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POINT WISE QUASICONTINUITY AND BAIRE SPACES

By

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Abstract

In this paper, it is proved that the notions of point wise semi-continuity and quasicontinuity are the same even when the mapping is not globally semi-continuous. The concept of removable quasidiscontinuity at a point is introduced with some of its applications [Theorem 4.1]. Finally, a set of sufficient conditions for a topological space to be a Baire space is formulated. In particular, it was shown that if every mapping from a topological space X to an infinite T_2 space is quasicontinuous then X is a Baire space.

Keyword and phrases : *baire space, topological space, quasicontinuity.*

সংক্ষিপ্তসার

এই গবেষণা পত্রে প্রমাণ করা হয়েছে যে বিন্দু ভিত্তিক অর্ধ - সন্তত এবং কার্যত - সন্ততের ধারণাগুলি একই যদিও চিত্রণটি সার্বিক অর্ধ-সন্তত নয়। একটি বিন্দুতে স্থানান্তরণ যোগ্য কার্যত - অসন্ততির ধারণাকে কিছু প্রয়োগসহ উপস্থাপন করা হয়েছে [উপপাদ্য – 4.1] শেষতঃ টপোলজীয় দেশটি যে একটি বেয়ার দেশ [Baire Space] তার জন্য এক সেট্ যথেষ্ট শর্তের সূত্র নির্ণয় করা হয়েছে। বিশেষতঃ এটা দেখানো হয়েছে, যে যদি প্রত্যেক চিত্রণ টপোলজীয় দেশ X থেকে অসীম T_2 দেশে কার্যত সন্তত হয় তবে X একটি বেয়ার দেশ (Baire Space) হবে।

1. Introduction

In this paper we define a point of semi-continuity of a function in a topological space and demonstrate some of its applications. We show that a function is globally semi-continuous [11] if and only if each point of the domain space is a point of semi-continuity. In [16], it was shown that a function is globally semi-continuous if and only if it is quasi-continuous [15] at each point in the domain space. Thus the notion of a point of semi-continuity of ours and a point of quasicontinuity of Neubrunn, coincide when the function is globally semi-continuous. In spite of this the bottom line of the inclusion of our definition is: A local property l_1 and another local property l_2 of a function may coincide when the function has a global property P; but l_1 and l_2 may differ when P does not hold. As an example, we cite [7], where it was shown that for a global continuous mapping all the points of continuity are s-points and conversely; but when the mapping is not globally continuous, a point of continuity may not be a s-point. Eventually, we find that for a map which is not globally semi-continuous, the notion of semi-continuity and quasi-continuity at a point are indeed the same. Next, with the help of the notion of removable quasidiscontinuity, we find when a function with closed graph will be quasicontinuous at a point. Finally we use the notion of point of quasicontinuity to get a set of sufficient conditions for a topological space to be a Baire space. Throughout the paper X, Y denote topological spaces, $(\Re, \mathfrak{U})$ denotes the usual topological space of real numbers and ϕ the empty set.

2. Known Definitions and results.

Definition 2.1[11]. A set is said to be semi-open if and only if there exists an open set O in X such that $O \subset A \subset Cl\ O$, where $Cl\ O$ is the closure of O in X.

Definition 2.2[1]. A set $A \subset X$ is said to be semi-closed if and only if there exists a closed set F in X such that $Int\ F \subset A \subset F$, where $Int\ A$ is the interior of F in X.

Definition 2.3[1]. The intersection of all semi-closed sets containing a set A is called the semi-closure of A and is denoted by $SCl\ A$.

Definition 2.4[2]. For $A \subset X$, the union of all semi-open sets contained in A is called the semi-interior of A and is denoted by $SInt\ A$.

Definition 2.5[11]. A mapping $f: X \to Y$ is said to be semi-continuous if and only if inverse of every open set in Y is semi-open in X.

Definition 2.6[8]. A mapping $f: X \to Y$ has at worst a removable discontinuity at $x \in X$ if there is a $y \in Y$ such that for each neighbourhood V of y, there is a neighbourhood U of x such that $f(U-\{x\}) \subset V$.

