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ON THE FLOW OF A VISCO-ELASTIC OLDROYDIAN FLUID IN A CIRCULAR PIPE

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Abstract

In this paper an attempt has been made to study unsteady flow of a visco-elastic Oldroydian fluid in a circular pipe. Using Laplace transformation technique the basic equations of motion and boundary conditions have been modified and using these modified equations and boundary conditions the solution of the problem has been derived.

Keyword and phrases : *visco elastic Oldroydion fluid, unsteady flow, circular pipe.*

সংক্ষিপ্তসার।

এই গবেষণা পত্রে অলড্রয়ডিয়ান সান্দ্র - স্থিতিস্থাপক প্রবাহী পদার্থের বৃত্তীয় নলে অস্থীর প্রবাহকে বিচার করা হয়েছে। ল্যাপলস রূপান্তর কৌশল প্রয়োগের সাহায্যে উক্ত সমস্যার সমাধান নির্ণয় করা হয়েছে।

1. Introduction

Sneddon [1] presented the motion of a heavy viscous fluid contained between two parallel planes inclined at certain angle with the horizon. Slow steady hydromagnetic flow between two porous walls was investigated by Sengupta and Ghosh [2]. Sengupta and Ghosh [3] also developed some problems of slow hydromagnetic flow in presence of periodic or radial magnetic field. The motion of a visco-elastic Maxwell fluid subjected to a uniform or periodic body force acting for a finite time was studied by Pal and Sengupta [4]. Motion of visco-elastic fluid of Maxwell type between two inclined parallel planes in presence of gravity and uniform magnetic field was considered by Panja and sengupta [5].

Adopting the similar procedure of Panja and Sengupta, the present author has endeavored to study the problem of the flow of visco-elastic Oldroydian fluid through a circular pipe. Initially the Oldroydian fluid is at rest and a constant pressure gradient is exerted upon the system. Laplace transformation technique has been used to solve the problem. These results are in fair agreement with the corresponding classical results.

2. Basic theory and equations of motion for Oldroydian fluid :

For slow motion, the rheological equations for Oldroydian visco-elastic fluid are

$$\tau_{ij} = -p'\delta_{ij} + \tau'_{ij}$$

where

$$\left(1+\lambda'_1 \frac{\partial}{\partial t}\right)\tau'_{ij} = 2\mu\left(1+\mu'_1 \frac{\partial}{\partial t}\right)e_{ij}$$

and

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

where τ_{ij} is the stress tensor, τ'_{ij} is the deviatoric stress tensor, e_{ij} is the rate of strain tensor, p' the pressure, λ'_1 is the material constant of relaxation time parameter, μ'_1 the strain rate retardation time parameter, δ_{ij} the metric tensor in Cartesian coordinates, μ the coefficient of viscosity and μ_i the velocity components.

We consider the unsteady flow of Oldroydian visco-elastic fluid through a circular pipe. Let (r',θ',z') be the cylindrical coordinates such that z' be along the axis of the pipe.

Let u',v',w' be the components of velocity given by

$$u' = 0, \qquad v' = 0, \qquad w' = w'(r',t')$$

Then the equations of motion are

$$-\frac{1}{\rho}\left(1+\lambda'_1 \frac{\partial}{\partial t'}\right)\frac{\partial p'}{\partial x'} = 0 \tag{1}$$

$$-\frac{1}{\rho}\left(1+\lambda'_1 \frac{\partial}{\partial t'}\right)\frac{\partial p'}{\partial y'} = 0 \tag{2}$$

$$\left(1+\lambda'_1 \frac{\partial}{\partial t}\right)\frac{\partial \omega'}{\partial t'} = -\frac{1}{\rho}\left(1+\lambda'_1 \frac{\partial}{\partial t}\right)\frac{\partial p'}{\partial z'} + \nu\left(1+\mu'_1 \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 w'}{\partial r'^2} + \frac{1}{r'}\frac{\partial w'}{\partial r'}\right) \tag{3}$$

and $\nu = \dfrac{\mu}{\rho}$ is the kinematic coefficient of viscosity.

