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GENERAL ANALYTICAL SOLUTION OF AN ELASTIC BEAM UNDER VARYING LOADS WITH VALIDATION

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Abstract

In this paper, we take into account the system of differential equations with boundary conditions of a fixed elastic beam model (EBM). Instead of finding a solution of EBM for a particularly specified load, which is the usual practice, we derive the general analytical solution of the model using techniques of integrations. The proposed general analytical solutions are not load-specific but can be used for any load without having to integrate successively again and again. We have considered load in a general polynomial form and obtained a general analytical solution for the deflection and slope parameters of EBM. Direct solutions have been determined under two types of loads: uniformly distributed load and linearly varying load. The formulation derived has been validated on the known cases of uniformly distributed load as appears frequently in the literature.

Keywords: Elastic beam, General analytical solution, Deflection, Slope.

I. Introduction

Beam theory has a long history, and several engineers, scientists, and others have created numerical schemes and tested various techniques to understand important structural and performance aspects of beams subjected to loads [I]. There are two theories of the beam, namely: the Timoshenko beam theory and the Euler-Bernoulli beam theory. The Timoshenko beam (TB) study was enlarged by S. Timoshenko and P. Ehrenfest early 20th century [VI]. TB model includes both shear deformation as well as rotational bending effects. In the case of the TB model, the beam is thick and the angle of the cross-section about the neutral line will change after deflection. The well-known Euler-Bernoulli beam (EBB) theory is a particular case of TB theory for finding load-carrying and deflection characters of a beam. In the EBB model, the beam has no change in the angle of cross-section about the neutral line before and after deflection. [VII].

Researchers commonly employ a numerical approximation of the beam model as a starting point to acquire a better understanding of the Reissner-Mindlin problem, which is more complex. When these problems are handled using the finite difference method or normal Galerkin finite methods, a negative behaviour known as the locking phenomenon [II] occurs. In [III] also, the authors worked out the finite element method's p and h-p versions for the TB model. Researchers have also discovered a precise analytical solution to the Timoshenko beam problem for both uniform and continuous loads in [V]. In [IV], authors proposed and applied a finite difference scheme to obtain a numerical solution of the Timoshenko beam under constant as well as variable load without facing locking phenomena and discretized system into algebraic sum.

From the present literature review, we have observed that the majority of researchers were concerned with numerical techniques while the exact analytical methods have been applied in rare cases. The demerit of numerical schemes is that these do not provide an exact answer and require a lot of time to reach a more accurate value. While available analytical methods require mathematical skills to find the solution of the beam model for each applied load. In this research paper, we derive the general analytical solution of an elastic beam model by applying techniques of integration on the general load function instead of a case-specific load. This general solution is able to provide a description of the slope and deflection parameters of the elastic beam model subject to any type of varying load just by performing a few simplifications and bypasses any need to apply direct integration or transformation technique followed by boundary conditions. The established general equations have been validated for the case of well-known relations available in the literature for the uniformly distributed loads.

II. Mathematical Model of Elastic Beam

The elastic beam model is usually described by a system of ordinary differential equations subject to initial/boundary conditions. We consider the following differential equation model representing an elastic beam, written as

$$\frac{d^2M}{dx^2} = w \qquad (1)$$

$$\frac{d\theta}{dx} = \frac{M}{EI} \qquad (2)$$

$$\frac{dv}{dx} = \theta \qquad (3)$$

Where w is the load function, x is the spatial variable denoting variable length of the beam, M is the bending moment, E is the modulus of elasticity, I is a moment of inertia, θ denotes slope of deflection of beam and v is the deflection of beam. The boundary conditions for a fixed beam take the form:

$\theta(0) = 0, \theta(L) = 0,$

$v(0) = 0, v(L) = 0$

III. Derivation of General Analytical Solution of Elastic Beam

Let the load function, w be defined generally as a polynomial:

$$w = \sum_{i=0}^{n} a_i x^i$$

Substiuting the load function in (1) gives:

$$\frac{d^2M}{dx^2} = \sum_{i=0}^{n} a_i x^i$$

Integratig throughout with respect to x leads to:

$$\frac{d\,M}{dx} = \sum_{i=0}^{n+1} \frac{a_i x^{i+1}}{i+1} + c_1$$

Integrating again with respect to x gives:

$$M(x) = \sum_{i=0}^{n+2} \frac{a_i x^{i+2}}{(i+1)(i+2)} + c_1\, x + c_2$$

Substituting the expression of $M(x)$ in (2), we have:

$$\frac{d\theta}{dx} = \frac{1}{EI}\left\{\sum_{i=0}^{n+2} \frac{a_i x^{i+2}}{(i+1)(i+2)} + c_1\, x + c_2\right\}$$

Integrating with respect to x, we have

$$\theta(x) = \frac{1}{EI}\left\{\sum_{i=0}^{n+3} \frac{a_i x^{i+3}}{(i+1)(i+2)(i+3)} + c_1\left(\frac{x^2}{2}\right) + c_2 x + c_3\right\} \quad (4)$$

Substituting expression of $\theta(x)$ in (3) gives:

$$\frac{d\,v}{dx} = \frac{1}{EI}\left\{\sum_{i=0}^{n+3} \frac{a_i x^{i+3}}{(i+1)(i+2)(i+3)} + c_1\left(\frac{x^2}{2}\right) + c_2 x + c_3\right\}$$

Integrating with respect to x both sides leads to:

$$v(x) = \frac{1}{EI}\left\{\sum_{i=0}^{n+4} \frac{a_i x^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + c_1\left(\frac{x^3}{6}\right) + c_2\left(\frac{x^2}{2}\right) + c_3 x + c_4\right\} \quad (5)$$

Applying boundary conditions in (4), we have:

$\theta(0) = 0, => c_3 = 0$

$\theta(L) = 0$, gives:

$$c_1 L + 2c_2 = -2\sum_{i=0}^{n+3} \frac{a_i L^{i+2}}{(i+1)(i+2)(i+3)} \quad (6)$$

Applying boundary conditions in (5), we have:

$$v(0) = 0, => c_4 = 0$$

$v(L) = 0$, gives:

$$c_1 L + 3c_2 = -6 \sum_{i=0}^{n+4} \frac{a_i L^{i+2}}{(i+1)(i+2)(i+3)(i+4)} \tag{7}$$

Subtracting (6) from (7), we have:

$$c_2 = 2 \sum_{i=0}^{n+3} \frac{a_i L^{i+2}}{(i+1)(i+2)(i+3)} - 6 \sum_{i=0}^{n+4} \frac{a_i L^{i+2}}{(i+1)(i+2)(i+3)(i+4)}$$

Substituting c_2 in (6) and simplifying leads to:

$$c_1 = 12 \sum_{i=0}^{n+4} \frac{a_i L^{i+1}}{(i+1)(i+2)(i+3)(i+4)} - 6 \sum_{i=0}^{n+3} \frac{a_i L^{i+1}}{(i+1)(i+2)(i+3)}$$

Thus, the general analytical solution for the deflection profile and slope of deflection, respectively, of a fixed elastic beam is summarized in (8)-(9) with determined constants in (10)-(11).

$$v(x) = \frac{1}{EI} \left\{ \sum_{i=0}^{n+4} \frac{a_i x^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + c_1 \left(\frac{x^3}{6}\right) + c_2 \left(\frac{x^2}{2}\right) \right\} \tag{8}$$

$$\theta(x) = \frac{1}{EI} \left\{ \sum_{i=0}^{n+3} \frac{a_i x^{i+3}}{(i+1)(i+2)(i+3)} + c_1 \left(\frac{x^2}{2}\right) + c_2 x \right\} \tag{9}$$

$$c_1 = 12 \sum_{i=0}^{n+4} \frac{a_i L^{i+1}}{(i+1)(i+2)(i+3)(i+4)} - 6 \sum_{i=0}^{n+3} \frac{a_i L^{i+1}}{(i+1)(i+2)(i+3)} \tag{10}$$

$$c_2 = 2 \sum_{i=0}^{n+3} \frac{a_i L^{i+2}}{(i+1)(i+2)(i+3)} - 6 \sum_{i=0}^{n+4} \frac{a_i L^{i+2}}{(i+1)(i+2)(i+3)(i+4)} \tag{11}$$

Equations (8)-(11) enable to directly get the slope and deflection parameters of a fixed elastic bean under any type of load, uniform or varying as a polynomial since most load functions exhibit polynomial behavior and the nonlinear ones can be approximated by power series in x.