Definition 2.7[15]. A mapping $f: X \to Y$ is quasicontinuous at $p \in X$ if for every U,V open such that $p \in U$, $f(p) \in V$ there exists a non-empty open set $G \subset U$ such that $f(G) \subset V$. It is called quasicontinuous if it is quasicontinuous at every $x \in X$.

Definition 2.8[6]. A topological space X is called R_0 if and only if for each $x \in X$ and open subset U, $x \in U$ implies $Cl\{x\} \subset U$. It is known ([13] cf [19, p 47]) that R_0 is weaker than T_1 and is independent of T_0. In fact, $T_1 = T_0 + R_0$. A Hausdorff space is therefore necessarily R_0.

Definition 2.9[17]. A topological space X is said to have an ascending chain of open sets if there are countably infinite open sets $O_1, O_2, O_3, \ldots, O_n, \ldots$. such that $O_1 \subsetneq O_2 \subsetneq O_3 \subsetneq \ldots \subsetneq O_n \subsetneq \ldots\ldots$, where $A \subsetneq B$ means A is a proper subset of B .

Definition 2.10[9]. A mapping $f: X \to Y$ is said to be almost-continuous at $x \in X$ if and only if for each neighbourhood V of $f(x)$, $Int\ Cl\, f^{-1}(V)$ is a neighbourhood of x. f is called almost-continuous on a subset A of X if it is almost-continuous at every $x \in A$.

Theorem 2.1[4]. A set $A \subset X$ is semi-closed if and only if $Scl\ A = A$.

Theorem 2.2[2]. If A, B are subsets of X, then $SCl\ (A \cap B) \subset SCl\ A \cap SCl\ B$.

Theorem 2.3[3]. If $A \subset B \subset X$, then (i) $SInt\ A \subset SInt\ B$ and (ii) $SCl\ A \subset SCl\ B$.

Theorem 2.4[1]. A mapping $f: X \to Y$ is semi-continuous if and only if for every closed set $F \subset Y$, $f^{-1}(F)$ is semi-closed in Y.

Theorem 2.5[16]. A single valued mapping $f: X \to Y$ is quasicontinuous if and only if it is semi-continuous.

Theorem 2.6[14]. Let $G(f)$ be the graph of the mapping $f: X \to Y$. Then $G(f)$ is closed if and only if for each $x \in X$ and $y \in Y$, where $y \neq f(x)$, there exist open sets U and V containing x and y respectively, such that $f(U) \cap V = \phi$.

Lemma 2.1[5]. A set $D \subset X$ is dense in X if and only if $SCl\, D = X$.

Lemma 2.2[5]. A set D is dense in X if and only if the complement of D has empty semi-interior.

Lemma 2.3[2]. A set A is nowhere dense in X if and only if $SInt\,(SCl\, A) = \phi$.

Lemma 2.4[17]. An infinite Hausdorff space is an R_0 space with an ascending chain of open sets. The converse of Lemma 2.4 is not true as shown in ([18], Example 1, p.199).

Lemma 2.5[18]. An infinite Hausdorff space has an ascending chain of regular open sets. Converse of Lemma 2.5 is not true as shown in ([19], Example 1).

Lemma 2.6[18]. Let X be an infinite space with an ascending chain of regular open sets. Then X contains a countably infinite discrete subspace.

3. Point wise semi-continuity and Point wise quasi-continuity of a function

Definition 3.1. If $f: X \to Y$ be a mapping from X into Y, then f is said to be semi-continuous at $x \in X$ if and only if $[x \in SCl\, A] \Rightarrow [f(x) \in Cl\, f(A)]$ for each $A \subset X$.

Remark 3.1. If f is continuous at $x \in X$, then f is semi-continuous at $x \in X$. But the converse is not true in general as shown by the following example.

Example 3.1. Let $f: (\Re, \mathfrak{U}) \to (\Re, \mathfrak{U})$ be the box function, viz., $f(x) = [x]$. It is easy to verify that f is semi-continuous at $x = 0, \pm 1, \pm 2, \ldots\ldots$; but f has discontinuities at these points.

