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ON BITOPOLOGICAL SPACES

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Abstract.

In this paper, we introduce weakly pairwise regular spaces and considering a weakly pairwise regular space, we prove a theorem on β-pairwise paracompactness as an analogue of Michael's characterization of paracompactness of regular spaces.

Keyword and phrases : *regular space, pairwise regular space. Paraconpactness.*

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এই গবেষণা পত্রে আমরা একটি যুগ্মভাবে দুর্বল দেশকে বিবেচনা করে যুগ্মভাবে দুর্বল সুষম দেশসমূহকে উপস্থাপন করছি। মাইকেলের বৈশিষ্ট্যযুক্ত সুষম দেশসমুহের আপা-ঘনত্বের সমগোত্রীয় হিসাবে বিটা - যুগ্মভাবে আপা - ঘনত্বের একটি উপপাদ্য প্রমাণ করেছি।

1. Introduction.

A set equipped with two topologies is called a bitopological space. Kelly [2] started the study of bitopological spaces. Then several authors have contributed to the development of the theory. The bitopological paracompactness was first introduced by Fletcher, Hoyle and Patty [1]. They called it pairwise paracompactness. But in presence of pairwise Hausdorffness, a pairwise paracompact space becomes a paracompact single topological space. Later on Raghavan and Reilly [4] introduced the notions of α-, β-, γ- and δ-pairwise paracompactness. They proved a δ-pairwise paracompactness version of Michael's characterization [3] of paracompactness of a regular topological space. They considered the space to be pairwise regular. In this paper, we introduce weakly pairwise regular space and prove the Michael's theorem for β-pairwise paracompactness considering the space to be weakly pairwise regular. We also give an example of a bitopological space which is weakly pairwise regular but not pairwise regular.

Definitions.

Let (X,τ_1,τ_2) be a bitopological space and let $\tau_1 \vee \tau_2$ denote the smallest topology on X containing both τ_1 and τ_2. For a set $A \subset X$, $(\tau_1 \vee \tau_2)ClA$ denotes the closure of A with respect to the topology $\tau_1 \vee \tau_2$. We denote the set of natural numbers and real numbers by N and R respectively.

Definition 1 (Kelly [2]). Let i, j = 1, 2. The bitopological space (X,τ_1,τ_2) is said to be *pairwise regular* if for every (τ_i)closed set F and $x \in X$ with $x \notin F$, there exist a τ_i – open set U and a (τ_j)open set V such that $i \neq j, x \in U, F \subset V$ and $U \cap V = \phi$.

In the sequel, we assume i = 1, 2.

Definition 2 (Raghavan and Reilly [4]). A collection $\mathscr{C}$ of subsets of X is said to be (τ_i)*locally finite* if for every $x \in X$, there exists a set $G \in \tau_i$ intersecting only a finite number of members of $\mathscr{C}$.

Definition 3 (Raghavan and Reilly [4]). The space (X,τ_1,τ_2) is said to be *β-pairwise paracompact* if every (τ_i)open cover of X has a $(\tau_1 \vee \tau_2)$open refinement which is (τ_i)locally finite.

We introduce the following definition.

Definition 4. The space (X,τ_1,τ_2) is said to be *weakly pairwise regular* if for any (τ_i)closed set F and any point $x\in X$ with $x\notin F$,there exist a $U\in\tau_i$ and a $V\in\tau_1\vee\tau_2$ such that $x\in U, F\subset V$ and $U\cap V=\phi$.

It is easy to see that (X,τ_1,τ_2) is weakly pairwise regular if and only if for any point x and any (τ_i)open set G with $x\in G$, there exists a $U\in\tau_i$ such that $x\in U\subset(\tau_1\vee\tau_2)clU\subset G$.

Result.

Theorem. *Let (X,τ_1,τ_2) be weakly pairwise regular. Then the following statements are equivalent:*

i) X is β-pairwise paracompact.

ii) Each (τ_i)open cover of X has a $(\tau_1\vee\tau_2)$open refinement that can be decomposed into an at most countable collection of (τ_i)locally finite families of $(\tau_1\vee\tau_2)$open sets.

iii) Each (τ_i)open cover of X has a (τ_i)locally finite refinement (not necessarily $(\tau_1\vee\tau_2)$open).

iv) Each (τ_i)open cover of X has a $(\tau_1\vee\tau_2)$closed (τ_i)locally finite refinement.

In the above theorem, we consider the space to be weakly pairwise regular. We now give an example of a bitopological space which is not pairwise regular but weakly pairwise regular.

Example. If τ_1 and τ_2 are the usual topology and the right order topology on R, then the bitopological space (R, τ_1, τ_2) is obviously weakly pairwise regular. In fact, if F is a (τ_1)closed set with $x \notin F$, then there exists an open interval (a, b) with $x \in (a, b) \subset R - F$. Let $a < \alpha < x < \beta < b$. Then $x \in (\alpha, \beta) = U \in \tau_1$, $F \subset (-\infty, \alpha) \cup (\beta, \infty) = V \in \tau_1 \subset \tau_1 \vee \tau_2$ and $U \cap V = \phi$. And if K is a (τ_2)closed set with $x \notin K$, then $K = (-\infty, \alpha]$ for some $\alpha \in R$ and $x > \alpha$. For $\beta \in (\alpha, x)$, $x \in (\beta, \infty) = G \in \tau_2$, $K \subset (-\infty, \beta) = H \in \tau_1 \subset \tau_1 \vee \tau_2$ and $G \cap H = \phi$. But it is not pairwise regular. In fact, if $x \in (a, b)$, then $x \notin B = (-\infty, a] \cup [b, \infty)$, a (τ_1)closed set but R is the only (τ_2)open set containing B.

