

TORSIONAL VIBRATION OF AN IN-HOMOGENEOUS ELASTIC CONE

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Abstract:

The object of this paper is to study the torsional vibration of an in-homogeneous elastic cone. For in-homogeneity of the material considered it is assumed that the elastic constants and the density of the material vary exponentially as the radial distance. Two broad cases of end condition have been taken into account. Displacements and stresses for a particular case have been obtained and are shown in tabular form and graphically for different values of radial distance r .

Keyword and phrases : torsion vibration, elastic cone, in-homogeneous, stress,

সংক্ষিপ্তসার

অসমসত্ত্ব স্থিতিস্থাপক শঙ্কুর ব্যাবর্তন স্পন্দনকে পুঙ্খানুপুঙ্খভাবে বিচার করাই এই গবেষণা পত্রের উদ্দেশ্য। বস্তুটির অসমসত্ত্বার কথা বিবেচনা করে বস্তুটির স্থিতিস্থাপক ধ্রুবকগুলি এবং ঘনত্বকে অরীয় (ব্যাসার্ধ) দূরত্বের সূচকীয় ভেদে বিদ্যমান হিসাবে ধরা হয়েছে। একটি বিশেষ ক্ষেত্রের জন্য সরণ এবং পীড়ণগুলিকে নির্ণয় করা হয়েছে এবং অরীয় দূরত্ব r এর বিভিন্ন মানের জন্য ইহাদের তালিকাকারে এবং লেখচিত্রের সাহায্যে দেখানো হয়েছে।

1. Introduction:

The problem of the torsional vibration of an elastic cone has been the object of intensive study for a pretty long year. For a cone of spherically anisotropic material torsional vibration problem has been studied Bhanja [4]. When the homogeneity breaks the corresponding problem for an isotropic material has been discussed by Mukherjee [11]. More generally when the cone is made of spherically an isotropic material the torsional vibration has been dealt with by the present author in details. Two cases of end conditions have been taken into account. Displacements and stresses in particular case are shown in tabular form and graphically for different values of r .

2. Ormulation and Solution of the Problem

We take the spherical exponential variable with the origin at the vertex of the cone and r-direction along the axis of the cone. The strain energy function of spherically anisotropic material can be written as Love [8].

$$2w = c_{33}' e_{rr}^2 + c_{11}' (e_{\theta\theta}^2 + e_{\varphi\varphi}^2) + 2c_{13}' (e_{\theta\theta} + e_{\varphi\varphi}) e_{rr} + 2c_{12}' e_{\theta\theta} e_{\varphi\varphi} + c_{66}' e_{\theta\varphi}^2 + c_{44}' (e_{r\varphi}^2 + e_{r\theta}^2) \text{-----(1)}$$

where $c_{12}' = c_{11}' - 2c_{66}'$ and c_{ij}' are elastic constants.

The stress-components are

$$\left. \begin{aligned} \hat{r}r &= c_{33}' e_{rr} + c_{13}' e_{\theta\theta} + c_{13}' e_{\varphi\varphi} \\ \hat{\theta}\theta &= c_{13}' e_{rr} + c_{11}' e_{\theta\theta} + c_{12}' e_{\varphi\varphi} \\ \hat{\varphi}\varphi &= c_{13}' e_{rr} + c_{12}' e_{\theta\theta} + c_{11}' e_{\varphi\varphi} \\ \hat{\theta}\varphi &= c_{66}' e_{\theta\varphi} \\ \hat{r}\varphi &= c_{44}' e_{r\varphi} \\ \hat{r}\theta &= c_{44}' e_{r\theta} \end{aligned} \right\} \text{-----(2)}$$

For torsional vibration, we assume the displacements for the present problem as

$$u_r = 0 = u_\theta \text{ and } u_\varphi = f(r) \sin\theta e^{i\omega t} \text{-----(3)}$$

