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ANDERSON'S ∇ – INTEGRAL INEQUALITY

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Abstract

Basically, time scale calculus is the theory of unification of traditional calculus with that calculus of difference i.e. discrete calculus. Time Scale Calculus is a field of discussion in the area of traditional analysis of mathematics. It focuses on the dynamic system which has a lot of applications in various fields of life. Calculus of time scales is a valuable field due to numerous applications in covid-19 disease cases. Notably, Time scale calculus has a long relation with mathematical inequalities that can be discussed with fractional calculus. The Anderson Integral Inequality, which provides a lower constraint for the integration of convex mapping in the form of the averages of each constituent, is described in this research paper on $\nabla -$ time-scale calculus. On ∇-time scale we formulated Anderson's integral inequality as given below:

$$\left[\left(\prod_{j=1}^{\alpha}\int_0^1 \varphi_i^{\rho}(\chi)\right)\left(2^{\alpha}\int_0^1[\rho(\chi)]^{\alpha}\nabla\chi\right)\right] \le \left(\prod_{j=1}^{\alpha}\int_0^1\varphi_i^{\rho}(\chi)\right)\left(\int_0^1(\rho(\chi)+\chi)^{\alpha}\nabla\chi\right)\le\int_0^1\left(\prod_{j=1}^{\alpha}\varphi_j(\chi)\right)\nabla\varphi$$

if $\varphi_j(j = 1, \ldots, \alpha)$ accomplish some appropriate cases.

Keywords: Time scales, Anderson's inequality, ∇ - differentiable

I. Introduction

In 1958, Anderson [1] showed the following very interesting inequality:

$$\int_0^1 \phi_1(x)\,\phi_2(x)\ldots\ldots\,\phi_p(x)dx \ge \frac{2p}{p+1}\left(\int_0^1 \phi_1(x)dx\right)\ldots\left(\int_0^1 \phi_\alpha(x)dx\right)\ (L_1)$$

If $\phi_j(0) = 0\ and\ \phi_1$ is convex increasing on the interval [0, 1] and $for\ all\ j = 1, 2, \ldots.\alpha$. Presently, Fink [I] refined the form of Anderson's integral inequality (L_1) as follows:

$$\int_0^1 \phi_1(x)\,\phi_2(x)\ldots\ldots\,\phi_p(x)dx \ge \frac{2^p}{p+1}\left(\int_0^1 \phi_1(x)dx\right)\ldots\left(\int_0^1 \phi_\alpha(x)dx\right)$$
(L_2)

For each $j = 1,2, \dots . \alpha$, If $\phi(0) = 0$ and $\frac{\phi_j(\chi)}{\chi}$ is increasing on (0, 1]. Further, It is discussed in Fink [I] that the state $\phi_j(0) = 0$ $(j = 1,2, \dots, \alpha)$ cannot be abandoned. The main aim of this article is to gain Anderson's integral inequality in the calculus of time scales. More related findings we can be read from the book [V] of Mitrinvic etc. Now, shortly

We introduce time scale theory now in a short way and refer readers to Hilger and Aulbach Hilger [II], Hilger [VII], and the books [VI] and [III] for additional study in detail.

Definition1$_A$: A time scale is randomly a closure i.e. nonempty set which is the subset of the set of real numbers ℝ, from the beginning of this article we consider that T is a time scale and it inherits from the real numbers ℝ that is the usual topology on ℝ. The shift operator which is forward can be defined at point $\chi \epsilon$ T, σ: T → T as

$$\sigma(\chi) := \inf\{\tau > \chi : \tau \in \mathrm{T}\} \in \mathrm{T},$$

and the backward shift operator ρ: T → T is defined as

$$\rho(\chi) := \sup\{\tau < \chi : \tau \in \mathrm{T}\} \in \mathrm{T}.$$

If $\sigma(\chi) > \chi, we\ say\ \chi\ is\ right\ scaterred, while\ if \rho(\chi) <$
$\chi, we\ say\ s\ is\ left\ scattered\ if\ \sigma(\chi) = \chi, we\ say\ t\ is\ right\ dense,$

$while\ if\ \rho(\chi) = \chi, we\ say\ '\chi'\ is\ left\ dense.$

Definition 1$_B$: The interval [a, b] in T is defined, for $a, b \in R$ with $a \leq b$, by

[a, b]:=$\{\chi \in \mathrm{T} : a \leq \chi \leq b\}$

Other sorts of intervals are defined in the same manner.

