ON SOME FIXED POINT CONVERGENCE THEOREMS FOR MANN ITERATIVE PROCESS

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Abstract

In this paper, we find a more generalized contractive mapping that is applied to prove some convergence theorems of Mann Iteration procedure. Our proof is comparatively easy. Actually, here we generalized some theorems of Rhoades[3], Qihou[1], Ganguly and Bandyopadhy[8], Kannan[12] to develop the concept on convergence of Mann Iteration procedure.

Keyword and phrases : *contractive mapping, convergence theorems, Mann Iteration procedure.*

সংক্ষিপ্তসার

আরও অধিক সাধারণীকৃত সংকোচনশীল চিত্রণ যাহা ম্যান (*Mann*) পুনরাবৃত্তিমূলক পদ্ধতির কতিপয় অভিসারি উপপাদ্যকে প্রমানের জন্য ব্যবহার করা হয়েছে তা আমরা উক্ত প্রবন্ধে নির্ণয় করেছি। আমাদের প্রমাণ গুলি তুলনামূলক ভাবে সহজ। প্রকৃত পক্ষে, এখানে আমরা ম্যান (*Mann*) পুনরাবৃত্তিমূলক পদ্ধতির অভিসারত্বের ধারণাকে উন্নত করার জন্য রোডিস্ (*Rhodes*), ক্বিহা (*Qihou*), গাঙ্গুলি এবং বন্দ্যোপাধ্যায়, কামান (*Kannan*) এর কতিপয় উপপাদ্যকে সাধারণীকরণ করেছি।

1.Introduction

Let S be a non-empty bounded closed convex Subset of a Banach space B. Let F be a mapping from S into S i.e., $F:S \to S$

Now, the contractive definition is

$$\| F(x)-F(y)\| \leq k \max\{\|x-y\|,[\|x-F(x)\|+\|y-F(y)\|],[\|x-F(y)\|+\|y-F(x)\|]\}$$

$$\text{for all } x,y \in S, \text{ where } k,c \geq 0, 0 \leq k < 1. \qquad (1)$$

The Mann iterative process is defined by

$$x_{n+1} = (1-a_n)x_n + a_n F(x_n), \quad x_0 \in S, \quad n \in N, \tag{2}$$

where, N is set of Natural numbers and $\{a_n\}$ is a sequence of non-negative numbers satisfying (i) $a_0 = 1$, (ii) $0 \le a_n \le 1$ & (iii) $\sum a_n = \infty$ $i.e.$, $\{a_n\}$ is divergent.

Qihou, L. [1] used the contractive definition (1) but he considered Ishikawa iterative process, which is defined as follows :

$$x_{n+1} = (1-a_n)x_n + a_n F[(1-b_n)x_n + b_n F(x_n)], \quad x_0 \in S, \quad n \in N, \tag{3}$$

where, $\{a_n\}$ and $\{b_n\}$ are sequences of non-negative numbers such that (i) $0 < a \le a_n \le 1$, $0 \le b_n \le 1$ (ii) $\limsup_{n \to \infty}(b_n) < 1$.

Kalishanker Tiwary[1] and S.C. Debnath[2] [2] used the Ishikawa iterative process (3) but they considered the following contractive definition:

$$\|F(x)-F(y)\| \le \max\{\|x-y\|, \tfrac{1}{2}[\|x-F(x)\|+\|y-F(y)\|], \tfrac{1}{2}[\|x-F(y)\|+\|y-F(x)\|]\}$$

for all $x, y \in B$. \tag{4}

Ganguly and Bandypadhyay [8] used the following contractive definition :

$$\|F(x)-F(y)\| \le a_1\|x-y\| + a_2\|x-F(x)\| + a_3\|y-F(y)\| + a_4\|x-F(y)\|$$

$$+ a_5\|y-F(x)\|, \text{ for all } x, y \in B, \text{ where, } a_i \ge 0 \text{ & } \sum_{i=1}^{5} a_i \le 1. \tag{5}$$

B.E. Rhoades [3] used (3) and (4).

Tiwary and Lahiri [19] used the contractive definition (5) but they considered the following iterative process:

$$x_{n+1} = \lambda x_n + (1-\lambda)F(x_{n-1}), \quad x_0 \in S, \quad n \in N, \quad 0 \le \lambda \le 1.$$

Kannan[12] considered the above iterative process for semi-non-expansive maps while Chakraborty and Lahari[7] used the contractive definition (5).

