum_{n}^{i=1} y_i}\right]+\delta\left[\frac{\left\{\frac{\Gamma(n+\theta+c+d)}{\Gamma(n+\theta+c)}\right\}^{\frac{1}{d}}}{\beta+\sum_{n}^{i=1} y_i}\right]$

Formulas in the table prepared and derived by researchers

V. Simulation and Results

For the simulation study, we chose n = 10, 25, and 50 as sample sizes, with different parameter values

$\mu = 0.5,1,2, (\mu, \theta, \beta) = (0.5,0.5,1), (0.5,1,2), (1,1,2), (1,2,3)(2,2,3),$ $(2,3,4), (3,3,4),$ and $(3,4,6)$ furthermore, we selected for the loss functions constants values $a = b = 0.5,2$, $(c, d) = (0.5,0.5), (2,2), (0.5,2), (2,0.5)$, for the ω values we chose $\omega = 0.2, 0.5, 0.7$. Note that we moved away from the default values which give equal results for loss functions such as a = 1, b = 1, c = 1, -2, and d = 1, -2. Simulations are repeated (5000) times to obtain the Bayesian estimators and their root mean square error (RMSE) as follows:

$$\hat{\mu} = \frac{\sum_{q=1}^{Q} \hat{\mu}_q}{Q} , RMSE(\hat{\mu}) = \sqrt{\frac{\sum_{q=1}^{Q} (\hat{\mu}_q - \mu)^2}{Q}} , q = 1,2, \dots , Q \ , Q = 5000$$

From tables (3), (4) below which includes Bayesian estimators and the RMSE values under Jeffrey prior also from the figures (1), (2) which include the RMSE values of the Bayesian estimators when $\mu = 2,3$, it is clear from the RMSE values that the best three estimators respectively are B1G2, BIW, and BIG1, while the estimators B1AS4, B1AB2 and B1AS2 have been obtained the highest for RMSEs respectively, also when using the balanced loss functions for the same experiments, the same above behavior appeared but with lower values for RMSE, where the lowest/highest for RMSEs respectively has been recorded when $\omega = 0.5$ and $\omega = 0.7$.

From table form 5 to10 which represents the different Bayesian estimators and their RMSE values under Gamma prior, furthermore figures (3), (4) represent the RMSE values of the Bayesian estimators when $\mu = \theta = 3, \beta = 4$, $\mu = 3, \theta = 4, \beta = 5$, we note that when $\mu = \theta = 0.5, \beta = 1$, the best three estimators are B2G2, B2M, and B2G1 respectively, while the highest RMSE values appeared respectively at B2AS4, B2AB2 and B2AS2 estimators. When $\mu = 0.5, \theta = 1, \beta = 2$, the smallest RMSEs have been obtained at B2W2, B2G2, and B2G1 estimators respectively, while the estimators B1AS4, B1AB2 and B1AS2 are given the highest RMSEs. When $\mu = 1, \theta = 1, \beta = 2$ and $\mu = 1, \theta = 2, \beta = 3$ the best three estimators are B2S2, B2K, and B2AS1 respectively, and the estimators B2AS4, B2AB2 and B2AS2 are obtained the highest RMSEs, while when $\mu = 2,3, \theta = 2,4, \beta = 3,5$ the best three estimators are B2AS2, B2AB2 and B2AS4 respectively, and the highest RMSEs appeared respectively at B2W, B2AG2 and B2M estimators, also RMSEs increase as ω increase, furthermore, the RMSEs of the BLF estimators are smaller than the RMSEs for the UBLFs.

Table 3: Bayes Estimators under Jeffery prior and when $\mu = 0.5$

LF	n	$\hat{\mu}$	RMSE	BLF	ω = 0.2		ω = 0.5		ω = 0.8	
					$\hat{\mu}$	RMSE	$\hat{\mu}$	RMSE	$\hat{\mu}$	RMSE
B1S	10	0.4085	0.0176	BB1S	0.5807	0.0137	0.4373	0.0082	0.4289	0.0106
	25	0.4177	0.0142		0.5726	0.0111	0.4436	0.0067	0.4360	0.0086
	50	0.4300	0.0103		0.5617	0.0080	0.4521	0.0048	0.4456	0.0062
B1W	10	0.4255	0.0112	BB1W	0.4370	0.0080	0.4551	0.0041	0.4468	0.0057
	25	0.4329	0.0091		0.4433	0.0065	0.4596	0.0033	0.4521	0.0046
	50	0.5570	0.0066		0.4518	0.0047	0.4657	0.0024	0.4593	0.0033
B1M	10	0.4174	0.0140	BB1M	0.4287	0.0105	0.4470	0.0058	0.4385	0.0078
	25	0.4257	0.0114		0.4358	0.0085	0.4523	0.0047	0.4446	0.0063
	50	0.5632	0.0082		0.4454	0.0061	0.4594	0.0034	0.4529	0.0046
B1K	10	0.5869	0.0157	BB1K	0.4241	0.0120	0.4424	0.0069	0.4338	0.0091
	25	0.5782	0.0127		0.4317	0.0105	0.4481	0.0058	0.4405	0.0078
	50	0.4335	0.0092		0.4420	0.0092	0.4559	0.0049	0.4494	0.0067
B1G1	10	0.4216	0.0126	BB1G1	0.4330	0.0092	0.4512	0.0049	0.4428	0.0067
	25	0.4294	0.0102		0.4397	0.0074	0.4561	0.0039	0.4485	0.0054
	50	0.5600	0.0073		0.4487	0.0054	0.4627	0.0028	0.4562	0.0039
B1G2	10	0.5708	0.0101	BB1G2	0.4408	0.0071	0.4587	0.0034	0.4505	0.0049
	25	0.5637	0.0084		0.4467	0.0057	0.4628	0.0028	0.4555	0.0040
	50	0.4458	0.0059		0.4547	0.0041	0.4684	0.0020	0.4621	0.0029
B1P	10	0.6014	0.0220	BB1P	0.5913	0.0179	0.4260	0.0117	0.5823	0.0145
	25	0.5912	0.0178		0.5822	0.0145	0.4334	0.0095	0.5740	0.0117
	50	0.5776	0.0129		0.5699	0.0104	0.4434	0.0069	0.5629	0.0085
B1AB1	10	0.3933	0.0246	BB1AB1	0.5972	0.0204	0.4195	0.0140	0.5884	0.0169
	25	0.4040	0.0199		0.5875	0.0165	0.4276	0.0113	0.5796	0.0137
	50	0.4184	0.0144		0.5743	0.0119	0.4384	0.0082	0.5677	0.0099
B1AB2	10	0.6245	0.0345	BB1AB2	0.6170	0.0305	0.6033	0.0238	0.6100	0.0270
	25	0.6120	0.0280		0.6053	0.0247	0.5930	0.0193	0.5990	0.0218
	50	0.5952	0.0202		0.5895	0.0179	0.5791	0.0139	0.5841	0.0158
B1AS1	10	0.4037	0.0197	BB1AS1	0.5859	0.0156	0.4319	0.0098	0.5765	0.0124
	25	0.4133	0.0159		0.5773	0.0127	0.4387	0.0080	0.5688	0.0100
	50	0.4263	0.0115		0.5657	0.0091	0.4479	0.0058	0.5585	0.0073
B1AS2	10	0.6310	0.0386	BB1AS2	0.6245	0.0349	0.6123	0.0284	0.6182	0.0315
	25	0.6179	0.0313		0.6120	0.0282	0.6011	0.0230	0.6064	0.0255
	50	0.6002	0.0226		0.5952	0.0204	0.5859	0.0166	0.5905	0.0184
B1AS3	10	0.3877	0.0275	BB1AS3	0.6034	0.0233	0.4125	0.0167	0.5951	0.0197
	25	0.3989	0.0223		0.5930	0.0189	0.4213	0.0135	0.5856	0.0160
	50	0.4141	0.0161		0.5791	0.0136	0.4331	0.0098	0.5728	0.0115
B1AS4	10	0.3818	0.0308	BB1AS4	0.6100	0.0266	0.5951	0.0199	0.6023	0.0230
	25	0.3936	0.0249		0.5990	0.0216	0.5856	0.0161	0.5920	0.0186
	50	0.4095	0.0180		0.5841	0.0156	0.5727	0.0116	0.5782	0.0135

