

TORSIONAL VIBRATION OF A LARGE THICK COMPOSITE PLATE UNDER SHEARING FORCES APPLIED ON THE FREE PLANE SURFACE

BY

P.C. BHATTACHARYYA

INSTITUTE OF MECHANICS OF CONTINUA AND MATHEMATICAL SCIENCES

P – 9/1, LIC Township, Madhyamgram, Kolkata – 700 129, W.B, India

E mail : prabirbhattacharyya@yahoo.com

Abstract.

In this paper dynamic stresses and displacements are calculated in a large thick composite plate due to torsional vibration under shearing forces applied on the free plane boundary while the other boundary being in contact with a rigid foundation. The applied shearing force on the free plane boundary is expressed in terms of Fourier – Bessel integrals; in particular, case of Gaussian load has been treated in details to find the distribution of stresses and displacements.

Keyword and phrases : torsional vibration, composite plate, shearing force, plane surface.

সংক্ষিপ্তসার

এই গবেষণা পত্রে সমতলের সীমানায় প্রযুক্ত কৃত্তন বলের অধীন ব্যাবর্তন কম্পনের জন্য বৃহৎ পুরু প্লেটে গভীর পীড়ন এবং সরণগুলিকে গণনা করা হয়েছে যখন অন্য সীমানাটি একটি সুদৃঢ় ভিতের সঙ্গে সংযুক্ত। মুক্ত সমতলের সীমানায় প্রযুক্ত কৃত্তন বলকে ফুরিয়ার – ব্যাসেল (Fourier – Bessel) সমাকলনের দ্বারা প্রকাশ করা হয়েছে। বিশেষতঃ পীড়ন ও সরণের বন্টনকে নির্ণয়ের জন্য গাউসীয় ভারের ক্ষেত্রে বিশদে ব্যবহার করা হয়েছে।

1. Introduction.

The problem of a large thick plate acted on by shearing couples on two opposite faces was investigated by Yu , Yi - Yuan (5). Following this work Sengupta (6) investigated the problem of torsion of a large thick composite plate under shearing forces on the free plane boundaries. This work was further extended to the case of composite plate consisting of different layers of materials by Sengupta (7). This is followed by a similar problem of two layered semi-infinite body of transversely isotopic material by the same author. Propagation of waves in a composite elastic layer was investigated by Bhattacharyya, P.C and Sengupta, P.R. (8). Effect of rotation on Rayleigh

surface waves under the linear theory of non-local elasticity was investigated by Acharya, D.P. and Mondal, A. (4). Sharma, M.D. (9) solved the problem of effect of initial stress on reflection at the free surface of anisotropic elastic medium. Gupta, S., Chattopadhyay, A. and Kumari, P.(10) find out the solution propagation of shear waves in anisotropic medium. Torsional waves propagation in an initially stressed dissipative cylinder was investigated by Selim, M.M.(11).

However, similar problems, as indicated above, in case of torsional vibration have not been investigated by a similar approach. The object of the present paper is to investigate torsional stresses due to torsional vibration of large thick composite plate under shearing forces applied on the free plane boundary and the other boundary being in contact with a rigid foundation.

Firstly, a general case of torsional shearing stress is applied on the free plane boundary, while the remaining plane boundary being rigidly fixed on a foundation. This is followed by the particular case of Gaussian load acting on the free plane boundary.

2. General Theory and Boundary Condition.

The stress equation of motion in cylindrical co-ordinates (r, θ, Z) are,

$$\frac{\partial \widehat{rr}}{\partial r} + \frac{1}{r} \cdot \frac{\partial \widehat{r\theta}}{\partial \theta} + \frac{\partial \widehat{rz}}{\partial z} + \frac{\widehat{rr} - \widehat{\theta\theta}}{r} + \rho F_r = \rho \frac{\partial^2 u_r}{\partial t^2} \quad (1.1)$$

$$\frac{\partial \widehat{r\theta}}{\partial r} + \frac{1}{r} \cdot \frac{\partial \widehat{\theta\theta}}{\partial \theta} + \frac{\partial \widehat{\theta z}}{\partial z} + \frac{2}{r} \cdot \widehat{r\theta} + \rho F_\theta = \rho \frac{\partial^2 u_\theta}{\partial t^2} \quad (1.2)$$

$$\frac{\partial \widehat{rz}}{\partial r} + \frac{1}{r} \cdot \frac{\partial \widehat{\theta z}}{\partial \theta} + \frac{\partial \widehat{zz}}{\partial z} + \frac{\widehat{rz}}{r} + \rho F_z = \rho \frac{\partial^2 u_z}{\partial t^2} \quad (1.3)$$