Thus we have strain components as [Love] [8]

$$e_{rr} = e_{\theta\theta} = e_{\varphi\varphi} = e_{r\theta} = e_{\theta\varphi} = 0$$

and

$$e_{r\varphi} = \left(f'(r) - \frac{f(r)}{r} \right) \sin\theta e^{i\omega t}$$

and the stress-components from (2.2) are

$$\hat{r}r = \hat{\theta}\theta = \hat{\varphi}\varphi = \hat{r}\theta = \hat{\theta}\varphi = 0$$

and

$$\hat{r}\varphi = c_{44}' \left(f'(r) - \frac{f(r)}{r} \right) \sin\theta e^{i\omega t}$$

For non-homogeneity of the material considered, we suppose,

$$c_{44}' = c_{44} e^{nr}, \quad \rho = \rho_0 e^{nr}$$

Thus we have,

$$\left. \begin{aligned} \hat{r}r = \hat{\theta}\theta = \hat{\varphi}\varphi = \hat{r}\theta = \hat{\theta}\varphi = 0 \\ \text{and} \\ \hat{r}\varphi = c_{44} e^{nr} \left(f'(r) - \frac{f(r)}{r} \right) \sin\theta e^{i\omega t} \end{aligned} \right\} \text{-----(4)}$$

Substituting (2.4), in the stress-equation equilibrium, in absence of body forces, the first two equations are found to be satisfied identically and third one viz.

$$\frac{\partial \hat{r}\varphi}{\partial r} + \frac{1}{r} \frac{\partial \hat{\theta}\varphi}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial \hat{\varphi}\varphi}{\partial \varphi} + \frac{1}{r} \left(3\hat{r}\varphi + 2\hat{\theta}\varphi \cot\theta \right) = \rho \frac{\partial^2 u_\varphi}{\partial t^2}$$

reduces to

$$r^2 \frac{d^2 f}{dr^2} + (nr^2 + 2r) \frac{df}{dr} + \{\lambda^2 r^2 - (nr + 2)\} f = 0 \text{ ----- (5)}$$

where

$$\lambda^2 = \frac{\rho_0 \omega^2}{c_{44}} \text{ ----- (6)}$$

The solution of equation (5) is

$$f(r) = A_1 f_1(r) + B_1 [\log r f_2(r) + f_3(r)] \text{ ----- (7)}$$

where

$$f_1(r) = r - \frac{\lambda^2}{10} r^3 + \frac{n\lambda^2}{90} r^4 - \left(\frac{n^2 \lambda^2}{840} - \frac{\lambda^4}{280} \right) r^5 + \left(\frac{n^3 \lambda^2}{8400} - \frac{\lambda^2}{2800} + \frac{n\lambda^4}{3600} \right) r^6 + \text{-----}$$

$$f_2(r) = \left(-\frac{2}{3} n\lambda^2 + \frac{1}{2} n^3 \right) r - \left(\frac{n\lambda^4}{60} + \frac{n^3 \lambda^2}{20} - \frac{n\lambda^2}{20} \right) r^3 + \left(\frac{n^2 \lambda^4}{135} - \frac{n^3 \lambda^2}{180} \right) r^4$$

$$+ \left(\frac{n^3 \lambda^4}{1008} + \frac{n^4 \lambda^2}{1680} + \frac{n\lambda^6}{1680} + \frac{n\lambda^4}{560} \right) r^5 + \text{-----}$$

$$f_3(r) = \left(-\frac{5}{36} n\lambda^2 + \frac{13}{6} n^3 - \frac{7}{6} n\lambda^2 \right) r + \left(\frac{23}{15} n\lambda^4 + \frac{8}{3} n^3 \lambda^2 + \frac{28}{15} n^2 \lambda^4 \right) r^3$$

$$- \left(\frac{51}{15} n^2 \lambda^4 + \frac{38}{15} n^4 \lambda^2 \right) r^4$$

$$- \left(\frac{704}{420} n^2 + \frac{307}{105} n^4 \lambda^2 - \frac{809}{420} n^2 \lambda^4 - \frac{293}{70} n\lambda^6 - \frac{1369}{420} n^3 \lambda^4 \right) r^5 + \text{-----}$$

and A_1, B_1 are arbitrary constants.