Table 4: Bayes Estimators under Jeffery prior and when $\mu = 1$

LF	n	$\hat{\mu}$	RMSE	BLF	$\omega = 0.2$		$\omega = 0.5$		$\omega = 0.8$	
					$\hat{\mu}$	RMSE	$\hat{\mu}$	RMSE	$\hat{\mu}$	RMSE
B1S	10	0.8923	0.0244	BB1S	1.0950	0.0190	1.0738	0.0114	1.0838	0.0147
	25	0.9030	0.0197		0.9145	0.0154	1.0664	0.0093	1.0754	0.0119
	50	0.9176	0.0143		0.9273	0.0111	1.0564	0.0067	1.0641	0.0086
B1W	10	0.9123	0.0156	BB1W	1.0742	0.0111	1.0529	0.0056	1.0627	0.0079
	25	0.9210	0.0126		1.0668	0.0090	1.0476	0.0046	1.0564	0.0064
	50	1.0671	0.0091		1.0568	0.0065	1.0404	0.0033	1.0479	0.0046
B1M	10	0.9028	0.0195	BB1M	1.0840	0.0145	1.0624	0.0080	1.0725	0.0108
	25	0.9125	0.0158		1.0756	0.0118	1.0562	0.0065	1.0652	0.0088
	50	1.0744	0.0114		1.0642	0.0085	1.0478	0.0047	1.0554	0.0063
B1K	10	1.1023	0.0218	BB1K	1.0893	0.0166	1.0679	0.0096	1.0779	0.0126
	25	1.0921	0.0176		1.0804	0.0145	1.0611	0.0080	1.0701	0.0108
	50	0.9217	0.0127		1.0683	0.0127	1.0519	0.0067	1.0596	0.0093
B1G1	10	0.9076	0.0174	BB1G1	1.0789	0.0127	1.0575	0.0067	1.0674	0.0093
	25	0.9169	0.0141		1.0710	0.0103	1.0517	0.0055	1.0606	0.0075
	50	1.0707	0.0102		1.0604	0.0074	1.0440	0.0039	1.0515	0.0054
B1G2	10	1.0834	0.0140	BB1G2	1.0697	0.0098	1.0486	0.0048	1.0583	0.0069
	25	1.0750	0.0114		1.0628	0.0080	1.0438	0.0039	1.0524	0.0056
	50	0.9362	0.0082		1.0534	0.0058	1.0372	0.0028	1.0446	0.0040
B1P	10	0.8806	0.0305	BB1P	1.1076	0.0248	1.0872	0.0163	0.9031	0.0201
	25	1.1074	0.0247		0.9032	0.0201	1.0785	0.0132	0.9128	0.0163
	50	1.0913	0.0178		0.9177	0.0145	1.0667	0.0095	0.9259	0.0117
B1AB1	10	1.1256	0.0341	BB1AB1	1.1144	0.0283	1.0947	0.0194	0.8959	0.0234
	25	0.8869	0.0276		0.8970	0.0229	1.0853	0.0157	0.9063	0.0190
	50	0.9039	0.0200		0.9125	0.0165	1.0725	0.0113	0.9203	0.0137
B1AB2	10	0.8535	0.0479	BB1AB2	0.8622	0.0423	0.8783	0.0330	0.8705	0.0374
	25	1.1319	0.0388		0.8760	0.0343	0.8905	0.0267	0.8835	0.0303
	50	1.1121	0.0280		0.8946	0.0248	0.9069	0.0193	0.9010	0.0219
B1AS1	10	1.1134	0.0273	BB1AS1	1.1011	0.0217	1.0802	0.0136	0.9099	0.0172
	25	0.8979	0.0221		0.9090	0.0176	1.0722	0.0110	0.9189	0.0139
	50	0.9133	0.0160		0.9227	0.0127	1.0613	0.0080	0.9311	0.0101
B1AS2	10	0.8457	0.0535	BB1AS2	0.8535	0.0483	0.8677	0.0394	0.8608	0.0436
	25	1.1388	0.0434		0.8681	0.0391	0.8810	0.0319	0.8747	0.0353
	50	1.1180	0.0313		0.8879	0.0283	0.8988	0.0230	0.8935	0.0255
B1AS3	10	1.1323	0.0381	BB1AS3	0.8783	0.0323	1.1030	0.0231	0.8880	0.0273
	25	0.8810	0.0309		0.8905	0.0262	1.0927	0.0187	0.8992	0.0221
	50	0.8988	0.0223		0.9069	0.0189	1.0788	0.0135	0.9143	0.0160
B1AS4	10	1.1392	0.0426	BB1AS4	0.8705	0.0369	0.8881	0.0276	0.8796	0.0319
	25	0.8747	0.0345		0.8835	0.0299	0.8993	0.0223	0.8916	0.0258
	50	0.8935	0.0250		0.9009	0.0216	0.9144	0.0161	0.9079	0.0187