It is shown that the contractive definition (1) is more general than contractive definitions (4) and (5). Ishikawa iterative process reduces to the Mann

iterative Process by setting $b_n = 0$ for all $n \in N$. In this paper, we consider Mann iterative process for the mapping satisfying (2).

Suppose, F be a mapping S into S. Now if $F(x) = x$ for all $x \in S$, then x is called the fixed point of F.

A Banach space B is said to be uniformly convex if,

$$\| x_n \| \leq 1, \| y_n \| \leq 1 \& \| x_n + y_n \| \to 2 \text{ as } n \to \infty \text{ imply } \| x_n - y_n \| \to 0, \quad \forall x_n, y_n \in B.$$

We call a set S in a topological space B relatively compact if its closure is complete.

2. The Main Results

Theorem A: Let S be a non-empty bounded closed convex subset of a Banach space B and F be a map from S into S satisfying the contractive definition (1). Let $\{x_n\}$ be a sequence in S defined by (2). Now, if $\{x_n\}$ converges, then it converges to a fixed point of F.

Proof:

Suppose that,

$\lim_{n \to \infty} x_n = r$, where r is any finite number.

Now, we are to show that r is the fixed point of F i.e., $F(r) = r$.

By (2) we have,

$$x_{n+1} = (1 - a_n)x_n + a_n F(x_n)$$

$$i.e., x_{n+1} - x_n = a_n [F(x_n) - x_n]$$

Since, $\lim_{n \to \infty} x_n = r$ and therefore,

$$\lim_{n \to \infty} a_n [F(x_n) - x_n] = 0, \quad i.e., \lim_{n \to \infty} [F(x_n) - x_n] = 0.$$

Which gives that $\| x_n - F(x_n) \| \to 0 \quad as \quad n \to \infty$.

By (1) we have,

$$\| F(x_n) - F(r) \| \leq k \max \{ \| x_n - r \|, [\| x_n - F(x_n) \| + \| r - F(r) \|], [\| x_n - F(r) \|$$

$+ \| r - F(x_n) \|] \}$

But,

$\| r - F(r) \| \leqq \| r - x_n \| + \| x_n - F(x_n) \| + \| F(x_n) - F(r) \|$

$\| x_n - F(r) \| \leqq \| x_n - F(x_n) \| + \| F(x_n) - F(r) \|$

$\| r - F(x_n) \| \leqq \| r - x_n \| + \| x_n - F(x_n) \| .$

Therefore,

$\| F(x_n) - F(r) \| \leq k \max \{ \| x_n - r \|, [2 \| x_n - F(x_n) \| + \| r - x_n \| + \| F(x_n) - F(r) \|],$

$[2 \| x_n - F(x_n) \| + \| r - x_n \| + \| F(x_n) - F(r) \|] \}$

$= k \max \{ c \| x_n - r \|, [2 \| x_n - F(x_n) \| + \| r - x_n \| + \| F(x_n) - F(r) \|] \}$

Now, since $x_n \to r$ and $\| x_n - F(x_n) \| \to 0 \quad as \quad n \to \infty$.

$\therefore \lim_{n \to \infty} \| F(x_n) - F(r) \| = 0$

Also we have,

$\| r - F(r) \| \leqq \| r - x_n \| + \| x_n - F(x_n) \| + \| F(x_n) - F(r) \| .$

So,

$\lim_{n \to \infty} \| r - F(r) \| = 0 .$

Which implies that, $r - F(r) = 0 .$

$\therefore F(r) = r .$

This completes the proof.

Theorem B: Let S be a non-empty closed convex subset of a uniformly convex Banach space B. Let $F : S \to S$ satisfying (1) and such that $F(S)$ is relatively compact. If $f(F)$ the fixed point set of S is non-empty, then Mann iterative process (2) with $\{ a_n \}$ satisfying (i), (ii) & (iii) converges to a fixed point of F.

