Krylov-Bogoliubov-Mitropolskii (KBM) Method for Fourth Order More Critically Damped Nonlinear Systems

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Abstract

Krylov-Bogoliubov-Mitropolskii (KBM) method has been extended for obtaining the solutions of fourth order more critically damped nonlinear systems. The results obtained by the presented KBM method show good coincidence with numerical results obtained by Runge-Kutta method. The method is illustrated by an example.

Keyword and phrases : *critically damped, non-linear system, KBM method, Runge-Kutta method.*

সংক্ষিপ্তসার

চতুর্থক্রমের আরও বেশী ক্রান্তিক অবমন্দিত অ-রৈখিকতন্ত্রের সমাধান নির্ণয়ের জন্য ক্রাইলভ-বোগোলিউভব-মিট্রোপোলস্কি পদ্ধতিকে সম্প্রসারিত করা হয়েছে। KBM পদ্ধতিতে নির্ণিত ফলগুলি রুঙ্গো - কুটা পদ্ধতির সাহায্যে নির্ণিত সাংখ্যমানের ফলগুলির সঙ্গে ভালভাবে মিলে যায়। এই পদ্ধতিটি উদাহরণের সাহায্যে ব্যাখ্যা করা হয়েছে।

1. Introduction

Krylov-Bogoliubov-Mitropolskii (KBM) [4, 5] is widely spread method to study nonlinear differential equations. Originally, the method was developed for obtaining the periodic solutions of a second order nonlinear differential equation with small nonlinearities. Later, this method has been extended by Popov [9] for nonlinear damped oscillatory system. Owing to physical importance Popov's results have been rediscovered by Mendelson [6]. Murty *et al.* [7] have been developed an asymptotic method based on the method of Bogoliubov to obtain the response of over-damped nonlinear systems. Murty [8] also presented a unified KBM method which covers the undamped, damped and over-damped cases. Sattar [12] has found an asymptotic solution of a second order critically damped nonlinear systems. Shamsul [15] has developed a new asymptotic solution for both over-damped and critically damped nonlinear systems.

Sattar [13] has studied a three-dimensional over-damped nonlinear systems. In article [19] Shamsul has presented a perturbation method for solving a third order over-damped system based on the KBM method when two roots of the linear equation are almost equal (rather than equal) and one root is small. Shamsul and Sattar [14] developed a perturbation technique based on the work of KBM for obtaining the solution of third order critically damped nonlinear equations. Shamsul [16] has investigated approximate solutions of third order critically nonlinear systems whose unequal eigenvalues are in integral multiple. In article [16] Shamsul has also investigated solutions of a third order more critically damped nonlinear system. Rokibul *et al.* [10] found a new technique for obtaining the solutions of third order critically damped systems.

In article [7], Murty *et al.* also extended the KBM method to solve fourth order over damped nonlinear systems. But their method is too much complex and laborious. Ali Akbar *et al.* [1] again presented an asymptotic method for fourth order over-damped nonlinear systems which is simple and easier than the method presented by [7] but the results obtained by [1] method is same as the results obtained by [7] method. Later Ali Akbar *et al.* [2] extended the method presented in [1] for fourth order damped oscillatory systems. Ali Akbar *et al.* [3] also presented a simple technique for obtaining certain over damped solutions of an *n*-th order nonlinear differential equation. Rokibul *et al.* [11] have extended the KBM method for fourth order critically damped nonlinear systems.

In the present paper, we have investigated solutions of fourth order more critically damped nonlinear systems under some conditions. The solutions obtained by the presented method show good coincidence with those obtained by numerical method.

2. The Method

Consider a fourth order weakly nonlinear ordinary differential equation

$$x^{(4)} + k_1\dddot{x} + k_2\ddot{x} + k_3\dot{x} + k_4 x = -\varepsilon f(x,\dot{x},\ddot{x},\dddot{x}) \qquad (1)$$

where $x^{(4)}$ denote the fourth derivative of x, over dots are denoted first, second and third derivatives with respect to t; k_1, k_2, k_3, k_4 are constants, ε is the small parameter and $f(x,\dot{x},\ddot{x},\dddot{x})$ is the given nonlinear function. As the equation is of fourth order so it has four eigenvalues. Since the system is more critically damped so the eigenvalues are real, negative and three of them are equal. Suppose the eigenvalues are $-\lambda,-\lambda,-\lambda,-\mu$. When $\varepsilon=0$, the equation becomes linear and the solution of the linear equation of (1) is

$$x(t,0)=(a_0+b_0\,t+c_0\,t^2)e^{-\lambda t}+d_0\,e^{-\mu t} \tag{2}$$

where a_0, b_0, c_0, d_0 are constants of integration.

When $\varepsilon \neq 0$, following [18] the solution of the equation (1) is sought in the form

$$x(t,\varepsilon)=(a+b\,t+ct^2)e^{-\lambda t}+d\,e^{-\mu t}+\varepsilon u_1(a,b,c,\ d,t)+\cdots \tag{3}$$

where a,b,c and d are slowly varying function of time t and satisfy the first order differential equation

$$\begin{aligned}\dot{a}(t)&=\varepsilon\ A_1(a,b,c,d,t)+\cdots\\ \dot{b}(t)&=\varepsilon\ B_1(a,b,c,d,t)+\cdots\\ \dot{c}(t)&=\varepsilon\ C_1(a,b,c,d,t)+\cdots\\ \dot{d}(t)&=\varepsilon\ D_1(a,b,c,d,t)+\cdots\end{aligned} \tag{4}$$

We only consider first few terms in the series expansion of (3) and (4), we evaluate the functions u_i and A_i, B_i, C_i, D_i, $i=1,2,\ldots\ldots\ldots,n$ such that $a,\ b,\ c,\ d$ appearing in (3) and (4) satisfy the given differential equation (1). In order to determine these unknown functions it is customary in KBM method that the correction terms, u_i, $i=1,2,\ldots\ldots n$ must exclude terms (known as secular terms) which make them large. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a lower order, usually the first Murty [8].