We may assume from the nature of the problem

$$u_r = 0, \quad u_\theta = u_\theta(r, z, t) = V(r, z)e^{i\omega t}, \quad u_z = 0 \quad (2)$$

and equation (1.1) and (1.3) are identically satisfied.
The non-zero components of stress are given by

$$\left. \begin{aligned} \widehat{\theta z} &= \mu \frac{\partial u_{\theta}(r, z, t)}{\partial z} \\ \widehat{r\theta} &= \mu \left[\frac{\partial u_{\theta}(r, z, t)}{\partial r} - \frac{u_{\theta}(r, z, t)}{r} \right] \end{aligned} \right\} \quad (3)$$

and the equation (1.2) takes the form.

$$\mu \left\{ \frac{\partial}{\partial r} \left(\frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r} \right) + \frac{\partial}{\partial z} \left(\frac{\partial u_{\theta}}{\partial z} \right) + \frac{2}{r} \left(\frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r} \right) \right\} = \rho \frac{\partial^2 u_{\theta}}{\partial t^2} \quad (4)$$

Let us put

$$V = RZ$$

Where R is a function of r alone and Z is a function of z alone.

Clearly, by using the equation (2) we have

$$\frac{1}{R} \left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} \right) + \left(\frac{\rho \omega^2}{\mu} - \frac{1}{r^2} \right) = -\frac{1}{Z} \cdot \frac{\partial^2 Z}{\partial z^2} = -m^2 \text{ (say)} \quad (5)$$

and this is satisfied if we assume,

$$\frac{\partial^2 Z}{\partial z^2} - m^2 Z = 0 \quad (6.1)$$

and

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + \left(\frac{\rho \omega^2}{\mu} + m^2 - \frac{1}{r^2} \right) R = 0 \quad (6.2)$$

in which m is an arbitrary constant of separation.

The solution of the differential equation (6.1) and (6.2) are

$$Z = \sum_{m=0}^{\infty} (A_1 e^{-mz} + B_1 e^{mz})$$

and

$$R = J_1(\alpha_m r), Y_1(\alpha_m r)$$

where

$$\alpha_m^2 = \frac{\rho \omega^2}{\mu} + m^2$$

and J_1 , Y_1 are ordinary Bessel's function of the first kind and second kind respectively and of order unity.

Clearly the solution of the differential equation (4) appropriate to this problem is

$$u_\theta = \sum_{m=0}^{\infty} (A_1 e^{-mz} + B_1 e^{mz}) J_1(\alpha_m r) e^{i\omega t} \quad (7)$$

It is supposed that the material above the plane satisfies the equation (5), (6) and (7). For the material beneath the middle plane the corresponding equations are

$$\left. \begin{aligned} \widehat{\theta Z}' &= \mu' \frac{\partial V'}{\partial z} e^{i\omega t} \\ r\theta' &= \mu' \left\{ \frac{\partial V'}{\partial r} - \frac{V'}{r} \right\} e^{i\omega t} \end{aligned} \right\} \quad (8)$$

and by the similar procedure the differential equation of motion takes the form

$$\frac{1}{R'} \left(\frac{\partial^2 R'}{\partial r^2} + \frac{1}{r} \frac{\partial R'}{\partial r} \right) + \left(\frac{\rho' \omega^2}{\mu'} - \frac{1}{r^2} \right) = -\frac{1}{Z'} \frac{\partial^2 Z'}{\partial z^2} = -m^2 \quad (9)$$

leading to the solution

$$u'_\theta = \sum_{m=0}^{\infty} (A_2 e^{-mz} + B_2 e^{mz}) J_1(\alpha_m r) e^{i\omega t}$$

and μ' is the shear modulus and ρ' , the density and V' , the displacement for the material.