IV. Results and discussion

Here, we first obtain expressions of the slope and deflection of an elastic beam under a uniformly distributed load to validate the approach. Finally, a case of the linearly varying load is also considered. The application of the method presented here can be extended for any varying load.

Case-1: For a Uniform distributed load $w(x) = a$

As per the devised notations earlier, here $n = 0,\ a_0 = a,\ a_1 = a_2 = \cdots a_n = 0$.

Using (10), we have:

$$c_1 = 12\sum_{i=0}^{0+4}\frac{a_iL^{i+1}}{(i+1)(i+2)(i+3)(i+4)} - 6\sum_{i=0}^{0+3}\frac{a_iL^{i+1}}{(i+1)(i+2)(i+3)}$$

$$c_1 = \frac{12a_0L}{(0+1)(0+2)(0+3)(0+4)} - \frac{6a_0L}{(0+1)(0+2)(0+3)}$$

$$c_1 = \frac{12aL}{24} - \frac{6aL}{6} = \frac{aL}{2} - aL = -\frac{aL}{2}$$

$$c_1 = -\frac{aL}{2}$$

From (11), we have:

$$c_2 = 2\sum_{i=0}^{0+3}\frac{a_iL^{i+2}}{(i+1)(i+2)(i+3)} - 6\sum_{i=0}^{0+4}\frac{a_iL^{i+2}}{(i+1)(i+2)(i+3)(i+4)}$$

$$c_2 = \frac{2a_0L^2}{(0+1)(0+2)(0+3)} - \frac{6a_0L^2}{(0+1)(0+2)(0+3)(0+4)}$$

$$c_2 = \frac{2aL^2}{6} - \frac{6aL^2}{24} = aL^2\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$c_2 = \frac{aL^2}{12}$$

Using c_1 and c_2 in (9) gives:

$$v(x) = \frac{1}{EI}\left\{\sum_{i=0}^{0+4}\frac{a_ix^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + c_1\left(\frac{x^3}{6}\right) + c_2\left(\frac{x^2}{2}\right)\right\}$$

$$v(x) = \frac{1}{EI}\left\{\frac{a_0x^{0+4}}{(0+1)(0+2)(0+3)(0+4)} + c_1\left(\frac{x^3}{6}\right) + c_2\left(\frac{x^2}{2}\right)\right\}$$

$$v(x) = \frac{1}{EI}\left\{\frac{ax^4}{24} + \left(-\frac{aL}{2}\right)\left(\frac{x^3}{6}\right) + \left(\frac{aL}{12}\right)\left(\frac{x^2}{2}\right)\right\}$$

$$v(x) = \frac{1}{EI}\left\{\frac{ax^4}{24} - \frac{aLx^3}{12} + \frac{aL^2x^2}{24}\right\}$$

$$v(x) = \frac{a}{24EI}\{x^4 - 2Lx^3 + L^2x^2\}$$

From (8), we have:

$$\theta(x) = \frac{1}{EI}\left\{\sum_{i=0}^{0+3}\frac{a_ix^{i+3}}{(i+1)(i+2)(i+3)} + c_1\left(\frac{x^2}{2}\right) + c_2x\right\}$$

$$\theta(x) = \frac{1}{EI}\left\{\frac{a_0x^{0+3}}{(0+1)(0+2)(0+3)} + \left(-\frac{aL}{2}\right)\left(\frac{x^2}{2}\right) + \left(\frac{aL^2}{12}\right)x\right\}$$

$$\theta(x) = \frac{1}{EI}\left\{\frac{ax^3}{6} - \frac{aLx^2}{4} + \frac{aL^2x}{12}\right\}$$

$$\theta(x) = \frac{a}{12EI}\{2x^3 - 3Lx^2 + L^2x]$$

One can readily verufy that the expressions for $\theta(x)$ and $v(x)$ clearly match with those found in books and literature for an elastic bean under uniformly distributed load.