Proof:

Suppose, $r \in f(F)$ then,

$\| F(x_n) - r \| = \| F(x_n) - F(r) \| \leq k \max \{ c \| x_n - r \|, [\| x_n - F(x_n) \| + \| r - F(r) \|],$

$[\| x_n - F(r) \| + \| r - F(x_n) \|] \}$

$\leq k \max \{ c \| x_n - r \|, [\| x_n - r \| + \| r - F(x_n) \| + \| r - F(r) \|],$

$[\| x_n - r \| + \| r - F(r) \| + \| r - F(x_n) \|] \}$

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$$= k \max\{c \parallel x_n - r \parallel, [\parallel x_n - r \parallel + \parallel r - F(x_n) \parallel],$$

$$[\parallel x_n - r \parallel + \parallel r - F(x_n) \parallel]\} \quad [\because F(r) = r]$$

Therefore, it follows that

$$\parallel F(x_n) - r \parallel = \parallel F(x_n) - F(r) \parallel \leq k \parallel x_n - r \parallel \tag{6}$$

Now,

$$\parallel x_{n+1} - r \parallel = \parallel (1 - a_n) x_n + a_n F(x_n) - r \parallel$$

$$= \parallel (1 - a_n)(x_n - r) + a_n (F(x_n) - r) \parallel$$

$$\leq (1 - a_n) \parallel x_n - r \parallel + a_n \parallel F(x_n) - r \parallel \tag{7}$$

Now, combining (6) & (7) we get,

$$\parallel x_{n+1} - r \parallel \leq (1 - a_n) \parallel x_n - r \parallel + k a_n \parallel F(x_n) - r \parallel$$

$$= \parallel (1 - a_n + k a_n) \parallel x_n - r \parallel$$

$$i.e., \parallel x_{n+1} - r \parallel \leq \parallel x_n - r \parallel \quad [\text{since } 0 \leq k < 1 \text{ and } 0 \leq a_n \leq 1] \tag{8}$$

Therefore , $\{\parallel x_n - r \parallel\}$ is a monotone decreasing positive sequence, and hence converges to a real number a.

Suppose, $a > 0$.

Since,

$$\parallel x_n - F(x_n) \parallel \leq \parallel x_n - r \parallel + \parallel F(x_n) - r \parallel \leq \parallel x_n - r \parallel + k \parallel x_n - r \parallel$$

$$= (1 + k) \parallel x_n - r \parallel$$

[Using (6)]

$$i.e., \parallel x_n - F(x_n) \parallel \leq (1 + k) \parallel x_n - r \parallel \tag{9}$$

So

$\{\parallel x_n - F(x_n) \parallel\}$ is a bounded sequence.

Now,

$$\parallel r - x_{n+1} \parallel = \parallel r - (1 - a_n) x_n - a_n F(x_n) \parallel$$

$$= \parallel (1 - a_n)(r - x_n) + a_n r - a_n F(x_n) \parallel$$

$$= \parallel (1 - a_n)(r - x_n) + (1 - a_n)(r - F(x_n)) + (2 a_n - 1)(r - F(x_n)) \parallel$$

$$\leq (1-a_n)\| (r-x_n)+r-F(x_n) \| +(2a_n -1)\| r-F(x_n) \|$$

$$\leq (1-a_n)\| (r-x_n)+(r-F(x_n)) \| +(2a_n -1)\| r-x_n \|$$

[Using (6)]

$$\leq (1-a_n)\| r-x_n \|.\frac{\| (r-x_n)+(r-F(x_n)) \|}{\| r-x_n \|}+(2a_n -1)\| r-x_n \| \qquad (10)$$

Now, define

$$L=\frac{r-x_n}{\| r-x_n \|} \,\& \, M=\frac{r-F(x_n)}{\| r-x_n \|}$$

Then,

$$\| L+M \|=\frac{\| (r-x_n)+(r-F(x_n)) \|}{\| r-x_n \|}$$

$$\| L \|=\frac{\| r-x_n \|}{\| r-x_n \|}\leq 1 \quad i.e., \quad \| L \|\leq 1$$

$$\| M \|=\frac{\| r-F(x_n) \|}{\| r-x_n \|}=\frac{|-1|.\| F(x_n)-r \|}{|-1|.\| x_n -r \|}\leq \frac{k\| x_n -r \|}{\| x_n -r \|}\leq 1 \quad i.e., \quad \| M \|\leq 1$$

[Using (6)]

and $\| L-M \|=\dfrac{\| r-x_n -r+F(x_n) \|}{\| r-x_n \|}=\dfrac{\| F(x_n)-x_n \|}{\| r-x_n \|}$