Now differentiating the equation (3) four times with respect to t, substituting the value of x and the derivatives $\dot{x},\ddot{x},\dddot{x},x^{(4)}$ in the original equation (1), utilizing the relations presented in (4) and finally equating the coefficients of ε, we obtain

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left\{\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1+t\left(\frac{\partial^2 C_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)+t^2\,\frac{\partial^2 C_1}{\partial t^2}\right\}$$
$$+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1+\left(\frac{\partial}{\partial t}+\lambda\right)^3\left(\frac{\partial}{\partial t}+\mu\right)u_1=-f^{(0)}(a,b,c,d,t) \tag{5}$$

where $f^{(0)}(a,b,c,d,t)=f(x_0,\dot{x}_0,\ddot{x}_0,\dddot{x}_0)$ and $x_0=(a+bt+ct^2)e^{-\lambda t}+d\,e^{-\mu t}$

Now we expand $f^{(0)}$ in the Taylor's series of the form

$$f^{(0)}=(bt+ct^2)^0\sum_{i,j=0}^{\infty}F_0(a,d)e^{-(i\lambda+j\mu)t}+(bt+ct^2)^1\sum_{i,j=0}^{\infty}F_1(a,d)e^{-(i\lambda+j\mu)t}$$
$$+(bt+ct^2)^2\sum_{i,j=0}^{\infty}F_2(a,d)e^{-(i\lambda+j\mu)t}+(bt+ct^2)^3\sum_{i,j=0}^{\infty}F_3(a,d)e^{-(i\lambda+j\mu)t}+\Lambda \tag{6}$$

Thus we can write

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left\{\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1+t\left(\frac{\partial^2 B_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)+t^2\,\frac{\partial^2 C_1}{\partial t^2}\right\}$$
$$+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1+\left(\frac{\partial}{\partial t}+\lambda\right)^3\left(\frac{\partial}{\partial t}+\mu\right)u_1$$
$$=-\left\{(bt+ct^2)^0\sum_{i,j=0}^{\infty}F_0(a,d)e^{-(i\lambda+j\mu)t}+(bt+ct^2)^1\sum_{i,j=0}^{\infty}F_1(a,d)e^{-(i\lambda+j\mu)t}\right.$$
$$\left.+(bt+ct^2)^2\sum_{i,j=0}^{\infty}F_2(a,d)e^{-(i\lambda+j\mu)t}+(bt+ct^2)^3\sum_{i,j=0}^{\infty}F_3(a,d)e^{-(i\lambda+j\mu)t}+\Lambda\right\} \tag{7}$$

We impose the condition that u_1 can not contain the fundamental terms of $f^{(0)}$, therefore equation (7) can be separated for unknowns functions u_1 and $A_1, B_1, C_1,\ D_1$ in the following way (see also [11, 14, 16, 17] for details).

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 C_1}{\partial t^2}=-c\sum_{i,j=0}^{\infty}F_1(a,d)e^{-(i\lambda+j\mu)t} \tag{8}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 B_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)=-b\sum_{i,j=0}^{\infty}F_1(a,d)e^{-(i\lambda+j\mu)t} \tag{9}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1\right)+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1$$
$$=-\sum_{i,j=0}^{\infty}F_0(a,d)e^{-(i\lambda+j\mu)t} \tag{10}$$

and $$\left(\frac{\partial}{\partial t}+\lambda\right)^3\left(\frac{\partial}{\partial t}+\mu\right)u_1 = -\sum_{i,j=0}^{\infty} F_2(a,d)\, e^{-(i\lambda+j\mu)t}(bt+ct^2)^2 - \Lambda \tag{11}$$

Solving the equation (8) we get the value

$$C_1 = \sum_{i,j=0}^{\infty} \frac{c\,F_1(a,d)\, e^{-((i-1)\lambda+j\mu)t}}{(i\lambda+(j-1)\mu)((i-1)\lambda+j\mu)^2} \tag{12}$$

Substituting the value of C_1 from (12) into equation (9), we obtain

$$\begin{aligned} e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 B_1}{\partial t^2} = &-6\sum_{i,j=0}^{\infty} \frac{c\,F_1(a,d)\, e^{-(i\lambda+j\mu)t}}{((i-1)\lambda+j\mu)(i\lambda+(j-1)\mu)} \\ &-\sum_{i,j=0}^{\infty} b\,F_1(a,d)\, e^{-(i\lambda+j\mu)t} \end{aligned} \tag{13}$$

Now solving equation (13), we obtain

$$B_1 = 6\sum_{i,j=0}^{\infty} \frac{c\,F_1(a,d)\, e^{-((i-1)\lambda+j\mu)t}}{((i-1)\lambda+j\mu)^3 (i\lambda+(j-1)\mu)^2} + \sum_{i,j=0}^{\infty} \frac{b\,F_1(a,d)\, e^{-((i-1)\lambda+j\mu)t}}{((i-1)\lambda+j\mu)^2 (i\lambda+(j-1)\mu)} \tag{14}$$