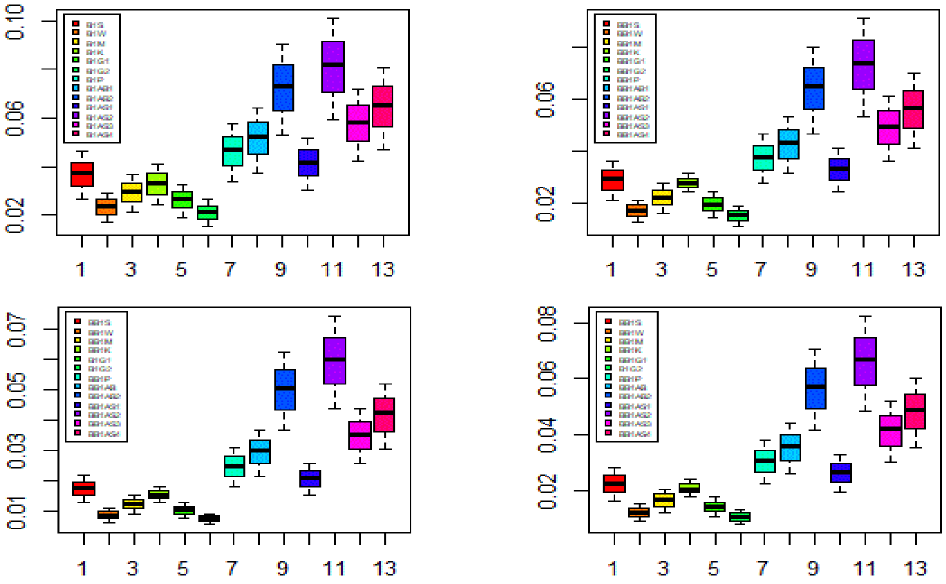

Fig.1. Bayes Estimators under Jeffery prior and when $\mu = 2$

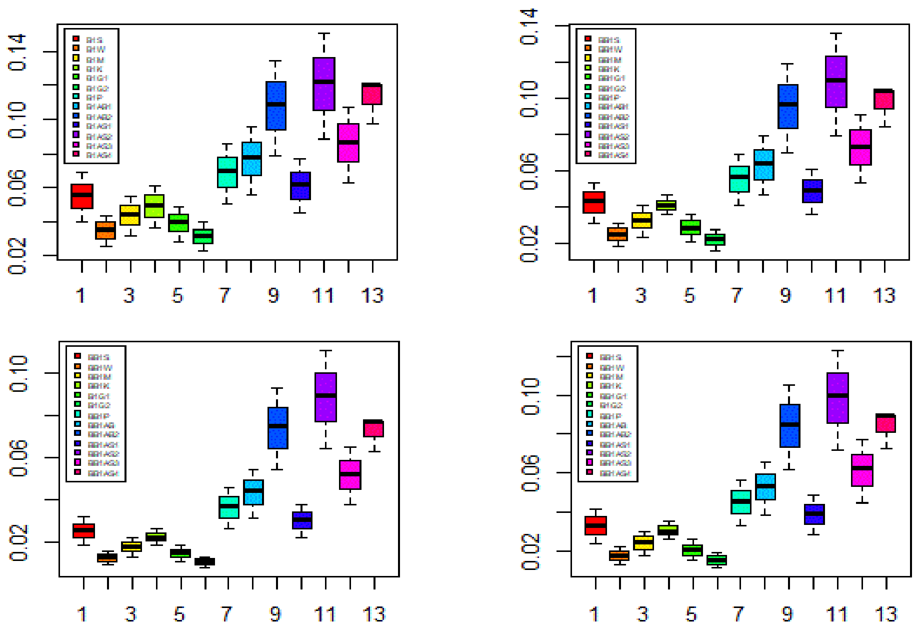

Fig. 2. Bayes Estimators under Jeffery prior and when $\mu = 3$

Table 5: Bayes Estimators under Gamma prior and when $\mu = \theta = 0.5, \beta = 1$

LF	n	$\hat{\mu}$	RMSE	BLF	$\omega = 0.2$		$\omega = 0.5$		$\omega = 0.8$	
					$\hat{\mu}$	RMSE	$\hat{\mu}$	RMSE	$\hat{\mu}$	RMSE
B2S	1	0.561	0.007	BB2S	0.569	0.010	0.578	0.013	0.410	0.016
	2	0.555	0.006		0.562	0.008	0.571	0.010	0.419	0.013
	5	0.546	0.004		0.559	0.007	0.560	0.007	0.431	0.009
B2W	1	0.547	0.004	BB2W	0.556	0.006	0.565	0.008	0.576	0.012
	2	0.542	0.003		0.550	0.005	0.559	0.007	0.569	0.009
	5	0.536	0.002		0.547	0.004	0.550	0.005	0.558	0.007
B2M	1	0.543	0.003	BB2M	0.552	0.005	0.561	0.007	0.572	0.010
	2	0.539	0.003		0.546	0.004	0.555	0.006	0.565	0.008
	5	0.533	0.002		0.544	0.004	0.547	0.004	0.555	0.006
B2K	1	0.556	0.006	BB2K	0.564	0.008	0.574	0.011	0.415	0.015
	2	0.550	0.005		0.558	0.007	0.566	0.010	0.423	0.012
	5	0.543	0.004		0.555	0.006	0.556	0.008	0.435	0.008
B2G1	1	0.551	0.005	BB2G1	0.560	0.007	0.569	0.010	0.580	0.013
	2	0.546	0.004		0.554	0.006	0.562	0.008	0.572	0.010
	5	0.539	0.003		0.551	0.005	0.553	0.005	0.561	0.007
B2G2	1	0.540	0.003	BB2G2	0.548	0.004	0.557	0.006	0.430	0.009
	2	0.536	0.002		0.543	0.003	0.552	0.005	0.437	0.007
	5	0.530	0.001		0.541	0.003	0.544	0.004	0.447	0.005
B2P	1	0.572	0.011	BB2P	0.580	0.013	0.589	0.017	0.599	0.021
	2	0.565	0.009		0.572	0.011	0.580	0.013	0.589	0.017
	5	0.555	0.006		0.568	0.010	0.568	0.010	0.575	0.012
B2AB1	1	0.578	0.013	BB2AB1	0.586	0.016	0.595	0.019	0.395	0.023
	2	0.570	0.010		0.577	0.013	0.585	0.015	0.406	0.019
	5	0.560	0.007		0.574	0.011	0.572	0.011	0.420	0.013
B2AB2	1	0.601	0.022	BB2AB2	0.607	0.025	0.614	0.029	0.621	0.033
	2	0.591	0.018		0.596	0.020	0.603	0.023	0.609	0.026
	5	0.577	0.013		0.591	0.018	0.587	0.017	0.593	0.019
B2AS1	1	0.566	0.009	BB2AS1	0.574	0.011	0.584	0.014	0.405	0.018
	2	0.559	0.007		0.567	0.009	0.575	0.012	0.415	0.015
	5	0.551	0.005		0.564	0.008	0.564	0.008	0.428	0.011
B2AS2	1	0.609	0.027	BB2AS2	0.615	0.030	0.621	0.033	0.628	0.036
	2	0.598	0.022		0.604	0.024	0.609	0.027	0.615	0.029
	5	0.584	0.015		0.598	0.022	0.593	0.019	0.598	0.021
B2AS3	1	0.585	0.015	BB2AS3	0.593	0.018	0.601	0.022	0.390	0.026
	2	0.577	0.012		0.583	0.015	0.591	0.018	0.401	0.021
	5	0.565	0.009		0.579	0.013	0.577	0.013	0.416	0.015
B2AS4	1	0.593	0.019	BB2AS4	0.600	0.022	0.607	0.025	0.384	0.029
	2	0.583	0.015		0.590	0.017	0.596	0.020	0.395	0.023
	5	0.571	0.011		0.585	0.016	0.582	0.014	0.411	0.017