Equation (3) is to be solved with the boundary conditions

$$\left.\begin{aligned} t' = 0 \ &: \ w' = 0 \\ t' > 0 \ &: \ w' = 0 \ \text{ when } \ r' = a \\ &\qquad = \text{finite when } r' = 0 \end{aligned}\right\} \tag{4}$$

We now introduce the following dimensionless quantities:

$$w = w'\frac{a}{\nu}, \qquad p = p'\frac{a^2}{\rho\nu^2}, \qquad t = t'\frac{\nu}{a^2}, \qquad r = \frac{r'}{a}$$

$$z = \frac{z'}{a}, \qquad \lambda_1 = \frac{\nu\lambda_1'}{a^2} \qquad \text{and} \qquad \mu_1 = \mu_1'\frac{\nu}{a^2}$$

Then equation (3) becomes

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial w}{\partial t} = \left(1+\lambda_1\frac{\partial}{\partial t}\right)p_0 + \left(1+\mu_1\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right) \tag{5}$$

where $-\dfrac{\partial p}{\partial z} = p_o$ (constant) for $t > 0$.

Equation (5) is to be solved under the boundary conditions

$$\left.\begin{aligned} t = 0 \ &: \ w = 0 \\ t > 0 \ &: \ w = 0 \ \text{ when } \ r = 1 \\ &\quad\ w = \text{finite when } r = 0 \end{aligned}\right\} \tag{6}$$

3. Method of solution

We define the Laplace transform of w(r, t) by

$$\overline{w}(r,s) = \int_0^\infty e^{-st} w(r,t)\,dt \tag{7}$$

and $\nu = \frac{\mu}{\rho}$ is the kinematic coefficient of viscosity.

Equation (3) is to be solved with the boundary conditions

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3. Method of solution

We define the Laplace transform of w(r, t) by

$$\overline{w}(r,s) = \int_0^\infty e^{-st} w(r,t)dt \tag{7}$$

Multiplying equation (5) by e^{-st} and integrating w. r. t. 't'. between the limits 0 to ∞ we get

$$\int_0^\infty e^{-st}\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial w}{\partial t}dt = \int_0^\infty e^{-st}\left(1+\lambda_1\frac{\partial}{\partial t}\right)p_0 dt$$
$$+\int_0^\infty e^{-st}\left(1+\mu_1\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)dt$$

or, $$\frac{d^2\overline{w}}{dr^2}+\frac{1}{r}\frac{d\overline{w}}{\partial r}-s\frac{(1+\lambda_1 s)}{1+\mu_1 s}\overline{w} = -\frac{p_o}{s(1+\mu_1 s)} \qquad (8)$$

Then general solution of equation (8) is

$$\overline{w} = AI_0(\beta r)+\frac{p_0}{s^2(1+\lambda_1 s)} \qquad (9)$$

where I_0 is the modified Bessel function of the first kind of order zero.

The boundary conditions of $\overline{w}$ are

$$\left.\begin{array}{ll} \overline{w}=0 & \text{when } r=1 \\ \overline{w}=\text{finite} & \text{when } r=0 \end{array}\right\} \qquad (10)$$

So the solution of equation (8) under conditions (10) is

$$\overline{w} = \frac{p_0}{s^2(1+\lambda_1 s)}\left[1-\frac{I_0(\beta r)}{I_0(\beta)}\right] \qquad (11)$$

where $\beta^2 = \dfrac{s(1+\lambda_1 s)}{1+\mu_1 s}$.

By inverse Laplace transform we obtain from equation (11)

$$w = \frac{P_0}{2\pi i}\int_{\lambda - i\alpha}^{\lambda + i\alpha} \frac{1}{s^2(1+\lambda_1 s}\left[1 - \frac{I_0(\beta r)}{I_0(\beta)}\right]e^{st}\,ds \tag{12}$$

where r is greater than the real part of the singularities of the integrand. We shall evaluate (12) by using Bromwich Contour.

The integrand of (12) is a single-valued function with poles at s = 0 and the roots of $I_0(\beta) = 0$. Performing contour integration in the usual way we get from (12)

$$w = p_o\left[\frac{1}{4}(1-r^2) - 2\sum_{n=1}^{\infty}\{\Delta_n^{(1)}e^{\beta_{1n}t} + \Delta_n^{(2)}e^{\beta_{2n}t}\}\frac{\alpha_n J_0(\alpha_n r)}{J_1(\alpha_n)}\right] \tag{13}$$

where β_{1n} and β_{2n} are the roots of the equation

$$\lambda_1 s^2 + (1 + \mu_1\alpha_n)s + \alpha_n = 0 \tag{14}$$

$$\Delta_n^{(i)} = \frac{(1+\mu_1\beta_{in})^2}{\beta_{in}^2(1+\lambda_1\beta_{in})(1+2\lambda_1\beta_{in} + \mu_1\lambda_1\beta_{in}^2)}, \quad i = 1, 2 \tag{15}$$

and $\alpha_n (n = 1,2,......)$ are the roots of the equation J0 $(\alpha) = 0$ and J_1 is the Bessel function of the first kind of order unity.

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