Definition 1$_C$: Consider x: T → ℝ and fix $\chi \in$ T; then with the property that there is a neighborhood U of χ, given any $\epsilon > 0$, we define $x^{\Delta}(\chi)$ (provided it exists) to be the number such that.

$$\left|[x(t) - x(\rho(\chi))] - x^{\nabla}(\chi)[(\chi) - \rho(\chi)]\right| < \epsilon\, |t - \rho(\chi)|,$$

For each $\chi \in U$. We call $x^{\nabla}(\chi)$the **nabla derivative** of $x(\chi)\ at\ \chi \in$ T and $x(\chi)$ nalba differentiable at χ. If at every point of T, $x(\chi)$ is nabla differentiable, then we have nabla derivative of $x(\chi)$ at $\chi \in$ T.

It can be defined, that for ∈ T , χ is left scattered and x: T → ℝ is continuous, then

$$x^{\nabla}(\chi) = \frac{x(\chi) - x(\rho(\chi))}{\chi - \rho(\chi)}$$

Definition 1_D: The mapping $\varphi\colon \mathrm{T}_k \to \mathbb{R}$ is an anti-derivative of the mapping $\varphi\colon \mathrm{T} \to \mathbb{R}$ if $\phi^{\nabla}(\chi) = \varphi(\chi)\ \forall\ \chi \in \mathrm{T}^k$. In this condition, the integral of φ is defined by

$$\int_t^{\chi} \varphi(\tau)\nabla\tau = \phi(s) - \phi(t)$$

For $t, \chi \in \mathrm{T}_k$, where

$$\mathrm{T}_k \coloneqq \begin{cases} \mathrm{T} - \{m\}, if\ \mathrm{T}\ has\ right - scattered\ minimal\ point\ m, \\ \mathrm{T}, \qquad otherwise \end{cases}$$

Definition 1_E: if $\varphi\colon \mathrm{T} \to \mathbb{R}$, then $\varphi^{\rho}\ \mathrm{T} \to \mathbb{R}$ is defined by

$$\varphi(\rho(\chi)) = \varphi^{\rho}(\chi)$$

$\forall\ \chi \in \mathrm{T}$

Definition 1_F: A mapping $\varphi\colon \mathrm{T} \to \mathbb{R}$ is left dense-continuous if it fulfills the understated two given cases

(A^*) the right-sided limit $lim_{t\to\chi}\ \varphi(t) = \varphi(\chi^+)$ exists each right-dense element $\chi \in \mathrm{T}$;

(B^*) φ is continuous at every left-dense or minimal point $\varphi \in \mathrm{T}$.

In this article we state

$C_{ld}[\mathrm{T}, \mathbb{R}] = \{\varphi \mid \varphi\colon \mathrm{T} \to \mathbb{R}$ is left dense-continuous mapping$\}$.

2. Main result

Lemma 2.1. Let $\varphi, \psi \in C_{ld}([0, 1], \mathbb{R})\ satisfy\ \varphi(0) = 0\ and\ \psi\ be\ icreasing\ on\ (0, 1].\ if\ \frac{\varphi(\rho(\chi))}{\chi+\rho(\chi)} is\ increasing\ on\ \{0,1\}$,

Then

$$\int_0^1 \psi^{\rho}(\chi)\varphi^*(\chi)\nabla\chi \leq \int_0^1 \psi^{\rho}\varphi(\chi)\nabla\chi$$

$where\ \varphi^*(\chi) \coloneqq (\chi + \rho(\chi)) \int_0^1 \varphi(u)\nabla u\ for\ \chi \in [0, 1]$

Proof. We define

$\omega(\chi) \coloneqq \int_0^1 [\,\varphi^*(\chi) - \varphi(\chi)]\nabla\chi$ on [0, 1].