Now substituting the value of C_1 from (12) and B_1 from (14) into equation (10), we obtain

$$\begin{aligned} e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 A_1}{\partial t^2} + e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1 = &-12\sum_{i,j=0}^{\infty} \frac{c\,F_1(a,d)\, e^{-(i\lambda+j\mu)t}}{((i-1)\lambda+j\mu)^2} \\ &-3\sum_{i,j=0}^{\infty} \frac{b\,F_1(a,d)\, e^{-(i\lambda+j\mu)t}}{((i-1)\lambda+j\mu)} - \sum_{i,j=0}^{\infty} F_0(a,d)\, e^{-(i\lambda+j\mu)t} \end{aligned} \tag{15}$$

Now we have only one equation (15) for obtaining the unknown functions A_1 and D_1. So we need to impose some restrictions. In this paper, we have used the restriction that the term $e^{-(i\lambda+j\mu)t}$ balance with A_1 if $i \ge j$ and the term $e^{-(i\lambda+j\mu)t}$ balance with D_1 if $j > i$. This restriction is important, since under this restriction the coefficient of A_1 and D_1 do not become large as well as this restriction is useful in the case of strongly more critically damping systems. This restriction is not used in previous papers ([10-12, 14, 16, 17]). Thus we shall be able to separate the equation (15) into two equations, one for A_1 and the other for D_1.

Since $\dot{a}, \dot{b}, \dot{c}, \dot{d}$ are proportional to small parameter ε, so they are slowly varying functions of time t and as a first approximation, we may consider them as constants in the right hand side of the equation (4). Now substituting the values of A_1, B_1, C_1 and D_1 into the equation (4) and integrating, we obtain

$$\begin{aligned} a &= a_0 + \varepsilon \int_0^t A_1(a_0, b_0, c_0, d_0, t)\, dt \\ b &= b_0 + \varepsilon \int_0^t B_1(a_0, b_0, c_0, d_0, t)\, dt \\ c &= c_0 + \varepsilon \int_0^t C_1(a_0, b_0, c_0, d_0, t)\, dt \\ d &= d_0 + \varepsilon \int_0^t D_1(a_0, b_0, c_0, d_0, t)\, dt \end{aligned} \tag{16}$$

Substituting the values of a, b, c, d and u_1 in the equation (3), we get the complete solution of (1).

Thus the determination of the first approximate solution is completed. The solution can be carried out for higher order systems in the same way.

3. Example

As an example of the above procedure consider a fourth order weakly nonlinear system governed by the ordinary differential equation

$$x^{(4)} + k_1 \dddot{x} + k_2 \ddot{x} + k_3 \dot{x} + k_4 x = -\varepsilon x^3 \tag{17}$$

Here $f = x^3$

Therefore,

$$\begin{aligned} f^{(0)} = {} & a^3 e^{-3\lambda t} + 3a^2 d\, e^{-(2\lambda+\mu)t} + 3a d^2\, e^{-(\lambda+2\mu)t} + d^3 e^{-3\mu t} \\ & + \left(3a^2 e^{-3\lambda t} + 6ad\, e^{-(2\lambda+\mu)t} + 3d^2\, e^{-(\lambda+2\mu)t}\right)(bt + ct^2) \\ & + \left(3a e^{-3\lambda t} + 3d\, e^{-(2\lambda+\mu)t}\right)(bt + ct^2)^2 + (bt + ct^2)^3 e^{-3\lambda t} \end{aligned}$$

Therefore, for equation (17), the equation (8)-(11) respectively become

$$e^{-\lambda t}\left(\frac{\partial}{\partial t} + \mu - \lambda\right)\frac{\partial^2 C_1}{\partial t^2} = -\left\{3a^2 c e^{-3\lambda t} + 6acd\, e^{-(2\lambda+\mu)t} + 3cd^2\, e^{-(\lambda+2\mu)t}\right\} \tag{18}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 B_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)=-\left\{3a^2be^{-3\lambda t}+6abd\,e^{-(2\lambda+\mu)t}+3bd^2e^{-(\lambda+2\mu)t}\right\} \quad (19)$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1\right)+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1$$
$$=-\left\{a^3e^{-3\lambda t}+3a^2d\,e^{-(2\lambda+\mu)t}+3ad^2\,e^{-(\lambda+2\mu)t}+d^3e^{-3\lambda_1 t}\right\} \quad (20)$$

and

$$\left(\frac{\partial}{\partial t}+\lambda\right)^3\left(\frac{\partial}{\partial t}+\mu\right)u_1=-\left\{b^3t^3e^{-3\lambda t}+6abc\,t^3e^{-3\lambda t}+3b^2c\,t^4e^{-3\lambda t}\right.$$
$$+3ac^2t^4e^{-3\lambda t}+3bc^2t^5e^{-3\lambda t}+c^3t^6e^{-3\lambda t}+6bcd\,t^3e^{-(2\lambda+\mu)t} \quad (21)$$
$$\left.+3c^2d\,t^4e^{-(2\lambda+\mu)t}+3ab^2t^2e^{-3\lambda t}+3db^2t^2e^{-(2\lambda+\mu)t}\right\}$$