Let the boundary planes of the large thick plate be $z = \pm c$ where c may be any finite value. At the boundary of the body torsional shearing forces are prescribed with their axes parallel to the Z - axis.

When $z = -c$, $V = 0$ (for a rigid foundation)

and

$$\widehat{\theta Z} = F(r) e^{i\omega t} \quad (10)$$

when

$$z = +c \quad (\text{on the free plane boundary})$$

Therefore, we have for the upper half $0 \leq z \leq c$

$$\begin{aligned}
 V &= \sum_{m=1}^{\infty} (A_1 e^{-mz} + B_1 e^{mz}) J_1(\alpha_m r) \\
 \widehat{\theta Z} &= \sum_{m=1}^{\infty} m \mu (-A_1 e^{-mz} + B_1 e^{mz}) J_1(\alpha_m r) e^{i\omega t} \\
 \widehat{r\theta} &= \sum_{m=1}^{\infty} \mu \alpha_m (A_1 e^{-mz} + B_1 e^{mz}) J_1(\alpha_m r) e^{i\omega t} \\
 &\quad - \sum_{m=1}^{\infty} \frac{\mu}{r} (A_1 e^{-mz} + B_1 e^{mz}) J_1(\alpha_m r) e^{i\omega t}
 \end{aligned} \tag{11}$$

Similarly for the lower half $-c \leq z \leq 0$

$$\begin{aligned}
 V' &= \sum_{m=1}^{\infty} (A_2 e^{-mz} + B_2 e^{mz}) J_1(\alpha_m r) \\
 \widehat{\theta z}' &= \sum_{m=1}^{\infty} m \mu' (-A_2 e^{-mz} + B_2 e^{mz}) J_1(\alpha_m r) e^{i\omega t} \\
 \widehat{r\theta}' &= \sum_{m=1}^{\infty} \mu' \alpha_m (A_2 e^{-mz} + B_2 e^{mz}) J_1(\alpha_m r) e^{i\omega t} \\
 &\quad - \sum_{m=1}^{\infty} \frac{\mu'}{r} (A_2 e^{-mz} + B_2 e^{mz}) J_1(\alpha_m r) e^{i\omega t}
 \end{aligned} \tag{12}$$

Using the boundary (10) we obtain.

$$V = 0 \text{ when } z = -c \text{ and } \widehat{\theta Z} = F(r) e^{i\omega t} \text{ when } z = +c$$

And we have

$$\begin{aligned}
 A_1 &= -\frac{e^{-mc}}{m\mu \cosh 2mc J_2^2(\alpha_m)} \int_0^1 r J_1(\alpha_m r) F(r) dr \\
 B_1 &= -\frac{e^{mc}}{m\mu \cosh mc J_2^2(\alpha_m)} \int_0^1 r J_1(\alpha_m r) F(r) dr
 \end{aligned} \tag{13}$$

Again, the following boundary conditions

$$[V]_{z=0} = [V']_{z=0}, [\widehat{\theta z}]_{z=0} = [\widehat{\theta z}']_{z=0} \quad (14)$$

yields the following equations to determine the unknown constants.

We have,

$$A_2 = A_1 p_1 \text{ and } B_2 = A_1 p_2 \quad (15)$$

Where,

$$p_1 = \frac{1}{2\mu'} [(\mu' + \mu) - (\mu' - \mu)e^{2mc}]$$

$$p_2 = \frac{1}{2\mu'} [(\mu' - \mu) - (\mu' + \mu)e^{2mc}]$$

Clearly for the region $0 \leq z \leq c$ and A_1 is indicated in equation (13) we have,