Case-2: For a Linear distributed load, $w(x) = 100x$

In this case, $a_0 = 0,\ a_1 = 100,\ a_2 = a_3 = \cdots a_n = 0$, and $n = 1$.

Using (10), we get:

$$c_1 = -15L^2$$

From (11), we obtain:

$$c_2 = \frac{10L^3}{3}$$

From (8)-(9), we have:

$$v(x) = \frac{5}{6EI}(x^5 - 3L^2x^3 + 2L^3x^2)$$

$$\theta(x) = \frac{5}{6EI}\{5x^4 - 9L^2x^2 + 4L^3x\}$$

For a realistic display, we consider the beam of length L = 30ft, with EI = 161111 unit, the deflection and slope parameters are shown in Figures I and II, respectively. We can verify that the maximum deflection is attained at $\frac{L}{2} = 15ft$ as expected. For case II, similar results are shown in Figures III and IV. Thus, the method devised in this study successfully leads to the expression of slope and deflection of an elastic beam under any type of varying load.

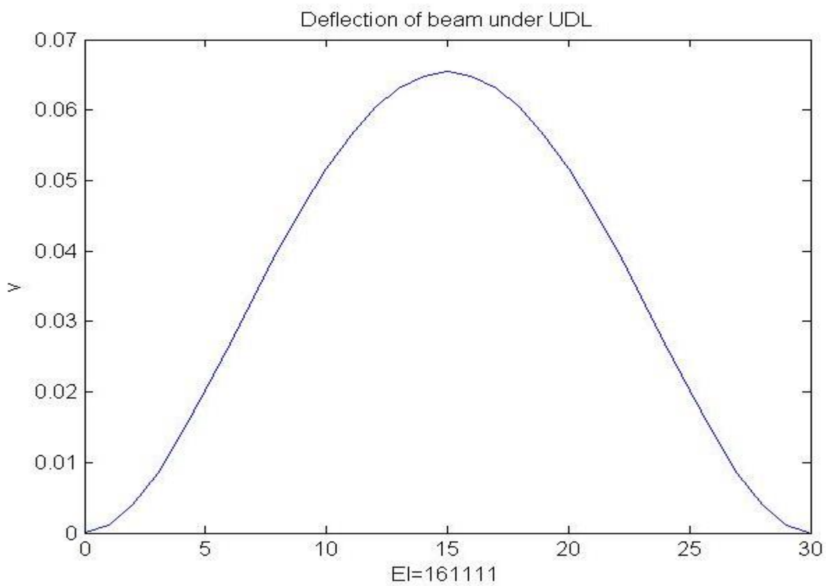

Fig.1. Deflection of elastic bean under applied uniformly distributed load

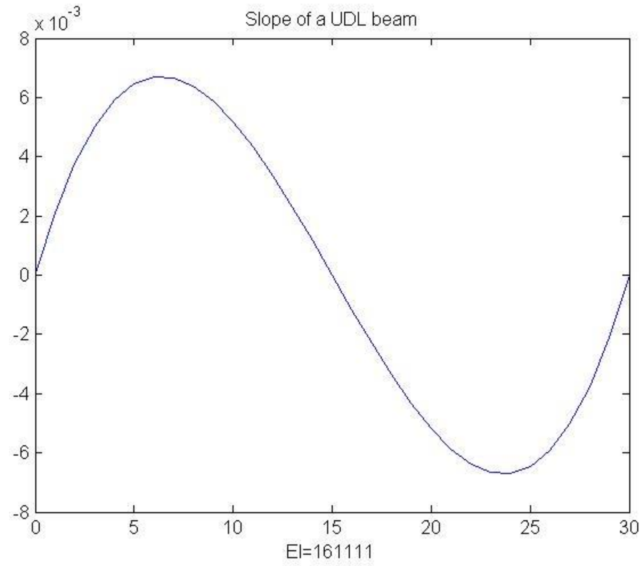

Fig.2. Slope of elastic bean under applied uniformly distributed load

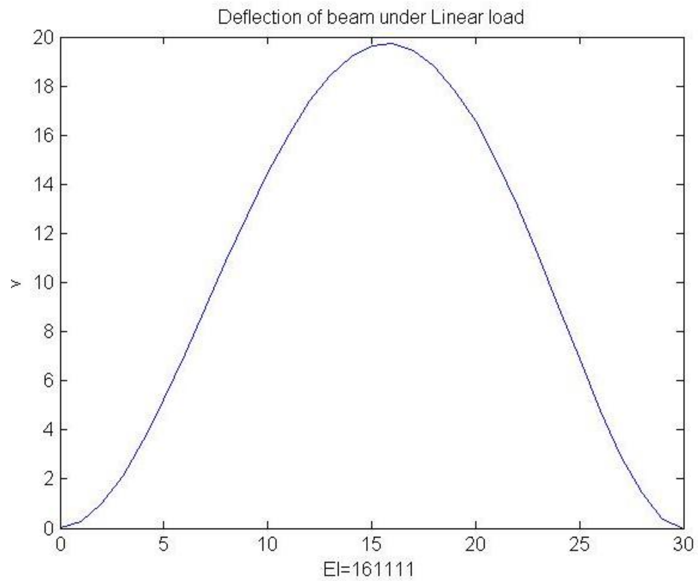

Fig.3. Deflection of elastic bean under applied varying linear load

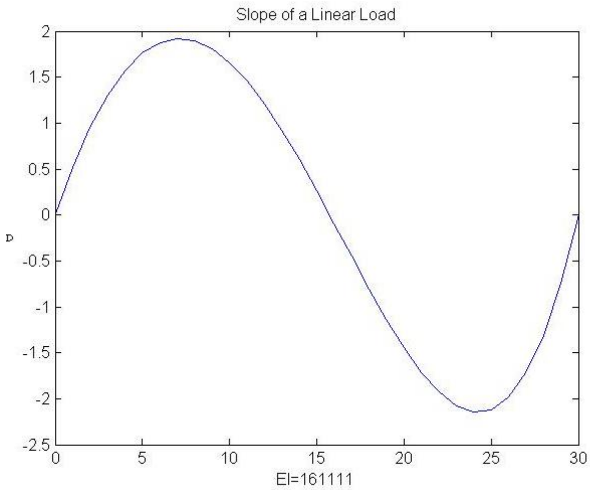

Fig.4. Slope of elastic bean under applied varying linear load

V. Conclusion

We have found a general analytical solution to the elastic beam model with defined boundary conditions in this study. The uniform distributed load and linear load are two different kinds of constant and variable loads, respectively, for which this general form has been validated as well. We have determined the deflection and slope of the elastic beam just by simplifications and validated the results with already existing solutions.