Then $\omega(0) = 0$ and

$$\omega(1) \coloneqq \int_0^1 [\,\big(\chi + \rho(\chi)\big)\nabla x - 1]\ \int_0^1 \varphi(u)\nabla u = [\langle \chi^2 \big|_0^1 \rangle - 1] \int_0^1 f(u)\nabla u = 0.$$

Moreover

$$0 \leq \omega^{\nabla}(0) = \varphi^{*}(0) - \varphi(0) = \rho(0) \int_0^1 \varphi(u)\nabla u = \varphi^{*}(0).$$

and

$$\omega^{\nabla}(\chi) = \varphi^{*}(\chi) - \varphi(\chi) = (\rho(\chi) + 1) \int_0^1 \varphi(u)\nabla u - \varphi(s).$$

$$= (\rho(\chi) + 1)\left\{\int_0^1 \varphi(u)\nabla u - \frac{\varphi(\rho(\chi))}{\chi+\rho(\chi)}\right\} on\ (0, 1].$$

Since $\frac{\varphi(\rho(\chi))}{\chi+\rho(\chi)}$ is increasing on the interval (0, 1], we obtain that $\frac{\omega^{\nabla}(\varphi)}{1+\rho(\varphi)}$ is decreasing on the open-closed interval (0, 1].

The next proof can be divided into two parts and we show that $\omega(\varphi) \geq 0\ on\ [0, 1]$.

(a^{*}) If there exists a point $\chi_0 \in (0, 1)$ such that $\frac{\omega^{\nabla}(\chi_0)}{\chi_0+\rho(\chi_0)} = 0, then$

$$\omega^{\nabla}(\chi) \leq 0 \quad on\ [\chi_0, 1]\ and\ \omega^{\nabla}(\chi) \geq 0 \quad on\ [0,\ \chi_0]$$

Hence if $\chi \in [0, \chi_0], then\ \omega(\chi) \geq \omega(0) = 0$ if $\chi \in [\chi_0, 1]$,then $\omega(s) \geq \omega(1) = 0$ on $[\chi_0, 1]$. Thus $\omega(s) \geq 0$ on [0,1].

(b^{*}) $\frac{\omega^{\nabla}(\chi)}{\chi+\rho(\chi)} > 0\ on\ (0, 1)$. In this state, $\omega(\chi)$ the function which is increasing on [0, 1). Hence $\omega(\chi) \geq G(0) = 0$ on [0, 1). It follows from $\omega(1) = 0$ that is $\omega(\chi) \geq \omega(0)$ on the interval [0, 1]. Hence, by Theorem 1.77 of [6].

$$\int_0^1 [\varphi(\chi) - \varphi^{*}(\chi) g^{\rho}(\chi)\nabla\chi = -\int_0^1 \omega^{\nabla}(\chi)\psi^{\rho}(\chi)\nabla\chi$$

$$= -\left\{\psi(\chi)\omega(\chi)\Big|_0^1 - \int_0^1 \omega(\chi) g^{\nabla}(\chi)\nabla\chi\right\}$$

$$= \int_0^1 \omega(\chi)\, \psi^{\Delta}(\chi)\nabla\chi$$

$$\geq 0$$

Because g is a function which is increasing and $0 \leq \omega(\chi)$ (and thus $0 \leq \psi^{\nabla}(\chi)$). Hence we have done our proof.

Now we are in a state to prove and write our actual result as given below:

Theorem 2.2 $Suppose\ that\ \chi \in$
$[0,1]\ does\ not\ satisfy\ the\ following\ two\ condtions$:

(1^{0}) $\chi\ is\ right\ dense\ and\ left\ scattered$,

(2^{0}) $\chi\ is\ left\ dense\ and\ rightg\ scattered$.

$Let\ \varphi_1, \varphi_2, , \ldots \ldots . \varphi_\alpha \in C_{ld}([0,1], \mathbb{R})\ with\ \varphi_j(0) =$
$0\ and\ \frac{\varphi_j(\rho(\chi))}{1+\rho(\chi)}\ increasing\ on\ (0,1], for\ j = 1,2, \ldots, \alpha; then$

$$\left[\left(\prod_{j=1}^{\alpha}\int_0^1 \varphi_i^{\rho}(\chi)\right)\left(2^{\alpha}\int_0^1 [\rho(\chi)]^{\alpha}\nabla\chi\right)\right]$$

$$\leq \left(\prod_{j=1}^{\alpha}\int_0^1 \varphi_i^{\rho}(\chi)\right)\left(\int_0^1 (\rho(\chi)+\chi)^{\alpha}\nabla\chi\right)$$

$$\leq \int_0^1\left(\prod_{j=1}^{\alpha}\varphi_j(\chi)\right)\nabla s \qquad (L_3)$$