Table 6 : Bayes Estimators under Gamma prior and when $\mu = 0.5, \theta = 1, \beta = 2$

10	n	$\hat{\mu}$	RMSE	BLF	ω = 0.2		ω = 0.5		ω = 0.8	
					$\hat{\mu}$	RMSE	$\hat{\mu}$	RMSE	$\hat{\mu}$	RMSE
B2S	1	0.549	0.005	BB2S	0.556	0.006	0.565	0.008	0.425	0.011
	2	0.544	0.004		0.551	0.005	0.558	0.007	0.432	0.009
	5	0.538	0.003		0.548	0.004	0.549	0.006	0.442	0.006
B2W	1	0.535	0.002	BB2W	0.542	0.003	0.551	0.005	0.439	0.007
	2	0.532	0.002		0.538	0.003	0.545	0.004	0.445	0.006
	5	0.527	0.001		0.536	0.002	0.539	0.003	0.453	0.004
B2M	1	0.553	0.006	BB2M	0.561	0.007	0.569	0.010	0.421	0.013
	2	0.548	0.004		0.555	0.006	0.562	0.008	0.429	0.010
	5	0.541	0.003		0.552	0.005	0.553	0.005	0.439	0.007
B2K	1	0.545	0.004	BB2K	0.553	0.005	0.561	0.007	0.571	0.010
	2	0.541	0.003		0.547	0.004	0.555	0.006	0.564	0.008
	5	0.534	0.002		0.545	0.004	0.546	0.004	0.554	0.006
B2G1	1	0.542	0.003	BB2G1	0.549	0.004	0.557	0.006	0.567	0.009
	2	0.537	0.002		0.544	0.004	0.551	0.005	0.560	0.007
	5	0.532	0.002		0.542	0.003	0.544	0.004	0.551	0.005
B2G2	1	0.538	0.003	BB2G2	0.545	0.004	0.554	0.005	0.564	0.008
	2	0.534	0.002		0.541	0.003	0.548	0.004	0.557	0.006
	5	0.529	0.001		0.539	0.003	0.541	0.003	0.549	0.004
B2P	1	0.563	0.008	BB2P	0.570	0.010	0.578	0.013	0.587	0.016
	2	0.557	0.007		0.563	0.008	0.570	0.010	0.578	0.013
	5	0.548	0.005		0.560	0.007	0.560	0.007	0.566	0.009
B2AB1	1	0.569	0.010	BB2AB1	0.576	0.012	0.583	0.015	0.408	0.018
	2	0.562	0.008		0.568	0.010	0.575	0.012	0.417	0.014
	5	0.553	0.006		0.565	0.009	0.564	0.008	0.429	0.010
B2AB2	1	0.588	0.017	BB2AB2	0.594	0.020	0.600	0.022	0.607	0.025
	2	0.580	0.014		0.585	0.016	0.590	0.018	0.596	0.020
	5	0.568	0.010		0.580	0.014	0.577	0.013	0.581	0.015
B2AS1	1	0.558	0.007	BB2AS1	0.565	0.009	0.573	0.011	0.417	0.014
	2	0.552	0.005		0.559	0.007	0.566	0.009	0.425	0.011
	5	0.544	0.004		0.556	0.006	0.556	0.006	0.436	0.008
B2AS2	1	0.596	0.021	BB2AS2	0.601	0.023	0.607	0.025	0.612	0.028
	2	0.587	0.017		0.591	0.018	0.596	0.020	0.601	0.023
	5	0.573	0.012		0.587	0.017	0.581	0.015	0.586	0.016
B2AS3	1	0.575	0.012	BB2AS3	0.581	0.014	0.589	0.017	0.403	0.020
	2	0.567	0.010		0.573	0.011	0.580	0.014	0.413	0.016
	5	0.557	0.007		0.570	0.010	0.568	0.010	0.426	0.011
B2AS4	1	0.581	0.014	BB2AS4	0.588	0.017	0.594	0.019	0.398	0.022
	2	0.573	0.011		0.579	0.013	0.585	0.016	0.408	0.018
	5	0.562	0.008		0.575	0.012	0.572	0.011	0.422	0.013

Table 7 : Bayes Estimators under Gamma prior and when $\mu = 1, \theta = 1, \beta = 2$