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Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

References

I. Babak Mansoori, Ashkan Torabi, Arash Totonch (2020). 'Numerical investigation of the reinforced concrete beams using cfrp rebar, steel sheets and gfrp'. *J. Mech. Cont.& math. Sci., vol.-15, no.-3, pp 195-204*.

II. Cheng XL, Han W, Huang HC (1997). Finite element methods for Timoshenko beam, circular arch and Reissner-Mindlin plate problems. *J. Comput. Appl. Math.,79(2): 215- 234.*

III. Li L (1990). 'Discretization of the Timoshenko beam problem by the p and h/p versions of the finite element method'. *Numer. Math., 57(1): 413-420.*

IV. Malik, Kamran, Shaikh, Abdul Wasim and Shaikh, Muhammad Mujtaba. (2021). "An efficient finite difference scheme for the numerical solution of Timoshenko beam model. *Journal of mechanics of continua and mathematical sciences", 16 (5): 76-88..*

V. Malik, Kamran, Shaikh, Muhammad Mujtaba and Shaikh, Abdul Wasim. (2021)."On exact analytical solutions of the Timoshenko beam model under uniform and variable loads. *Journal of mechanics of continua and mathematical sciences", 16 (5): 66-75.*

VI. Timoshenko SP (1921). On the correction for shear of the differential equation for transverse Vibrations of prismatic bars, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 41(245): 744-746.

VII. Timoshenko, S. (1953). History of strength of materials. New York: McGraw-Hill Charles V. Jones, "The Unified Theory of Electrical Machines", London, 1967.