The solution of the equation (18) is

$$C_1=l_1\,a^2ce^{-2\lambda t}+l_2\,acd\,e^{-(\lambda+\mu)t}+l_3\,cd^2e^{-2\mu t} \quad (22)$$

Where $l_1=\dfrac{3}{4\lambda^2(3\lambda-\mu)}$, $l_2=\dfrac{6}{2\lambda(\lambda+\mu)^2}$, $l_3=\dfrac{3}{4\mu^2(\mu+\lambda)}$

Putting the value of C_1 from the equation (22) in the equation (19), we obtain

$$B_1=m_1\,a^2ce^{-2\lambda t}+m_2\,acd\,e^{-(\lambda+\mu)t}+m_3\,cd^2e^{-2\mu t}$$
$$+m_4\,a^2be^{-2\lambda t}+m_5abd\,e^{-(\lambda+\mu)t}+m_6bd^2e^{-2\mu t} \quad (23)$$

where $m_1=\dfrac{9}{4\lambda^3(3\lambda-\mu)}$, $m_2=\dfrac{18}{\lambda(\lambda+\mu)^3}$, $m_3=\dfrac{9}{4\mu^3(\mu+\lambda)}$,
$m_4=\dfrac{3}{4\lambda^2(3\lambda-\mu)}$, $m_5=\dfrac{3}{\lambda(\lambda+\mu)^2}$, $m_6=\dfrac{3}{4\mu^2(\mu+\lambda)}$

To separate the equation (20) for determining unknown functions A_1 and D_1, in this paper we impose the restriction that term $e^{-(i\lambda+j\mu)t}$ balance with A_1 if $i\geq j$ and D_1 if $j>i$. Under this restriction, we obtain

$$
\begin{aligned}
e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 A_1}{\partial t^2} &= 6m_1\,\lambda(\mu-3\lambda)a^2 c\, e^{-3\lambda t} \\
&-12\,\lambda(\lambda+\mu)m_2\, a c d\, e^{-(2\lambda+\mu)t}+6\lambda(\mu-3\lambda)m_4\, a^2 b e^{-3\lambda t} \\
&-6\,\lambda(\lambda+\mu)m_5 a b d\, e^{-(2\lambda+\mu)t}-6\,(\mu-3\lambda)l_1 a^2 c\, e^{-3\lambda t} \\
&+24\,\lambda l_2 a c d\, e^{-(2\lambda+\mu)t}-a^3 e^{-3\lambda t}-3a^2 d\, e^{-(2\lambda+\mu)t}
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1 &= -6\,\mu(\lambda+\mu)m_3\, c d^2\, e^{-(\lambda+2\mu)t}-6\,\mu(\lambda+\mu)m_6 b d^2\, e^{-(\lambda+2\mu)t} \\
&+6(\lambda+\mu)l_3\, c d^2\, e^{-(\lambda+2\mu)t}-3a d^2\, e^{-(\lambda+2\mu)t}-d^3 e^{-3\mu t}
\end{aligned} \tag{25}
$$

The particular solutions of (24) and (25) respectively become

$$
\begin{aligned}
A_1 &= n_1\, a^2 c\, e^{-2\lambda t}+n_2\, a c d\, e^{-(\lambda+\mu)t} \\
&+n_3\, a^2 b e^{-2\lambda t}+n_4\, a b d\, e^{-(\lambda+\mu)t}+n_5\, a^2 c\, e^{-2\lambda t} \\
&+n_6\, a c d\, e^{-(\lambda+\mu)t}+n_7\, a^3 e^{-2\lambda t}+n_8\, a^2 d\, e^{-(\lambda+\mu)t}
\end{aligned} \tag{26}
$$

$$
D_1 = p_1\, c d^2\, e^{-(\lambda+\mu)t}+p_2\, b d^2\, e^{-(\lambda+\mu)t}+p_3\, a d^2\, e^{-(\lambda+\mu)t}+p_4\, d^3 e^{-2\mu t} \tag{27}
$$

where $n_1=\dfrac{27}{8\lambda^4(3\lambda-\mu)}$, $n_2=\dfrac{18}{\lambda(\mu+\lambda)^4}$, $n_3=\dfrac{9}{8\lambda^3(3\lambda-\mu)}$,

$n_4=\dfrac{9}{\lambda(\mu+\lambda)^3}$, $n_5=\dfrac{9}{8\lambda^4(\mu-3\lambda)}$, $n_6=-\dfrac{18}{\lambda(\mu+\lambda)^4}$,

$n_7=\dfrac{1}{4\lambda^2(\mu-3\lambda)}$, $n_8=\dfrac{3}{2\lambda(\mu+\lambda)^2}$, $p_1=\dfrac{9}{4\mu^5}$,

$p_2=\dfrac{9}{16\mu^4}$, $p_3=\dfrac{3}{8\mu^3}$ $p_4=\dfrac{1}{(3\mu-\lambda)^3}$