LF	n	$\hat{\mu}$	RMSE	BLF	ω = 0.2		ω = 0.5		ω = 0.8	
					$\hat{\mu}$	RMSE	$\hat{\mu}$	RMSE	$\hat{\mu}$	RMSE
B2S	10	1.0464	0.0044	BB2S	0.9444	0.0062	0.9334	0.0089	1.0796	0.0128
	25	0.9582	0.0035		0.9499	0.0051	0.9401	0.0072	1.0716	0.0104
	50	1.0355	0.0025		0.9524	0.0046	0.9491	0.0052	1.0609	0.0075
B2W	10	1.0765	0.0124	BB2W	0.9140	0.0157	0.9035	0.0197	1.1082	0.0248
	25	0.9311	0.0101		0.9226	0.0127	0.9132	0.0160	1.0974	0.0201
	50	1.0585	0.0073		0.9265	0.0115	0.9262	0.0115	1.0828	0.0145
B2M	10	1.0983	0.0211	BB2M	0.8931	0.0249	0.8838	0.0294	1.1262	0.0347
	25	1.0885	0.0171		0.9038	0.0202	0.8955	0.0238	1.1136	0.0281
	50	1.0752	0.0123		0.9086	0.0182	0.9111	0.0172	1.0966	0.0203
B2K	10	1.0504	0.0051	BB2K	0.9402	0.0072	0.9292	0.0101	0.9163	0.0142
	25	0.9546	0.0042		0.9462	0.0059	0.9363	0.0082	0.9246	0.0115
	50	1.0386	0.0030		0.9489	0.0053	0.9458	0.0059	0.9359	0.0083
B2G1	10	1.0596	0.0073	BB2G1	0.9309	0.0098	0.9199	0.0132	0.9072	0.0177
	25	0.9464	0.0059		0.9378	0.0080	0.9279	0.0107	0.9165	0.0144
	50	1.0456	0.0043		0.9409	0.0072	0.9387	0.0077	0.9290	0.0104
B2G2	10	1.0904	0.0177	BB2G2	0.9006	0.0213	0.8908	0.0258	1.1199	0.0311
	25	1.0814	0.0143		0.9105	0.0173	0.9017	0.0209	1.1079	0.0252
	50	1.0692	0.0103		0.9150	0.0156	0.9165	0.0151	1.0917	0.0182
B2P	10	1.0648	0.0087	BB2P	0.9257	0.0115	0.9147	0.0151	1.0977	0.0198
	25	0.9417	0.0073		0.9331	0.0098	0.9233	0.0132	1.0879	0.0161
	50	1.0496	0.0061		0.9364	0.0084	0.9348	0.0116	1.0747	0.0116
B2AB1	10	1.0704	0.0104	BB2AB1	0.9201	0.0134	0.9093	0.0173	1.1028	0.0222
	25	0.9366	0.0084		0.9280	0.0109	0.9184	0.0140	1.0925	0.0180
	50	1.0539	0.0061		0.9316	0.0098	0.9306	0.0101	1.0786	0.0130
B2AB2	10	1.1161	0.0301	BB2AB2	0.8764	0.0340	0.8685	0.0385	0.8601	0.0436
	25	1.1045	0.0244		0.8888	0.0276	0.8817	0.0312	0.8741	0.0353
	50	1.0888	0.0176		0.8944	0.0249	0.8994	0.0226	0.8930	0.0255
B2AS1	10	1.0548	0.0061	BB2AS1	0.9357	0.0084	0.9247	0.0116	0.9119	0.0159
	25	0.9507	0.0050		0.9421	0.0068	0.9322	0.0094	0.9207	0.0128
	50	1.0419	0.0036		0.9450	0.0062	0.9424	0.0068	0.9326	0.0093
B2AS2	10	1.1262	0.0358	BB2AS2	0.8671	0.0397	0.8601	0.0440	0.8528	0.0488
	25	1.1136	0.0290		0.8804	0.0322	0.8741	0.0356	0.8675	0.0395
	50	1.0966	0.0210		0.8864	0.0290	0.8930	0.0258	0.8874	0.0285
B2AS3	10	1.0832	0.0148	BB2AS3	0.9076	0.0183	0.8974	0.0225	0.8861	0.0278
	25	1.0749	0.0120		0.9168	0.0148	0.9076	0.0183	0.8975	0.0225
	50	1.0636	0.0087		0.9210	0.0134	0.9215	0.0132	0.9129	0.0163
B2AS4	10	1.1068	0.0251	BB2AS4	0.8851	0.0291	0.8764	0.0336	1.1329	0.0388
	25	1.0962	0.0203		0.8966	0.0235	0.8888	0.0272	1.1196	0.0315
	50	1.0817	0.0147		0.9017	0.0212	0.9055	0.0197	1.1016	0.0227

Table 8 : Bayes Estimators under Gamma prior and when $\mu = 1, \theta = 2, \beta = 3$

LF	n	$\hat{\mu}$	RMSE	BLF	ω = 0.2		ω = 0.5		ω = 0.8	
					$\hat{\mu}$	RMSE	$\hat{\mu}$	RMSE	$\hat{\mu}$	RMSE
B2S	10	1.0408	0.0034	BB2S	0.9511	0.0048	0.9414	0.0069	1.0700	0.0099
	25	1.0368	0.0027		1.0440	0.0039	1.0527	0.0056	1.0630	0.0080
	50	0.9688	0.0020		1.0418	0.0035	0.9552	0.0041	1.0536	0.0058
B2W	10	1.0674	0.0096	BB2W	0.9244	0.0121	0.9151	0.0153	1.0952	0.0192
	25	1.0606	0.0078		1.0681	0.0098	0.9236	0.0124	1.0857	0.0156
	50	1.0515	0.0056		0.9353	0.0089	0.9350	0.0089	1.0729	0.0113
B2M	10	1.0865	0.0163	BB2M	0.9060	0.0193	0.8978	0.0228	1.1111	0.0269
	25	1.0778	0.0132		0.9154	0.0156	0.9080	0.0185	1.1000	0.0218
	50	1.0662	0.0095		0.9196	0.0141	0.9218	0.0133	1.0850	0.0157
B2K	10	1.0444	0.0040	BB2K	0.9474	0.0056	0.9377	0.0078	0.9263	0.0110
	25	1.0400	0.0032		1.0474	0.0045	1.0561	0.0064	0.9337	0.0089
	50	0.9660	0.0023		1.0450	0.0041	0.9523	0.0046	0.9436	0.0064
B2G1	10	1.0524	0.0057	BB2G1	0.9392	0.0076	0.9295	0.0102	0.9183	0.0137
	25	1.0472	0.0046		1.0548	0.0062	1.0635	0.0083	0.9265	0.0111
	50	1.0401	0.0033		1.0520	0.0056	0.9461	0.0060	0.9375	0.0080
B2G2	10	1.0796	0.0137	BB2G2	0.9125	0.0165	0.9039	0.0199	1.1055	0.0241
	25	1.0716	0.0111		0.9213	0.0134	0.9135	0.0162	1.0950	0.0195
	50	1.0609	0.0080		0.9252	0.0121	0.9265	0.0117	1.0807	0.0141
B2P	10	1.0570	0.0068	BB2P	0.9346	0.0089	0.9250	0.0117	1.0859	0.0154
	25	1.0513	0.0057		1.0589	0.0076	1.0675	0.0102	1.0774	0.0124
	50	1.0436	0.0047		0.9441	0.0065	0.9426	0.0090	1.0658	0.0090
B2AB1	10	1.0620	0.0081	BB2AB1	0.9296	0.0104	0.9202	0.0134	1.0905	0.0172
	25	1.0558	0.0065		1.0633	0.0084	1.0718	0.0108	1.0814	0.0139
	50	1.0474	0.0047		0.9398	0.0076	0.9389	0.0078	1.0692	0.0101
B2AB2	10	1.1022	0.0233	BB2AB2	0.8913	0.0264	0.8843	0.0298	0.8769	0.0338
	25	1.0920	0.0189		0.9021	0.0214	0.8959	0.0242	0.8892	0.0274
	50	1.0782	0.0136		0.9070	0.0193	1.0885	0.0175	0.9058	0.0198
B2AS1	10	1.0483	0.0047	BB2AS1	0.9434	0.0065	0.9337	0.0090	0.9224	0.0123
	25	1.0434	0.0038		1.0509	0.0053	1.0597	0.0073	0.9302	0.0099
	50	1.0369	0.0028		1.0484	0.0048	0.9493	0.0052	0.9407	0.0072
B2AS2	10	1.1111	0.0278	BB2AS2	0.8831	0.0308	0.8769	0.0341	0.8704	0.0378
	25	1.1000	0.0225		0.8948	0.0249	0.8892	0.0276	0.8834	0.0306
	50	1.0850	0.0162		0.9000	0.0225	1.0942	0.0199	0.9009	0.0221
B2AS3	10	1.0732	0.0115	BB2AS3	0.9187	0.0142	0.9097	0.0175	0.8998	0.0215
	25	1.0659	0.0093		1.0732	0.0115	0.9187	0.0141	0.9098	0.0174
	50	1.0560	0.0067		0.9305	0.0104	0.9309	0.0102	0.9233	0.0126
B2AS4	10	1.0940	0.0194	BB2AS4	0.8989	0.0225	0.8913	0.0260	1.1169	0.0301
	25	1.0846	0.0158		0.9090	0.0182	0.9021	0.0211	1.1052	0.0244
	50	1.0719	0.0114		0.9135	0.0164	1.0832	0.0152	1.0894	0.0176