Thus using (7) in (3) and (4) we get

$$u_{\varphi} = \left[A_1 f_1(r) + B_1 \{ \log r f_2(r) + f_3(r) \} \right] \sin \theta e^{i\omega t} \text{ ----- (8)}$$

and

$$\hat{r} \varphi = c_{44} e^{nr} \left[A_1 \varphi_1(r) + B_1 \{ \log r \varphi_2(r) + \varphi_3(r) \} \right] \sin \theta e^{i\omega t} \text{ ----- (9)}$$

where

$$\varphi_1(r) = f_1'(r) - \frac{1}{r} f_1(r)$$

$$\varphi_2(r) = f_2'(r) - \frac{1}{r} f_2(r)$$

$$\varphi_3(r) = f_3'(r) - \frac{1}{r} (f_3(r) - f_2(r))$$

3. Boundary Conditions

Case-I:

Let the end caps be fixed ,so that we have

$$u_{\varphi} = 0 \text{ on } r = a \text{ and } r = b \text{ ----- (10)}$$

Then from (8) and (9), we get

$$\left. \begin{aligned} A_1 f_1(a) + B_1 \{ \log a f_2(a) + f_3(a) \} &= 0 \\ A_1 f_1(b) + B_1 \{ \log b f_2(b) + f_3(b) \} &= 0 \end{aligned} \right\} \text{----- (11)}$$

Eliminating A_1 and B_1 from (11) we obtain the frequency equation as

$$\frac{f_2(a) \log a + f_3(a)}{f_2(b) \log b + f_3(b)} = \frac{f_1(a)}{f_1(b)} \text{ (12)}$$

Case-II:

Let the end caps free, so that we have at the ends

$$\hat{r} \varphi = 0 \text{ on } r = a \text{ and } r = b \text{ -----(13)}$$

and the frequency equation for this case be

$$\frac{\phi_2(a) \log a + \phi_3(a)}{\phi_2(b) \log b + \phi_3(b)} = \frac{\phi_1(a)}{\phi_1(b)} \text{(14)}$$

4. Particular Case

When $n = 0$

In this case $c_{44} = c_{44}$, $\rho = \rho_0$

then the stress components transferred to

$$\left. \begin{aligned} \hat{r}r = \hat{\theta}\theta = \hat{\varphi}\varphi = \hat{r}\theta = \hat{\theta}\varphi = 0 \\ \text{and} \\ \hat{r}\varphi = c_{44} \left(f'(r) - \frac{f(r)}{r} \right) \sin\theta e^{i\omega t} \end{aligned} \right\} \text{-----(15)}$$

Substituting (15) in the equation of equilibrium Love [8], we get,

$$r^2 \frac{d^2 f}{dr^2} + 2r \frac{df}{dr} - 2f = 0 \text{ -----(16)}$$

The solution of the ordinary differential equation (16) is

$$f(r) = A_2 r^1 + B_2 r^{-2}$$

where, A_2 and B_2 are arbitrary constants.

Thus from (3) and (15) we get,

$$u_\varphi = (A_2 r^1 + B_2 r^{-2}) \sin\theta e^{i\omega t} \text{ -----(17)}$$

$$\hat{r}\varphi = -c_{44} B_2 r^{-3} \sin\theta e^{i\omega t} \text{ -----(18)}$$

Boundary conditions:

Let the end caps be fixed, we take

$$u_{\varphi} = s_1 \sin \theta e^{i \omega t} \quad \text{at } r=a \quad \text{and} \quad u_{\varphi} = s_2 \sin \theta e^{i \omega t} \quad \text{at } r=b \} \dots\dots\dots(19)$$