The solution of the equation (21) for u_1 is

$$
\begin{aligned}
u_1 &= \left(r_1 t^3+r_2 t^2+r_3 t+r_4\right)\left(b^3+6abc\right)e^{-3\lambda t}+\left(r_5 t^4+r_6 t^3+r_7 t^2+r_8 t+r_9\right) \\
&\times(b^2 c+ac^2)e^{-3\lambda t}+\left(r_{10}t^5+r_{11}t^4+r_{12}t^3+r_{13}t^2+r_{14}t+r_{15}\right)bc^2 e^{-3\lambda t} \\
&+\left(r_{16}t^6+r_{17}t^5+r_{18}t^4+r_{19}t^3+r_{20}t^2+r_{21}t+r_{22}\right)c^3 e^{-3\lambda t} \\
&+\left(r_{23}t^3+r_{24}t^2+r_{25}t+r_{26}\right)bcde^{-(\mu+2\lambda)t} \\
&+\left(r_{27}t^4+r_{28}t^3+r_{29}t^2+r_{30}t+r_{31}\right)c^2 de^{-(\mu+2\lambda)t} \\
&+\left(r_{32}t^2+r_{33}t+r_{34}\right)ab^2 e^{-3\lambda t}+\left(r_{35}t^2+r_{36}t+r_{37}\right)b^2 de^{-(\mu+2\lambda)t}
\end{aligned} \tag{28}
$$

where $r_1 = \dfrac{1}{8\lambda^3(\mu-3\lambda)}$ $\quad r_2 = r_1\left\{-\dfrac{3}{(\mu-3\lambda)}+\dfrac{9}{2\lambda}\right\}$

$$r_3 = r_1\left\{\frac{6}{(\mu-3\lambda)^2}-\frac{9}{\lambda(\mu-3\lambda)}+\frac{9}{\lambda^2}\right\}$$

$$r_4 = r_1\left\{-\frac{6}{(\mu-3\lambda)^3}+\frac{9}{\lambda(\mu-3\lambda)^2}-\frac{9}{\lambda^2(\mu-3\lambda)}+\frac{15}{2\lambda^3}\right\}$$

$$r_5 = \frac{3}{8\lambda^3(\mu-3\lambda)} \qquad r_6 = r_5\times\left\{-\frac{4}{(\mu-3\lambda)}+\frac{6}{\lambda}\right\}$$

$$r_7 = r_5\times\left\{\frac{12}{(\mu-3\lambda)^2}-\frac{18}{\lambda(\mu-3\lambda)}+\frac{18}{\lambda^2}\right\}$$

$$r_8 = r_5\times\left\{-\frac{24}{(\mu-3\lambda)^3}+\frac{36}{\lambda(\mu-3\lambda)^2}-\frac{36}{\lambda^2(\mu-3\lambda)}+\frac{30}{\lambda^3}\right\}$$

$$r_9 = r_5\times\left\{\frac{24}{(\mu-3\lambda)^4}-\frac{36}{\lambda(\mu-3\lambda)^3}+\frac{36}{\lambda^2(\mu-3\lambda)^2}-\frac{30}{\lambda^3(\mu-3\lambda)}+\frac{45}{2\lambda^4}\right\}$$

$$r_{10} = \frac{3}{8\lambda^3(\mu-3\lambda)} \qquad r_{11} = r_{10}\times\left\{-\frac{5}{(\mu-3\lambda)}+\frac{15}{2\lambda}\right\}$$

$$r_{12} = r_{10}\times\left\{\frac{20}{(\mu-3\lambda)^2}-\frac{30}{\lambda(\mu-3\lambda)}+\frac{30}{\lambda^2}\right\}$$

$$r_{13} = r_{10}\times\left\{-\frac{60}{(\mu-3\lambda)^3}+\frac{90}{\lambda(\mu-3\lambda)^2}-\frac{90}{\lambda^2(\mu-3\lambda)}+\frac{75}{\lambda^3}\right\}$$

$$r_{14} = r_{10}\times\left\{\frac{120}{(\mu-3\lambda)^4}-\frac{180}{\lambda(\mu-3\lambda)^3}+\frac{180}{\lambda^2(\mu-3\lambda)^2}-\frac{150}{\lambda^3(\mu-3\lambda)}+\frac{225}{2\lambda^4}\right\}$$

$$r_{15} = r_{10}\times\left\{-\frac{120}{(\mu-3\lambda)^5}+\frac{180}{\lambda(\mu-3\lambda)^4}-\frac{180}{\lambda^2(\mu-3\lambda)^3}+\frac{150}{\lambda^3(\mu-3\lambda)^2}-\frac{150}{\lambda^4(\mu-3\lambda)}+\frac{315}{4\lambda^5}\right\}$$

$$r_{16} = \frac{1}{8\lambda^3(\mu-3\lambda)} \qquad r_{17} = r_{16}\times\left\{-\frac{6}{(\mu-3\lambda)}+\frac{9}{2\lambda}\right\}$$

$$r_{18} = r_{16} \times \left\{ \frac{30}{(\mu - 3\lambda)^2} - \frac{45}{\lambda(\mu - 3\lambda)} + \frac{45}{\lambda^2} \right\}$$

$$r_{19} = r_{16} \times \left\{ -\frac{120}{(\mu - 3\lambda)^3} + \frac{180}{\lambda(\mu - 3\lambda)^2} - \frac{180}{\lambda^2(\mu - 3\lambda)} + \frac{150}{\lambda^3} \right\}$$

$$r_{20} = r_{16} \times \left\{ \frac{360}{(\mu - 3\lambda)^4} - \frac{540}{\lambda(\mu - 3\lambda)^3} + \frac{540}{\lambda^2(\mu - 3\lambda)^2} - \frac{450}{\lambda^3(\mu - 3\lambda)} + \frac{675}{2\lambda^4} \right\}$$