Table 9 : Bayes Estimators under Gamma prior and when $\mu = 2, \theta = 2, \beta = 3$

LF	n	$\hat{\mu}$	RMSE	BLF	ω = 0.2		ω = 0.5		ω = 0.8	
					$\hat{\mu}$	RMSE	$\hat{\mu}$	RMSE	$\hat{\mu}$	RMSE
B2S	1	2.112	0.026	BB2S	2.124	0.033	2.138	0.041	1.846	0.050
	2	2.100	0.021		2.112	0.026	1.875	0.033	2.138	0.040
	5	2.085	0.015		1.893	0.024	2.105	0.024	2.117	0.029
B2W	1	2.144	0.045	BB2W	2.154	0.052	2.166	0.061	2.179	0.070
	2	2.129	0.036		2.139	0.042	1.850	0.049	1.838	0.057
	5	2.110	0.026		1.867	0.038	1.872	0.035	1.863	0.041
B2M	1	2.170	0.065	BB2M	2.179	0.072	2.188	0.079	2.198	0.088
	2	2.153	0.052		2.161	0.058	1.830	0.064	2.178	0.071
	5	2.130	0.038		1.846	0.052	1.855	0.046	2.151	0.051
B2K	1	2.121	0.032	BB2K	2.133	0.038	2.147	0.046	2.161	0.056
	2	2.109	0.026		2.120	0.031	1.867	0.037	1.854	0.045
	5	2.093	0.018		1.885	0.028	2.112	0.027	1.876	0.033
B2G1	1	2.132	0.038	BB2G1	2.144	0.045	2.156	0.053	2.170	0.063
	2	2.119	0.031		2.129	0.036	1.859	0.043	1.846	0.051
	5	2.101	0.022		1.876	0.033	2.119	0.031	1.869	0.036
B2G2	1	2.156	0.054	BB2G2	2.166	0.061	2.177	0.070	2.188	0.079
	2	2.140	0.044		2.149	0.050	1.840	0.056	2.169	0.064
	5	2.119	0.032		1.857	0.045	1.864	0.041	2.144	0.046
B2P	1	1.905	0.018	BB2P	2.107	0.024	2.122	0.031	2.138	0.040
	2	1.914	0.015		2.097	0.019	1.890	0.025	1.875	0.032
	5	2.072	0.011		1.907	0.017	2.093	0.018	1.894	0.023
B2AB1	1	1.912	0.015	BB2AB1	2.100	0.020	2.114	0.027	2.131	0.036
	2	1.921	0.013		2.090	0.017	1.896	0.024	1.881	0.029
	5	2.066	0.011		1.914	0.015	2.087	0.021	1.899	0.021
B2AB2	1	1.932	0.009	BB2AB2	1.919	0.013	1.904	0.018	2.112	0.025
	2	1.938	0.007		2.072	0.010	1.914	0.014	2.101	0.020
	5	1.948	0.005		2.068	0.009	2.073	0.010	2.086	0.015
B2AS1	1	1.896	0.022	BB2AS1	2.115	0.028	2.130	0.035	2.145	0.045
	2	1.907	0.018		2.104	0.023	1.883	0.029	1.868	0.036
	5	2.078	0.013		1.900	0.020	2.099	0.021	1.888	0.026
B2AS2	1	1.937	0.007	BB2AS2	1.925	0.011	1.910	0.016	1.892	0.023
	2	1.943	0.006		2.067	0.009	1.919	0.013	1.903	0.018
	5	1.952	0.004		2.064	0.008	2.068	0.009	1.918	0.013
B2AS3	1	1.919	0.013	BB2AS3	2.093	0.017	2.108	0.024	2.125	0.032
	2	1.927	0.010		2.083	0.014	1.902	0.019	2.112	0.026
	5	2.061	0.007		1.920	0.013	2.082	0.014	2.095	0.018
B2AS4	1	1.926	0.011	BB2AS4	1.913	0.015	1.898	0.021	2.118	0.028
	2	1.933	0.009		2.078	0.012	1.908	0.017	2.106	0.023
	5	2.056	0.006		1.925	0.011	2.077	0.012	2.090	0.016

Table 10 : Bayes Estimators under Gamma prior and when $\mu = 2, \theta = 3, \beta = 4$