Using (19) solving the equation (17) and (18) we get,

$$A_2 = \frac{s_2 b^2 - s_1 a^2}{b^3 - a^3} \quad , \quad B_2 = \frac{a^2 b^2 (s_1 b - s_2 a)}{b^3 - a^3} \dots\dots\dots(20)$$

Substitution the values of A_2 and B_2 from (20) in (17) and (18) we get

$$u_{\varphi} = \left(\frac{s_2 b^2 - s_1 a^2}{b^3 - a^3} r^1 + \frac{a^2 b^2 (s_1 b - s_2 a)}{b^3 - a^3} r^{-2} \right) \sin \theta e^{i \omega t}$$

and

$$r\varphi = -c_{44} \frac{a^2 b^2 (s_1 b - s_2 a)}{b^3 - a^3} r^{-3} \sin \theta e^{i \omega t}$$

5. Numerical results and discussions:

We numerically evaluated the displacements and stresses for different values of $r(1 \leq r \leq 2)$ and $s_1 = 3s_2$, $a = 1$, $b = 2$ on the surface of the cone.

The values of

$$R \left[= \frac{u_{\varphi}}{s_2 \sin \theta e^{i \omega t}} \right]$$

and

$$P \left[= -\frac{2}{5} \frac{r\varphi}{s_2 c_{44} \sin \theta e^{i \omega t}} \right]$$

for different values of r are shown in the Table-1

TABLE-1

Displacements and stresses for different values of r on the surface of the cone :

r	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
R	1	.840	.719	.625	.553	.495	.448	.411	.380	.354	.333
P	1.14	.858	.661	.520	.416	.338	.279	.233	.196	.167	.143

The corresponding values of displacement R and the stresses P are shown graphically in Fig. - 1.

6. Conclusion:

We see that as r increases the corresponding values of the displacement (R) as well as the stresses (P) both gradually decreases. Moreover stresses P decreases more rapidly than the displacements R .

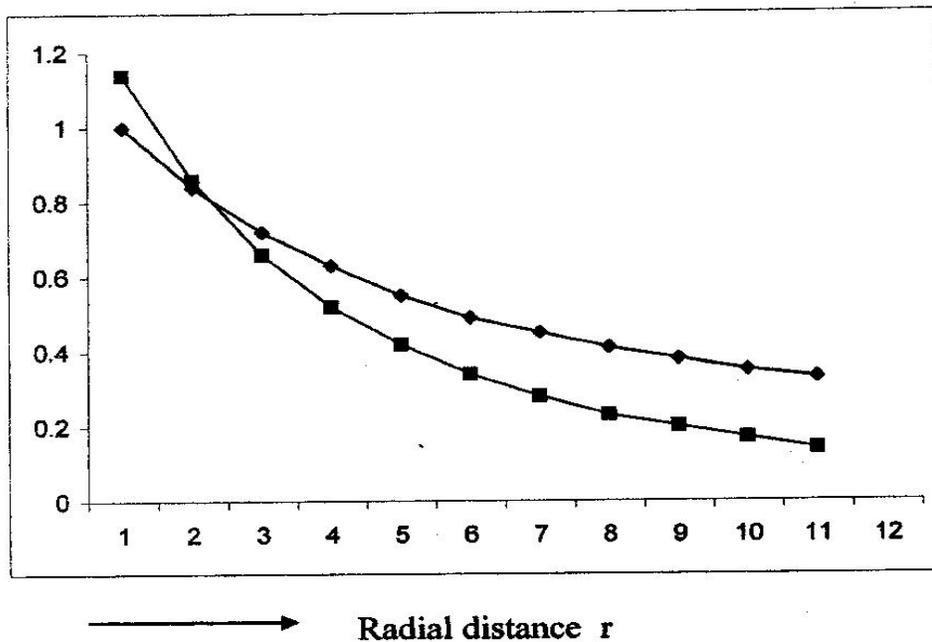


FIG. 1: Curves showing the variation of stress and displacement with radial distance.

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