$$r_{21} = r_{16} \times \left\{ -\frac{720}{(\mu - 3\lambda)^5} + \frac{1080}{\lambda(\mu - 3\lambda)^4} - \frac{1080}{\lambda^2(\mu - 3\lambda)^3} + \frac{900}{\lambda^3(\mu - 3\lambda)^2} - \frac{675}{\lambda^4(\mu - 3\lambda)} + \frac{945}{2\lambda^5} \right\}$$

$$r_{22} = r_{16} \times \left\{ \frac{720}{(\mu - 3\lambda)^6} - \frac{1080}{\lambda(\mu - 3\lambda)^5} + \frac{1080}{\lambda^2(\mu - 3\lambda)^4} - \frac{900}{\lambda^3(\mu - 3\lambda)^3} \right.$$
$$\left. + \frac{675}{\lambda^4(\mu - 3\lambda)^2} - \frac{945}{2\lambda^5(\mu - 3\lambda)} + \frac{315}{\lambda^5} \right\}$$

$$r_{23} = -\frac{3}{2\lambda(\lambda + \mu)^3} \qquad r_{24} = r_{23} \left\{ \frac{3}{2\lambda} + \frac{9}{(\lambda + \mu)} \right\}$$

$$r_{25} = r_{23} \left\{ \frac{3}{2\lambda^2} + \frac{9}{\lambda(\lambda + \mu)} + \frac{9}{(\lambda + \mu)^2} \right\}$$

$$r_{26} = r_{23} \left\{ \frac{3}{4\lambda^3} + \frac{9}{2\lambda^2(\lambda + \mu)} + \frac{9}{\lambda(\lambda + \mu)^2} + \frac{60}{(\lambda + \mu)^3} \right\}$$

$$r_{27} = -\frac{3}{2\lambda(\lambda + \mu)^3} \qquad r_{28} = r_{27} \times \left\{ \frac{2}{\lambda} + \frac{12}{(\lambda + \mu)} \right\}$$

$$r_{29} = r_{27} \times \left\{ \frac{3}{\lambda^2} + \frac{18}{\lambda(\lambda + \mu)} + \frac{72}{(\lambda + \mu)^2} \right\}$$

$$r_{30} = r_{28} \times \left\{ \frac{3}{\lambda^3} + \frac{18}{\lambda^2(\lambda + \mu)} + \frac{72}{\lambda(\lambda + \mu)^2} + \frac{240}{(\lambda + \mu)^3} \right\}$$

$$r_{31} = r_{28} \times \left\{ \frac{3}{2\lambda^4} + \frac{9}{\lambda^3(\lambda + \mu)} + \frac{36}{\lambda^2(\lambda + \mu)^2} + \frac{120}{\lambda(\lambda + \mu)^3} + \frac{360}{(\lambda + \mu)^4} \right\}$$

$$r_{32} = \frac{3}{8\lambda^3(\mu - 3\lambda)}, \qquad r_{33} = r_{32} \times \left\{ \frac{3}{\lambda} - \frac{2}{(\mu - 3\lambda)} \right\},$$

$$r_{34} = r_{32} \times \left\{ \frac{2}{2\lambda^2} - \frac{3}{\lambda(\mu - 3\lambda)} + \frac{2}{(\mu - 3\lambda)^2} \right\}$$

$$r_{35} = -\frac{3}{2\lambda(\lambda+\mu)^3}, \qquad r_{36} = r_{35} \times \left\{\frac{6}{(\lambda+\mu)} + \frac{1}{\lambda}\right\}$$

$$r_{37} = r_{35} \times \left\{\frac{12}{(\mu+\lambda)^2} + \frac{3}{\lambda(\mu+\lambda)} + \frac{1}{2\lambda^2}\right\}$$

Substituting the values of A_1, B_1, C_1, D_1 from the equations (26), (23), (22) and (27) into equation (16), we obtain

$$\begin{aligned} a = a_0 + \varepsilon \Bigg\{ & \frac{n_1 a_0^2 c_0 \left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{n_2 a_0 c_0 d_0 \left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} \\ & + \frac{n_3 a_0^2 b_0 \left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{n_4 a_0 b_0 d_0 e^{-(\lambda+\mu)t}}{(\lambda+\mu)} + \frac{n_5 a_0^2 c_0 \left(1 - e^{-2\lambda t}\right)}{2\lambda} \\ & + \frac{n_6 a_0 c_0 d_0 \left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} + \frac{n_7 a_0^3 \left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{n_8 a_0^2 d_0 \left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} \Bigg\} \end{aligned} \tag{29}$$

$$\begin{aligned} b = b_0 + \varepsilon \Bigg\{ & \frac{m_1 a_0^2 c_0 \left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{m_2 a_0 c_0 d_0 \left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} \\ & + \frac{m_3 c_0 d_0^2 \left(1 - e^{-2\mu t}\right)}{2\mu} + \frac{m_4 a_0^2 b_0 \left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{m_5 a_0 b_0 d_0 \left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} \\ & + \frac{m_6 b_0 d_0^2 \left(1 - e^{-2\mu t}\right)}{2\mu} \Bigg\} \end{aligned} \tag{30}$$

$$c = c_0 + \varepsilon \left\{ \frac{l_1 a_0^2 c_0 \left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{l_2 a_0 c_0 d_0 e^{-(\lambda+\mu)t}}{(\lambda+\mu)} + \frac{l_3 c_0 d_0^2 e^{-2\mu t}}{2\mu} \right\} \tag{31}$$

$$d = d_0 + \varepsilon \left\{ \frac{p_1 c_0 d_0^2 e^{-(\lambda+\mu)t}}{(\lambda+\mu)} + \frac{p_2 b_0 d_0^2 e^{-(\lambda+\mu)t}}{(\lambda+\mu)} + \frac{p_3 a_0 d_0^2 e^{-(\lambda+\mu)t}}{(\lambda+\mu)} + \frac{p_4 d_0^3 e^{-2\mu t}}{2\mu} \right\} \tag{32}$$

Therefore, we obtain the first approximate solution of the equation (17) as

$$x(t,\varepsilon) = (a + b\,t + c\,t^2) e^{-\lambda t} + d\, e^{-\mu t} + \varepsilon u_1(a, b, c, d, t) \tag{33}$$

where a, b, c, d are given by the equations (29)-(32) and u_1 given by (28).