$$\left. \begin{aligned} V &= \sum_{m=1}^{\infty} \frac{2 \sinh m(z+c) J_1(\alpha_m r)}{m \mu \cosh 2mc J_2^2(\alpha_m)} \int_0^1 r J_1(\alpha_m r) F(r) dr \\ \widehat{\theta z} &= \sum_{m=1}^{\infty} \frac{2 \cosh m(z+c) J_1(\alpha_m r)^{i\omega t}}{\cosh 2mc J_2^2(\alpha_m)} \int_0^1 r J_1(\alpha_m r) F(r) dr \\ r\widehat{\theta} &= \sum_{m=1}^{\infty} \frac{2 \sinh m(z+c) \alpha_m J_1'(\alpha_m r)^{e^{i\omega t}}}{m \cosh 2mc J_2^2(\alpha_m)} \int_0^1 r J_1(\alpha_m r) F(r) dr \\ &+ \sum_{m=1}^{\infty} \frac{2 \sinh m(z+c) J_1'(\alpha_m r)^{e^{i\omega t}}}{m \cosh 2mc J_2^2(\alpha_m)} \int_0^1 r J_1(\alpha_m r) F(r) dr \\ &+ \sum_{m=1}^{\infty} \frac{2 \sinh m(z+c) J_1'(\alpha_m r)^{e^{i\omega t}}}{m \cosh 2mc J_2^2(\alpha_m)} \int_0^1 r \alpha_m r J_1'(\alpha_m r) F(r) dr \\ &+ \sum_{m=1}^{\infty} \frac{2 \sinh m(z+c) J_1'(\alpha_m r)^{e^{i\omega t}}}{m \cosh 2mc J_2^2(\alpha_m)} \int_0^1 r J_1(\alpha_m r) F'(r) dr \\ &- \sum_{m=1}^{\infty} \frac{2 \sinh m(z+c) J_1'(\alpha_m r)}{rm \cosh 2mc J_2^2(\alpha_m)} \int_0^1 r J_1(\alpha_m r) F(r) dr \end{aligned} \right\} \quad (16)$$

Similarly for the region $-c \leq z \leq c$, we have,

$$\left. \begin{aligned}
 V' &= - \sum_{n=1}^{\infty} \frac{(p_1 e^{-nz} + p_2 e^{nz}) e^{-nc} J_1'(\alpha_n r)}{m \mu \cosh 2mc J_2^2(\alpha_n)} \int_0^1 r J_1(\alpha_n r) F(r) dr \\
 \widehat{\theta Z}' &= \sum_{n=1}^{\infty} \frac{\mu' (p_1 e^{-nz} - p_2 e^{nz}) e^{-nc} J_1(\alpha_n r) e^{i\omega t}}{\mu \cosh 2mc J_2^2(\alpha_n)} \int_0^1 r J_1(\alpha_n r) F(r) dr \\
 \widehat{r\theta}' &= - \sum_{n=1}^{\infty} \frac{\mu' (p_1 e^{-nz} + p_2 e^{nz}) e^{-nc} \alpha_n J_1'(\alpha_n r) e^{i\omega t}}{\mu \cosh 2mc J_2^2(\alpha_n)} \int_0^1 r J_1(\alpha_n r) F(r) dr \\
 &\quad - \sum_{n=1}^{\infty} \frac{\mu' (p_1 e^{-nz} + p_2 e^{nz}) e^{-nc} J_1(\alpha_n r) e^{i\omega t}}{m \mu \cosh 2mc J_2^2(\alpha_n)} \int_0^1 J_1(\alpha_n r) F(r) dr \\
 &\quad - \sum_{n=1}^{\infty} \frac{\mu' (p_1 e^{-nz} + p_2 e^{nz}) e^{-nc} J_1(\alpha_n r) e^{i\omega t}}{m \mu \cosh 2mc J_2^2(\alpha_n)} \int_0^1 r \alpha_n J_1'(\alpha_n r) F(r) dr \\
 &\quad - \sum_{n=1}^{\infty} \frac{\mu' (p_1 e^{-nz} + p_2 e^{nz}) e^{-nc} J_1(\alpha_n r) e^{i\omega t}}{m \mu \cosh 2mc J_2^2(\alpha_n)} \int_0^1 r J_1(\alpha_n r) F'(r) dr \\
 &\quad + \sum_{n=1}^{\infty} \frac{\mu' (p_1 e^{-nz} + p_2 e^{nz}) e^{-nc} J_1(\alpha_n r) e^{i\omega t}}{r m \mu \cosh 2mc J_2^2(\alpha_n)} \int_0^1 r J_1(\alpha_n r) F'(r) dr
 \end{aligned} \right\} \quad (17)$$

3. Distribution of shearing stress on the free plane boundary on a large thick composite plate acted on by Gaussian load.