So, from (10) we get

$$\| r-x_{n+1} \|\leq (1-a_n)\| r-x_n \|.\| L+M \|+(2a_n -1)\| r-x_n \| \qquad (11)$$

Now,·

$$\lim_{n\to\infty} \| L+M \|=\lim_{n\to\infty}\frac{\| (r-x_n)+(r-F(x_n)) \|}{\| r-x_n \|}$$

$$\leq \lim_{n\to\infty}\frac{\| r-x_n \|+\| r-F(x_n) \|}{\| r-x_n \|} \quad \text{[By triangle inequality]}$$

$$=\lim_{n\to\infty}\frac{\| r-x_n \|}{\| r-x_n \|}+\lim_{n\to\infty}\frac{\| r-F(x_n) \|}{\| r-x_n \|}$$

$$=\lim_{n\to\infty}\frac{\| r-x_n \|}{\| r-x_n \|}+\lim_{n\to\infty}\frac{|-1|.\| F(x_n)-r \|}{\| x_n -r \|}$$

$$\leq \lim_{n\to\infty} \frac{\| r - x_n \|}{\| r - x_n \|} + \lim_{n\to\infty} \frac{k\| x_n - r \|}{\| x_n - r \|}$$

[Using (6)]

$$\leq 2$$

i.e., $\quad \lim_{n\to\infty} \| L + M \| \leq 2 \quad$ i.e., $\quad \| L + M \| \to 2 \quad$ as $\quad n \to \infty$

Therefore, from (11) we get

$$\| r - x_{n+1} \| \leq \| r - x_n \| \tag{12}$$

Hence, we can also say that $\{\| r - x_n \|\}$ is a monotone decreasing positive

sequence.

Since, B is uniformly convex and $\| L \| \leq 1, \| M \| \leq 1 \& \| L + M \| \to 2$.

Then, we get $\| L - M \| \to 0$

i.e., $\quad \lim_{n\to\infty} \| L - M \| = 0 \Rightarrow \lim_{n\to\infty} \frac{\| x_n - F(x_n) \|}{\| r - x_n \|} = 0$

But, $\lim_{n\to\infty} \| r - x_n \| \neq 0$. Because, if $\lim_{n\to\infty} \| r - x_n \| = 0$ then, $x_n \to r$ and which

contradicts the definition of a.

Hence,

$$\lim_{n\to\infty} \| x_n - F(x_n) \| = 0$$

Since, $F(S)$ is relatively compact, there exists a subsequence $\{F(x_{n_i})\}$ of $\{F(x_n)\}$

Such that

$\lim_i [F(x_{n_i})] = d \in B, \text{(Say)}$

But,

$$\| x_{n_i} - d \| \leq \| x_{n_i} - F(x_{n_i}) \| + \| F(x_{n_i}) - d \| \to 0 \text{ as } i \to \infty$$

Now,

$$\| F(d) - d \| \leq \| F(d) - F(x_{n_i}) \| + \| F(x_{n_i}) - d \|$$

$$\leq k \max \{c \| d - x_{n_i} \|, [\| x_{n_i} - F(x_{n_i}) \| + \| d - F(d) \|], [\| x_{n_i} - F(d) \| + \| d - F(x_{n_i}) \|]\}$$

$$+ \| F(x_{n_i}) - d \|$$

$$\leq k \max \{c \| d - x_{n_i} \|, [\| x_{n_i} - F(x_{n_i}) \| + \| d - F(d) \|], [\| x_{n_i} - d \| + \| d - F(d) \|$$

$$+ \| d - F(x_{n_i}) \|]\} + \| F(x_{n_i}) - d \|$$

So,

$$\| F(d) - d \| \leq k \max \{\| d - F(d) \|, \| d - F(d) \|\} \quad \text{as} \quad i \to \infty.$$

Hence,

$\parallel F(d) - d \parallel \le k \parallel d - F(d) \parallel$.

Which is a contradiction, because $0 \le k < 1$ and $F(d) = d$.

Hence d is a fixed point of F *i.e.*, $d \in f(F)$.

Now, replacing r by d in (8), it follows that $\{\parallel x_n - d \parallel\}$ is monotone decreasing sequence in n.

Since, $\lim_i [x_{n_i}] = d$, which implies that $\lim_n [x_n] = d$.

Hence, we can say that $\{x_n\}$ converges to a fixed point of F.

This completes the proof.

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