4. Result and Discussion

By means of extended KBM method an asymptotic solution of a fourth order more critically damped nonlinear system has been found in this paper. It is usual to compare the perturbation solutions to the numerical solutions to test the accuracy of the approximate solutions. With regard to such a comparison concerning the presented KBM method of this paper, we refer to work of Murty *et al.* [7]. In the present paper, we have compared the solutions obtained by (33) to those obtained by Runge-Kutta method for different values of λ and μ as well as different set of initial conditions.

First of all $x(t,\varepsilon)$ has been computed by (33) in which a, b, c, d are calculated by (29)-(32) with initial conditions $a_0 = 0.5$, $b_0 = 0.0$, $c_0 = 0.4$, $d_0 = 0.1$ for different values of λ and μ say (i) $\lambda = 4$, $\mu = 3.5$ (ii) $\lambda = 4.5$, $\mu = 3$ (iii) $\lambda = 4$, $\mu = 2$ (iv) $\lambda = 2.5$, $\mu = 1$ when $\varepsilon = 0.1$. The corresponding numerical solution has been computed by Runge-Kutta method and percentage errors are calculated. All the results are presented in Table 1 and Table 2.

Table 1

T	x_1	x_1^*	Error%	x_2	x_2^*	Error%
0.0	0.599998	0.599998	0.00000	0.599999	0.599999	0.00000
0.5	0.098580	0.098577	0.00304	0.085554	0.085553	0.00116
1.0	0.019504	0.019497	0.03590	0.014977	0.014972	0.03339
1.5	0.003995	0.003991	0.10022	0.002750	0.002747	0.10921
2.0	0.000796	0.000794	0.25188	0.000507	0.000506	0.19762
2.5	0.000152	0.000152	0.00000	0.000094	0.000094	0.00000
3.0	0.000028	0.000028	0.00000	0.000018	0.000018	0.00000
3.5	0.000005	0.000005	0.00000	0.000004	0.000004	0.00000
4.0	0.000001	0.000001	0.00000	0.000001	0.000001	0.00000
4.5	0.000000	0.000000	0.00000	0.000000	0.000000	0.00000
5.0	0.000000	0.000000	0.00000	0.000000	0.000000	0.00000

Initial conditions are $a_0 = 0.5$, $b_0 = 0.0$, $c_0 = 0.4$, $d_0 = 0.1$ and $\varepsilon = 0.1$

x_1 is computed by (33) using (i) $\lambda = 4$, $\mu = 3.5$

x_2 is computed by (33) using (ii) $\lambda = 4.5$, $\mu = 3$

x_1^* and x_2^* are corresponding numerical solutions.

Table 2

T	x_1	x^*_1	*Error%*	x_2	x^*_2	*Error%*
0.0	0.599998	0.599998	0.00000	0.599852	0.599852	0.00000
0.5	0.117994	0.117990	0.00339	0.232841	0.232761	0.03437
1.0	0.030020	0.030006	0.04665	0.110893	0.110643	0.22595
1.5	0.008450	0.008439	0.13034	0.055381	0.055059	0.58482
2.0	0.002536	0.002531	0.19755	0.027769	0.027473	1.07742
2.5	0.000810	0.000808	0.24752	0.014051	0.013823	1.64942
3.0	0.000273	0.000272	0.36764	0.007276	0.007117	2.23408
3.5	0.000096	0.000095	1.05263	0.003893	0.003788	2.77191
4.0	0.000034	0.000034	0.00000	0.002156	0.002088	3.25676
4.5	0.000012	0.000012	0.00000	0.001229	0.001187	3.53833
5.0	0.000005	0.000005	0.00000	0.000717	0.000691	3.76266

Initial conditions are $a_0 = 0.5$, $b_0 = 0.0$, $c_0 = 0.4$, $d_0 = 0.1$ and $\varepsilon = 0.1$.

x_3 is computed by (33) using (iii) $\lambda = 4$, $\mu = 2$

x_4 is computed by (33) using (iv) $\lambda = 2.5$, $\mu = 1$

x^*_3 and x^*_4 are corresponding numerical solutions.

From Table 1, we see that when the ratio of λ and μ is $O(1)$, the errors are greater than the errors when the ratio of λ and μ is 1.5. Again from Table 2, we see that when the ratio of λ and μ is $O(3)$ the errors are much greater than the error when the ratio of λ and μ is 2. Thus we see that when the ratio of the eigenvalues λ and μ lies between $O(1)$ and $O(3)$, the errors occur much smaller than 1%.