Proof. From Lemma 2.1 it follows, the increasing property and $\chi \geq \rho(\chi)$ of $\varphi_j, j = 1,2, \ldots, \alpha$, that for φ_α^* defined as in Lemma 2.1,

$$\int_0^1 \left(\prod_{j=1}^{\alpha}\varphi_j(\chi)\right)\nabla\chi \geq \left(\left(\int_0^1 \prod_{j=1}^{\alpha-1}\varphi_i^{\rho}(\chi)\right)\varphi_j(\chi)\right)\nabla\chi$$

$$\geq \int_0^1 \varphi_\alpha^*(\rho(\chi))\left(\prod_{j=1}^{\alpha}\varphi_j(\chi)\right)\nabla\chi$$

$$= \left(\int_0^1 \varphi_\alpha(\rho(\chi))\nabla\chi\right)\left(\int_0^1(\chi + \rho(\chi)\right)\left(\prod_{j=1}^{\alpha-1}\varphi_j(\chi)\nabla\chi\right)$$

$$= \left(\int_0^1 \varphi_\alpha(\rho(\chi))\nabla\chi\right)\int_0^1(\sigma(\chi) + \chi)^{\alpha}\left(\prod_{j=1}^{\alpha-1}\varphi_j(\chi)\right)\nabla\chi \qquad \text{(using } \rho(\sigma(\chi)) = \chi\text{)}$$

$$\geq \left(\int_0^1 \varphi_\alpha(\rho(\chi))\nabla\chi\right)\int_0^1 \varphi_{\alpha-1}(\rho(\chi))\left\{(\sigma(\chi) + \chi)^{\rho}\prod_{j=1}^{\alpha-2}\varphi_j(\chi)\right\}\nabla\chi$$

$$\geq \left(\int_0^1 \varphi_\alpha(\rho(\chi))\nabla\chi\right)\int_0^1 \varphi_{\alpha-1}^*(\rho(\chi))\left\{(\sigma(\chi) + \chi)^{\rho}\prod_{j=1}^{\alpha-2}\varphi_j(\chi)\right\}\nabla\chi$$

$$\geq \left(\int_0^1 \varphi_\alpha(\rho(\chi))\nabla\chi\right)\left(\int_0^1 \varphi_{\alpha-1}(\rho(\chi))\nabla\chi\right)\int_0^1(\rho(\chi) + \chi)\left\{(\chi + \sigma(\chi))^{\rho}\prod_{j=1}^{\alpha-2}\varphi_j(\chi)\right\}\nabla\chi$$

$$= \left(\int_0^1 \varphi_\alpha(\rho(\chi))\nabla\chi\right)\left(\int_0^1 \varphi_{\alpha-1}(\rho(\chi))\nabla\chi\right)\int_0^1\left\{[(\sigma(\chi) + \chi)^{\rho}]^2\prod_{j=1}^{\alpha-2}\varphi_j(\chi)\right\}\nabla\chi$$

$$\geq \left(\int_0^1 \varphi_\alpha(\rho(\chi))\nabla\chi\right)\left(\int_0^1 \varphi_{\alpha-1}(\rho(\chi))\nabla\chi\right)\int_0^1 \varphi_{\alpha-2}(\rho(\chi))\left\{[(\sigma(\chi)+\chi)^\rho]^2 \prod_{j=1}^{\alpha-3}\varphi_j(\chi)\right\}\nabla\chi$$

$$\geq \cdots$$

$$= \left(\prod_{j=1}^{\alpha}\int_0^1 \varphi_j(\rho(\chi))\nabla\chi\right)\left(\int_0^1 [(\sigma(\chi)+\chi)^\rho]^\alpha \nabla\chi\right)$$

$$= \left(\prod_{j=1}^{\alpha}\int_0^1 \varphi_j(\rho(\chi))\nabla\chi\right)\left(\int_0^1 (\chi+\rho(\chi))^\alpha \nabla\chi\right)$$

$$\geq \left(2^\alpha \int_0^1 [\rho(\chi)]^p \nabla\chi\right)\left(\prod_{j=1}^{\alpha}\int_0^1 \varphi_i(\rho(\chi))\nabla\chi\right).$$

Now, we have proof of theorem 2.2.

Conflict of Interest:

The authors declare that no conflict of interest to report the present study.

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