The distribution of shearing stress on the free plane boundary due to Gaussian load may be given by

$$[\widehat{\theta z}]_{z=c} = F(r) e^{i\omega t} = p_0 e^{-q^2 r^2 + i\omega t} \quad (18)$$

where p_0 and q_0 are constants.

The problem explains the facts that applied shearing stress decreases with the increase of the distance from any arbitrary axis or from a centre. The shearing stress is maximum at the centre or at the mentioned point but stress is negligible at a large distance from the centre or at the mentioned point. According to this type of distribution of applied shearing stress unknown constants and displacements and stresses in different layers are given below with the help of the equation (16) and (17).

For the region $0 \leq z \leq c$, we get

$$\begin{aligned}
 V &= \sum_{m=1}^{\infty} \frac{2 \sinh m(z+c) \cdot J_1(\alpha_m r)}{m \mu \cosh 2mc J_2^2(\alpha_m)} \int_0^1 r J_1(\alpha_m r) p_o e^{-q^2 r^2} dr \\
 \widehat{\theta Z} &= \sum_{m=1}^{\infty} \frac{2 \cosh m(z+c) \cdot J_1(\alpha_m r) e^{i\omega t}}{\cosh 2mc J_2^2(\alpha_m)} \int_0^1 r J_1(\alpha_m r) p_o e^{-q^2 r^2} dr \\
 \widehat{r\theta} &= \sum_{m=1}^{\infty} \frac{2 \sinh m(z+c) \cdot \alpha_m J_1(\alpha_m r) e^{i\omega t}}{\cosh 2mc J_2^2(\alpha_m)} \int_0^1 r J_1(\alpha_m r) p_o e^{-q^2 r^2} dr \\
 &+ \sum_{m=1}^{\infty} \frac{2 \sinh m(z+c) J_1(\alpha_m r) e^{i\omega t}}{m \cosh 2mc J_2^2(\alpha_m)} \int_0^1 J_1(\alpha_m r) p_o e^{-q^2 r^2} dr \\
 &+ \sum_{m=1}^{\infty} \frac{2 \sinh m(z+c) \cdot J_1(\alpha_m r) e^{i\omega t}}{m \cosh 2mc J_2^2(\alpha_m)} \int_0^1 r \alpha_m J_1'(\alpha_m r) p_o e^{-q^2 r^2} dr \\
 &- \sum_{m=1}^{\infty} \frac{4 \sinh m(z+c) \cdot J_1(\alpha_m r) e^{i\omega t}}{m \cosh 2mc J_2^2(\alpha_m)} \int_0^1 r^2 J_1(\alpha_m r) p_o e^{-q^2 r^2} dr \\
 &- \sum_{m=1}^{\infty} \frac{2 \sinh m(z+c) \cdot J_1(\alpha_m r) e^{i\omega t}}{rm \cosh 2mc J_2^2(\alpha_m)} \int_0^1 r J_1(\alpha_m r) p_o e^{-q^2 r^2} dr
 \end{aligned} \tag{19}$$