LF	n	$\hat{\mu}$	RMSE	BLF	ω = 0.2		ω = 0.5		ω = 0.8	
					$\hat{\mu}$	RMSE	$\hat{\mu}$	RMSE	$\hat{\mu}$	RMSE
B2S	1	1.901	0.020	BB2S	1.890	0.025	1.878	0.031	2.135	0.039
	2	2.088	0.016		1.901	0.020	1.890	0.025	2.121	0.031
	5	2.075	0.012		2.093	0.018	2.093	0.018	2.103	0.022
B2W	1	1.873	0.035	BB2W	1.863	0.040	2.146	0.047	1.842	0.054
	2	1.886	0.028		2.122	0.033	1.868	0.038	1.858	0.044
	5	2.096	0.020		2.116	0.029	1.887	0.027	1.879	0.032
B2M	1	1.850	0.050	BB2M	2.157	0.055	2.165	0.061	2.174	0.068
	2	1.865	0.040		2.141	0.045	1.850	0.050	2.157	0.055
	5	2.114	0.029		2.134	0.040	1.873	0.036	2.133	0.040
B2K	1	1.892	0.024	BB2K	1.882	0.030	2.129	0.036	1.857	0.043
	2	2.096	0.020		2.106	0.024	1.883	0.029	1.872	0.035
	5	2.082	0.014		2.100	0.021	2.099	0.021	1.891	0.025
B2G1	1	1.883	0.029	BB2G1	1.873	0.035	2.137	0.041	1.850	0.048
	2	1.895	0.024		2.114	0.028	1.876	0.033	1.865	0.039
	5	2.089	0.017		2.108	0.025	2.105	0.024	1.885	0.028
B2G2	1	1.862	0.042	BB2G2	1.853	0.047	2.155	0.054	2.165	0.061
	2	1.876	0.034		2.131	0.038	1.859	0.043	2.149	0.049
	5	2.105	0.024		2.125	0.035	1.880	0.031	2.126	0.035
B2P	1	1.916	0.014	BB2P	1.905	0.018	1.892	0.024	1.878	0.031
	2	2.075	0.011		1.914	0.015	2.096	0.019	1.890	0.025
	5	2.063	0.008		2.081	0.013	2.082	0.014	1.906	0.018
B2AB1	1	1.923	0.012	BB2AB1	1.911	0.016	1.898	0.021	1.884	0.027
	2	2.069	0.010		1.920	0.013	2.091	0.018	1.895	0.022
	5	2.058	0.008		2.075	0.011	2.077	0.016	1.911	0.016
B2AB2	1	2.059	0.007	BB2AB2	1.929	0.010	1.916	0.014	2.099	0.019
	2	2.053	0.005		1.936	0.008	2.075	0.011	2.089	0.016
	5	1.954	0.004		1.939	0.007	2.064	0.008	2.076	0.011
B2AS1	1	1.909	0.017	BB2AS1	1.898	0.022	1.885	0.027	1.871	0.034
	2	2.081	0.014		1.908	0.017	1.897	0.022	1.884	0.028
	5	2.069	0.010		2.087	0.016	2.087	0.016	1.901	0.020
B2AS2	1	2.055	0.006	BB2AS2	1.934	0.008	1.921	0.012	1.905	0.018
	2	2.049	0.005		1.940	0.007	2.071	0.010	1.915	0.014
	5	1.957	0.003		1.943	0.006	2.060	0.007	1.927	0.010
B2AS3	1	2.070	0.010	BB2AS3	1.918	0.013	1.905	0.018	2.110	0.024
	2	2.063	0.008		1.926	0.011	2.085	0.015	2.099	0.020
	5	2.054	0.006		1.929	0.010	2.072	0.010	2.084	0.014
B2AS4	1	2.065	0.008	BB2AS4	1.923	0.011	1.910	0.016	2.104	0.022
	2	2.058	0.007		1.931	0.009	2.080	0.013	2.094	0.018
	5	2.049	0.005		1.934	0.008	2.068	0.009	2.080	0.013

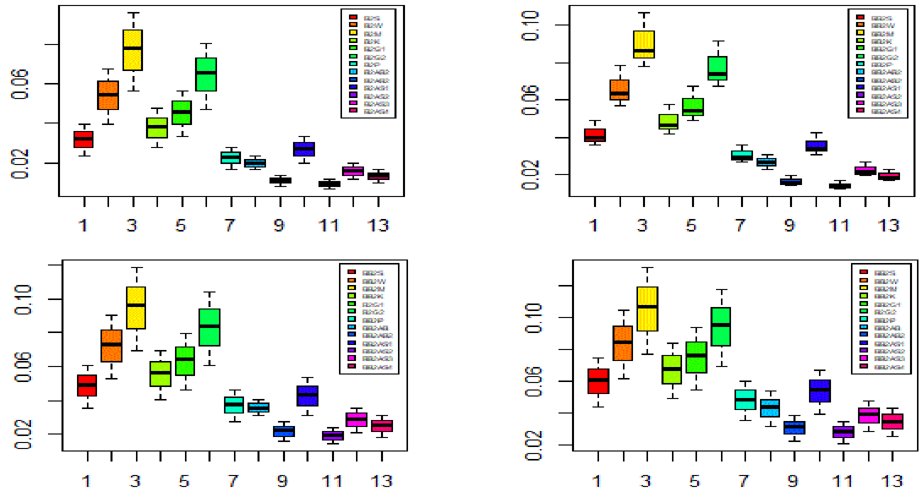

Figure (3) Bayes Estimators under Gamma prior and when $\mu = 3, \theta = 3, \beta = 4$

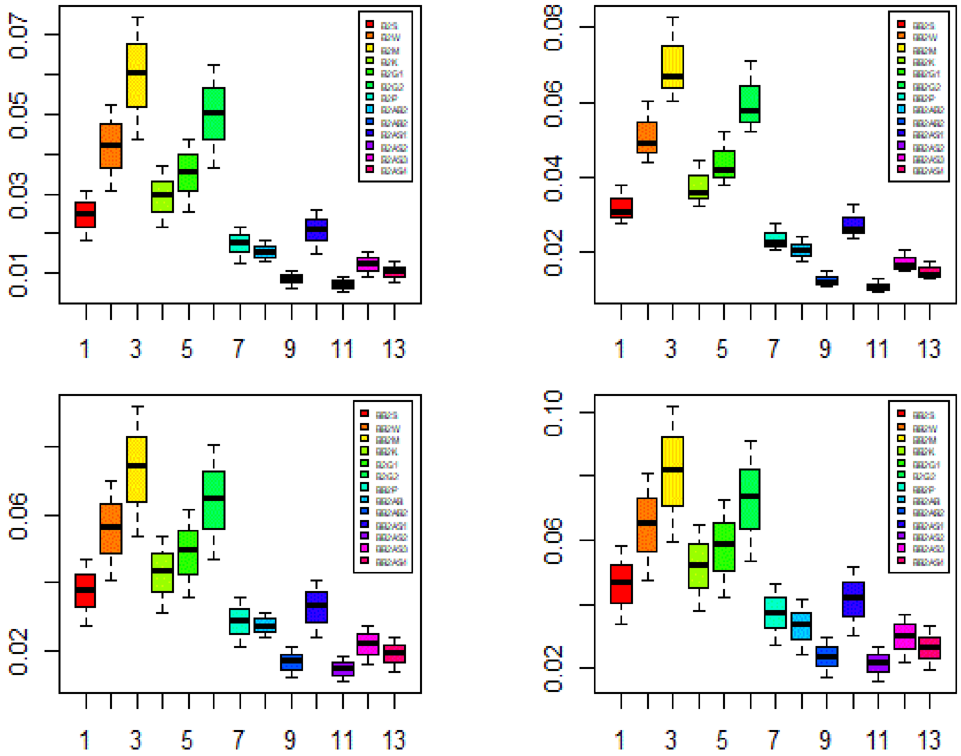

Fig. 4. Bayes Estimators under Gamma prior and when $\mu = 3, \theta = 3, \beta = 4$

The Bayesian estimators ranking under different loss functions explained in table (11) below:

Table 11: Bayesian estimators ranking under different loss functions

$\mu = 0.5,1,2,3$		$\mu = 0.5, \theta = 1, \beta = 2$		$\mu = 0.5, \theta = 1, \beta = 2$		$\mu = \theta = 1, \beta = 2$		$\mu = 0.5, \theta = 1, \beta = 2$	
UBLF	BLF	UBLF	BLF	UBLF	UBLF	BLF	UBLF	UBLF	UBLF
B1G2 B1W B1G1 B1M B1K B1S B1AS1 B1P B1AB1 B1AS3 B1AS4 B1AB2 B1AS2	BB1G2 BB1W BB1G1 BB1M BB1K BB1S BB1AS1 BB1P BB1AB1 BB1AS3 BB1AS4 BB1AB2 BB1AS2	B2G2 B2W B2G1 B2M B2K B2S B2AS1 B2P B2AB1 B2AS3 B2AS4 B2AB2 B2AS2	BB2G2 BB2W BB2G1 BB2M BB2K BB2S BB2AS1 BB2P BB2AB1 BB2AS3 BB2AS4 BB2AB2 BB2AS2	B2G2 B2W B2G1 B2M B2K B2S B2AS1 B2P B2AB1 B2AS3 B2AS4 B2AB2 B2AS2	B2G2 B2W B2G1 B2M B2K B2S B2AS1 B2P B2AB1 B2AS3 B2AS4 B2AB2 B2AS2	BB2G2 BB2W BB2G1 BB2M BB2K BB2S BB2AS1 BB2P BB2AB1 BB2AS3 BB2AS4 BB2AB2 BB2AS2	B2G2 B2W B2G1 B2M B2K B2S B2AS1 B2P B2AB1 B2AS3 B2AS4 B2AB2 B2AS2	B2AS2 B2AB2 B2AS4 B2AS3 B2P B2AB1 B2AS1 B2S B2AK B2M B2G1 B2G2 B2M	B2AS2 B2AB2 B2AS4 B2AS3 B2P B2AB1 B2AS1 B2S B2AK B2M B2G1 B2G2 B2M
		$\mu = \theta = 2, \beta = 3$		$\mu = 2, \theta = 3, \beta = 4$		$\mu = \theta = 3, \beta = 4$		$\mu = 3, \theta = 4, \beta = 5$	
		UBLF	BLF	UBLF	BLF	UBLF	BLF	UBLF	BLF
		B2AS2 B2AB2 B2AS4 B2AS3 B2P B2AB1 B2AS1 B2S B2AK B2M B2G1 B2G2 B2M	BB2AS2 BB2AB2 BB2AS4 BB2AS3 BB2P BB2AB1 BB2AS1 BB2S BB2AK BB2M BB2G1 BB2G2 BB2M	B2AS2 B2AB2 B2AS4 B2AS3 B2P B2AB1 B2AS1 B2S B2AK B2M B2G1 B2G2 B2M	BB2AS2 BB2AB2 BB2AS4 BB2AS3 BB2P BB2AB1 BB2AS1 BB2S BB2K BB2M BB2G1 BB2G2 BB2M	B2AS2 B2AB2 B2AS4 B2AS3 B2P B2AB1 B2AS1 B2S B2AK B2M B2G1 B2G2 B2M	BB2AS2 BB2AB2 BB2AS4 BB2AS3 BB2P BB2AB1 BB2AS1 BB2S BB2K BB2M BB2G1 BB2G2 BB2M	B2AS2 B2AB2 B2AS4 B2AS3 B2P B2AB1 B2AS1 B2S B2AK B2M B2G1 B2G2 B2M	BB2AS2 BB2AB2 BB2AS4 BB2AS3 BB2P BB2AB1 BB2AS1 BB2S BB2K BB2M BB2G1 BB2G2 BB2M

VI. Conclusions

From the simulations results**,** we can conclude the following:

- Estimating the exponential distribution scale parameter by using Gamma prior is better than estimating it using the Jeffrey prior for all balanced and unbalanced loss functions.
- The performance of the Bayesian estimators under the Jeffrey prior is improved by using the balanced loss functions, especially when the weight factor value is equal to 0.5. Therefore, it is possible to use initial estimators as the maximum likelihood estimator in the absence of sufficient information about the parameter to be estimated.
- Increasing the initial information about the parameter to be estimated is useless action, therefore when using Gamma prior Bayesian estimations performance under unbalanced loss functions is better than the performance under the balanced loss functions.
- The error squared loss function and the K loss function are close in the nature of their performance, they work well when the scale parameter value for the exponential distribution is equal to 1, and their performance declines when the scale parameter value is equal to 0.5, and the loss functions are worse when the scale parameter is equal to 2 or 3
- The AL-Bayyati and AL-Sayyes loss function are convergent in nature, and they work under very well Gamma prior when the scale parameter value for the exponential distribution is equal to 2 or 3, and their performance decreases when the scale parameter value is equal to 1, and the worse performance of those loss functions are when the scale parameter is 0.5 also under Jeffery prior.
- The modified error squared loss function and the general entropy loss function (constant=2) are convergent in nature, and they work well when the scale parameter value for the exponential distribution is equal to 0.5, and their performance decreases when the scale parameter value is equal to 1, and this performance decreases when the scale parameter value is equal to 2 or 3
- Under Jeffrey prior, the performance of the general entropy loss function when its constant is equal to 2 is better than the performance of this function with the constant equal to 0.5, but under Gamma prior, the performance of the general entropy loss function when the value of its constant is equal to 2 is better than the performance of this function with a constant equal to 0.5 if the scale parameter value for the exponential distribution is less than 1, and vice versa if the value of the scale parameter is greater than or equal to 1. In general, this context is true under balanced and unbalanced loss functions.
- With the two cases balanced and unbalanced, the weighted squared error loss function works well under Jeffrey prior, and it has poor performance under Gamma prior especially if the scale parameter value is greater than or equal to 2.
- Regarding the modified square error loss function, it seems that it succeeds in its work under Jeffrey prior and its performance under Gamma prior is less than when the value of the scale parameter is less than 1. In general, the performance of this loss function, whether it is balanced or unbalanced, deteriorates when the scale value parameter is greater than or equal to 2.

- Under Jeffrey and Gamma priors, and in both the balanced and unbalanced cases, it appears that the precautionary loss function does not outperform, and despite that, it is more stable than the performance of the rest loss functions, and the best performance of this loss function is under Gamma prior when the value of the two scale parameters of the exponential and Gamma distributions are equal to 1, in general, the performance of this loss function improves by increasing the scale parameter value under a Gamma prior.
- The performance Al-Bayyati loss function with both constants under Jeffrey prior is not successful, and the performance of this function when the constant value is equal to 0.5 is better than its performance when the constant value is equal to 2, and the same behavior of this function occurs under the Gamma prior, that is when the scale parameter value for the exponential distribution is equal to 0.5. the performance of this loss function improves by increasing the scale parameter value for the exponential distribution specifically when the value of the constant is equal to 1.
- As for the AL-Sayyad loss function, it works perfectly under Gamma prior also when the value of the scale parameter value for the exponential distribution is greater than or equal to 2, and it is at its best if the two constants values of this function is equal to 2, but under Jeffrey prior, the performance of this function declines significantly, especially if the two constants values of this function is equal to 2.
- In general, it can be said that the determination of the best loss function depends on the parameters values of the prior distribution and the parameter value itself, so the best Bayesian estimator for the scale parameter of the exponential distribution is the balanced loss functions estimator under Gamma prior and explained in the table (11).

Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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