Proof of the Theorem.

i) $\Rightarrow$ *ii)*: Straightforward.

ii) $\Rightarrow$ *iii)*: Let $\mathscr{U}$ be a (τ_i)open cover of X. Then there exists a $(\tau_1 \vee \tau_2)$open refinement $\mathscr{V}$ of $\mathscr{U}$ such that $\mathscr{V} = \bigcup_{n \in N} \mathscr{V}_n$, where the sub-collection $\mathscr{V}_n$ of $\mathscr{V}$ is (τ_i) locally finite for each n. Suppose $\mathscr{V}_n = \{V_{n\alpha} \mid \alpha \in A\}$. Since $\mathscr{V}$ is a refinement of $\mathscr{U}$, there exists a set $U_{n\alpha} \in \mathscr{U}$ such that $V_{n\alpha} \subset U_{n\alpha}$. We write $G_n = \bigcup_n U_{n\alpha}$. Then $\mathscr{G} = \{ G_n \mid n \in \mathrm{N}\}$ is a (τ_i)open cover of X. Let $E_n = G_n - \bigcup_{i<n} G_i$. If $x \in X$ and n_0 is the smallest n such that $x \in G_{n_0}$, then $x \in E_{n_0}$. Therefore $\{E_n \mid n \in N\}$ is a cover of X and is a refinement of $\mathscr{G}$. It is also (τ_i)locally finite. In fact, G_{n_0} is a (τ_i) open nbd of x which does not intersect any E_n for $n > n_0$. The collection $\{ E_n \cap V_{n\alpha} \mid n \in N, \alpha \in A \}$ is then a (τ_i)locally finite refinement of $\mathscr{V}$ and hence of $\mathscr{U}$.

iii) $\Rightarrow$ *iv)*: Let $\mathscr{U}$ be a (τ_i)open cover of X. For $x \in X$, we choose a $U_x \in \mathscr{U}$ such that $x \in U_x$. Since X is weakly pairwise regular and U_x is (τ_i)open, there exists a (τ_i)open set V_x such that $x \in V_x \in (\tau_1 \vee \tau_2)clV_x \subset U_x$. Then the collection $\mathscr{V} = \{V_x \mid x \in X\}$ forms a (τ_i)open cover of X and so by *iii*), there exists a (τ_i)locally finite refinement $\{A_\gamma \mid \gamma \in \Gamma\}$ of $\mathscr{V}$ and hence of $\mathscr{U}$. If $A_\gamma \subset V_x \subset U_x$, then $(\tau_1 \vee \tau_2)clA_\gamma \subset U_x$. Therefore by lemma 20.4 (Willard [5]), $\{(\tau_1 \vee \tau_2)clA_\gamma\}$ is a $(\tau_1 \vee \tau_2)$ closed (τ_i) locally finite refinement of $\mathscr{U}$

iv) $\Rightarrow$ *i*): Let $\mathscr{U}$ be a (τ_i)open cover of X and $\mathscr{V}$ be a (τ_i)locally finite refinement of $\mathscr{U}$. For each $x \in X$, let W_x be a (τ_i)open set such that $x \in W_x$ and W_x intersects a finite number of sets $\in \mathscr{V}$. Then $\mathscr{W} = \{W_x \mid x \in X\}$ is a (τ_i) open cover of X. Therefore $\mathscr{W}$ has a (τ_i)locally finite $(\tau_1 \vee \tau_2)$closed refinement $\mathscr{F}$ For each $V \in \mathscr{V}$, we write

$$V^* = X - \cup\{F \in \mathscr{F} \mid F \cap V = \phi\}.$$

Since $\mathscr{F}$ is (τ_i)locally finite, it is $(\tau_1 \vee \tau_2)$locally finite and hence by lemma 20.5 (Willard [5]), it follows that the collection $\{V^* \mid V \in \mathscr{V}\}$ is a $(\tau_1 \vee \tau_2)$ open cover of X. We now show that it is also (τ_i)locally finite. Let H_x be a (τ_i) open nbd of x intersecting $F_1, F_2, \ldots, F_n$ of $\mathscr{F}$. But

$$H_x \cap V^* \neq \phi,$$

$$\Rightarrow F_k \cap V^* \neq \phi \text{ for some } k = 1, 2, \ldots, n,$$

$$\Rightarrow F_k \cap V \neq \phi \text{ for some } k = 1, 2, \ldots, n.$$

Again each F_k can intersect only a finite number of $V \in \mathscr{V}$. Therefore H_x can intersect a finite number of V^*. Thus $\{V^* \mid V \in \mathscr{V}\}$ is (τ_i)locally finite.

For every $V \in \mathcal{V}$, let $U(V) \in \mathcal{U}$ satisfy the condition $V \subset U(V)$. Then the collection $\{ V^* \cap U(V)\}$ is a $(\tau_1 \vee \tau_2)$open (τ_i)locally finite refinement of $\mathcal{U}$ Therefore X is β-pairwise paracompact.

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