and for the region $-c \leq z \leq 0$, we have

$$\left. \begin{aligned}
 V' &= -\sum_{n=1}^{\infty} \frac{(p_1 e^{-nz} + p_2 e^{nz}) e^{-nc} J_1(\alpha_n r)}{m \mu \cosh 2mc J_2^2(\alpha_n)} \int_0^1 r J_1(\alpha_n r) p_0 e^{-q^2 r^2} dr \\
 \widehat{\theta Z}' &= \sum_{n=1}^{\infty} \frac{\mu (p_1 e^{-nz} - p_2 e^{nz}) e^{-nc} J_1(\alpha_n r) e^{i\alpha r}}{\mu \cosh 2mc J_2^2(\alpha_n)} \int_0^1 r J_1(\alpha_n r) p_0 e^{-q^2 r^2} dr \\
 \widehat{r\theta}' &= -\sum_{n=1}^{\infty} \frac{\mu (p_1 e^{-nz} + p_2 e^{nz}) e^{-nc} \alpha_n J_1'(\alpha_n r) e^{i\alpha r}}{m \mu \cosh 2mc J_2^2(\alpha_n)} \int_0^1 r J_1(\alpha_n r) p_0 e^{-q^2 r^2} dr \\
 &\quad - \sum_{n=1}^{\infty} \frac{\mu (p_1 e^{-nz} + p_2 e^{nz}) e^{-nc} J_1(\alpha_n r) e^{i\alpha r}}{m \mu \cosh 2mc J_2^2(\alpha_n)} \int_0^1 J_1(\alpha_n r) p_0 e^{-q^2 r^2} dr \\
 &\quad - \sum_{n=1}^{\infty} \frac{\mu (p_1 e^{-nz} + p_2 e^{nz}) e^{-nc} J_1(\alpha_n r) e^{i\alpha r}}{m \mu \cosh 2mc J_2^2(\alpha_n)} \int_0^1 r \alpha_n J_1'(\alpha_n r) p_0 e^{-q^2 r^2} dr \\
 &\quad + \sum_{n=1}^{\infty} \frac{2\mu (p_1 e^{-nz} + p_2 e^{nz}) e^{-nc} J_1(\alpha_n r) e^{i\alpha r}}{m \mu \cosh 2mc J_2^2(\alpha_n)} \int_0^1 r^2 J_1(\alpha_n r) p_0 q^2 e^{-q^2 r^2} dr \\
 &\quad + \sum_{n=1}^{\infty} \frac{\mu (p_1 e^{-nz} + p_2 e^{nz}) e^{-nc} J_1(\alpha_n r) e^{i\alpha r}}{m \mu \cosh 2mc J_2^2(\alpha_n)} \int_0^1 r J_1(\alpha_n r) p_0 e^{-q^2 r^2} dr
 \end{aligned} \right\} \quad (20)$$

4. Discussion

The integrals corresponding to the values of $v, \widehat{r\theta}, \widehat{\theta Z}$ for the upper half when $z = -c$ and $v', \widehat{r\theta}'$ and $\widehat{\theta Z}'$ for the lower half when $z = +c$ as represented in the equations (19) and (20) are all convergent. These may be evaluated numerically by numerical computational techniques. Firstly the definite integral is calculated by considering as many terms as needed for desired accuracy of the problem. For numerical computation, the value of the infinite series are to be obtained for a particular value of z , implying the variations $v, \widehat{r\theta}, \widehat{\theta Z}$ and also $v', \widehat{r\theta}', \widehat{\theta Z}'$ on a particular plane parallel to the plane boundary.

REFERENCE

- 1) Chatterjee, P.P (1957), Proc. Th. and App. Mech P. 129.
- 2) Love, A.E.H. (1994), A Treatise on the Mathematical theory of Elasticity.
- 3) Timoshenko and Goodier (1951), theory of Elasticity P. 307.
- 4) Acharya, D.P. and Mondal, A, Effect of rotation on Rayleigh surface waves under the linear theory of non-local elasticity, Ind.J.Theor.phys 52(1) (2004)
- 5) Yu, Yi-Yuan (1954), Quar. Journ. of Mech. And App. Math., Vol.7.p.287.
- 6) Sengupta, P.R. (1964). Bull. Cal. Math. Soc. Vol. 56, No.-1, March,1967. pp. 33-40.
- 7) Sengupta, P.R. (1965). Bull. Cal. Math. Soc. Vol. 57. No.2 & 3, June and September, (1965). pp. 69-70
- 8) Bhattacharyya, P.C. and Sengupta, P.R., propagation of waves in composite elastic layer, Jour.Sci. res. Vol. 3. No. 3. (1981) pp 211-214
- 9) Sharma M.D, Effect of initial stress on reflection at the free surface of anisotropic elastic medium, J.Earth Syst. Sci. 116 (2007), pp 537-557
- 10) Gupta, S., Chattopadhyaya, A and Kumari, P., propagation of share waves in anisotropic medium., Appl. Math. Sci., 1 (2007) pp. 2699 – 2706
- 11) Selim. M.M., Torsional waves propagation in an initially stressed dissipative cylinder., Appl. Math. Sci. 1 (2007). pp. 1419-1427