Again $x(t,\varepsilon)$ has been computed by (33) for the same eigenvalues (i) $\lambda = 4$, $\mu = 3.5$ (ii) $\lambda = 4.5$, $\mu = 3$ (iii) $\lambda = 4$, $\mu = 2$ (iv) $\lambda = 2.5$, $\mu = 1$ with another set of initial conditions $a_0 = 0.4$, $b_0 = 0.0$, $c_0 = 0.3$, $d_0 = 0.1$ when $\varepsilon = 0.25$. The corresponding numerical solutions are computed by Runge-Kutta method and are presented in Table 3 and Table 4

Table 3

T	x_1	x_1^*	*Error%*	x_2	x_2^*	*Error%*
0.0	0.499998	0.499998	0.00000	0.499999	0.499999	0.00000
0.5	0.081663	0.081659	0.00489	0.072379	0.072376	0.00414
1.0	0.015841	0.015831	0.06316	0.012755	0.012748	0.05491
1.5	0.003190	0.003184	0.18844	0.002370	0.002366	0.16906
2.0	0.000628	0.000626	0.31948	0.000445	0.000444	0.22522
2.5	0.000119	0.000119	0.00000	0.000085	0.000085	0.00000
3.0	0.000022	0.000022	0.00000	0.000017	0.000016	6.25000
3.5	0.000004	0.000004	0.00000	0.000003	0.000003	0.00000
4.0	0.000001	0.000001	0.00000	0.000001	0.000001	0.00000
4.5	0.000000	0.000000	0.00000	0.000000	0.000000	0.00000
5.0	0.000000	0.000000	0.00000	0.000000	0.000000	0.00000

Initial Conditions are $a_0 = 0.4$, $b_0 = 0.0$, $c_0 = 0.3$, $d_0 = 0.1$ and $\varepsilon = 0.25$.

x_1 is computed by (33) using (i) $\lambda = 4$, $\mu = 3.5$

x_2 is computed by (33) using (ii) $\lambda = 4.5$, $\mu = 3$

x_1^* and x_2^* are corresponding numerical solutions.

Table 4

T	x_1	x_1^*	*Error%*	x_2	x_2^*	*Error%*
0.0	0.499997	0.499997	0.00000	0.499809	0.499809	0.00000
0.5	0.101082	0.101072	0.00989	0.197367	0.197186	0.09179
1.0	0.026358	0.026337	0.07973	0.094747	0.094200	0.58067
1.5	0.007645	0.007629	0.20972	0.047913	0.047218	1.47189
2.0	0.002369	0.002361	0.33883	0.024507	0.024875	2.64712
2.5	0.000777	0.000774	0.38759	0.012715	0.012231	3.95715
3.0	0.000267	0.000266	0.37593	0.006762	0.006424	5.26151
3.5	0.000095	0.000094	1.06382	0.003707	0.003484	6.40068
4.0	0.000034	0.000034	0.00000	0.002093	0.001951	7.27831
4.5	0.000012	0.000012	0.00000	0.001210	0.001122	7.84313
5.0	0.000005	0.000005	0.00000	0.000712	0.000658	8.20668

Initial conditions are $a_0 = 0.4$, $b_0 = 0.0$, $c_0 = 0.3$, $d_0 = 0.1$ and $\varepsilon = 0.25$.

x_3 is computed by (33) using (iii) $\lambda = 4$, $\mu = 2$

x_4 is computed by (33) using (iv) $\lambda = 2.5$, $\mu = 1$

x_3^* and x_4^* are corresponding numerical solutions.

We again see from Table 3 that when the ratio of λ and μ is $O(1)$, the errors are greater than the errors when the ratio of λ and μ is 1.5. Again from Table 4, we observed that when the ratio of λ and μ is $O(3)$, the errors are much greater than the errors when the ratio of λ and μ is 2. Again when the ratio of λ and μ are greater than 3 the errors occur much smaller than 1% (see Table 5).

Table 5

T	x_1	x^*_1	*Error%*
0.0	0.599995	0.599995	0.00000
0.5	0.141914	0.141922	0.00563
1.0	0.053305	0.053312	0.01313
1.5	0.025802	0.025805	0.01162
2.0	0.014249	0.014251	0.01403
2.5	0.008351	0.008352	0.01197
3.0	0.005008	0.005009	0.01996
3.5	0.003027	0.003027	0.00000
4.0	0.001834	0.001834	0.00000
4.5	0.001112	0.001112	0.00000
5.0	0.000674	0.000674	0.00000

Initial conditions are $a_0 = 0.5$, $b_0 = 0.0$, $c_0 = 0.4$, $d_0 = 0.1$ and $\varepsilon = 0.1$

x_1 is computed by (33) using $\lambda = 4.0$, $\mu = 1.0$

x_1^* is corresponding numerical solutions.

Thus from all five tables we see that if the ratio of the eigenvalues λ and μ is less than or greater than 3, the solution (33) gives desired results. If the ratio of λ and μ is $O(1)$, the system undergoes strongly more critically damping. In this case the errors are also smaller than 1%. Therefore, the solution (33) is also usable for strongly more critically damped systems. When $\lambda = 3\mu$ the solution (33) breakdown and in this case the treatment will be different.

5. Conclusion

In the presence of different critically damping effect, a formula has been presented for obtaining the solutions of more critically-damped non-linear systems governed by the fourth order ordinary differential equation. For different set of initial conditions as well as for different set of eigenvalues the solutions obtained by this method show good coincidence with the corresponding numerical solutions. The solutions are also useful for strongly more critically system.

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