Theorem 3.1. Let $f: X \to Y$ be a mapping from X into Y and let $x \in X$. Then the following statements are equivalent:

(a). f is semi-continuous at x.

(b). $[x \in SCl\, A] \Rightarrow [x \in f^{-1}(Cl\, f(A))]$ for each $A \subset X$.

(c). $[f(x) \in Int\, B] \Rightarrow [x \in SInt\, f^{-1}(B)]$ for each $B \subset Y$.

(d). $[x \in f^{-1}(Int\, B)] \Rightarrow [x \in SInt\, f^{-1}(B)]$ for each $B \subset Y$.

(e). $[x \in SCl\, f^{-1}(B)] \Rightarrow [x \in f^{-1}(Cl\, B)]$ for each $B \subset Y$.

(f). For each open set V containing $f(x)$, there is a semi-open set U containing x such that $f(U) \subset V$.

Proof. The proof is routine and hence omitted.

Theorem3.2. Let $f: X \to Y$ be a mapping from X into Y. Then f is semi-continuous (in the global sense [11]) if and only if f is semi-continuous at every point of X.

Proof. The proof is routine and hence omitted.

Theorem 3.3. A function $f: X \to Y$ is quasi-continuous at $p \in X$ if and only if $x \in Cl(Int\ f^{-1}(V))$ for every open set V containing $f(x)$.

Proof. First we suppose that f is quasi-continuous at x. Let V be any open set containing $f(x)$ and A be any open set containing x. Since f is quasicontinuous at x, there is a non-empty open set G such that $G \subset A$ and $f(G) \subset V$. So, $\phi \neq G \subset Int\, f^{-1}(V)$ as G is open. Now if $x \in G$, then $x \in Cl\,(Int\, f^{-1}(V))$. Again if $x \neq G$, then $\phi \notin G \subset (A\text{-}\{x\}) \cap (Int\, f^{-1}(V))$ and hence $x \in Cl\,(Int\, f^{-1}(V))$.

Next, let $x \in Cl\,(Int\, f^{-1}(V))$ for every open set V containing $f(x)$. Let A be any open set containing x. Then $A \cap (Int\, f^{-1}(V)) \neq \phi$. We take $G = A \cap (Int\, f^{-1}(V))$. Then G is open, non-empty such that $G \subset A$ and $f(G) \subset f(f^{-1}(V)) \subset V$.

Theorem 3.4. A function $f: X \to Y$ is quasi-continuous at a point $x \in X$ if and only if for each open set V containing $f(x)$, there is a semi-open set U containing x such that $f(U) \subset V$.

Proof. Taking $U = (Int f^{-1}(V)) \cup \{x\}$, the proof follows from Theorem 3.3.

Theorem 3.5. A function $f: X \to Y$ is quasicontinuous at a point $x \in X$ if and only if it is semi-continuous at the point x.

Proof. The proof follows from Theorem 3.1(f) and Theorem 3.4.

Remark 3.2. From Theorem 3.5 it follows that the notion of semi-continuity at a point and the notion of quasicontinuity at a point are the same even when the function is not globally semi-continuous [11] (or equivalently quasicontinuous [15]). Hence from now on we use the phrase *quasicontinuity* instead of *semi-continuity*.

4. Removable quasidis continuity

In this section we introduce a property of a function at a point which generalizes both of the notions: removable discontinuity [8] and point wise quasicontinuity [15]. Furthermore, under what additional restriction this property will be quasicontinuity is studied.

Definition 4.1. Let $f: X \to Y$ be a mapping from X into Y. Then f has at worst a removable quasidiscontinuity at $x \in X$ if there is a $y \in Y$ such that for each open set V containing y, there is a semi-open set U containing x such that $f(U - \{x\}) \subset V$.

Remark 4.1. Every removable discontinuity [8] is a removable quasidiscontinuity. But the converse is not true in general as shown by the following example.

Example 4.1. Let f: $(\Re,\mathfrak{U}) \to (\Re,\mathfrak{U})$ be defined by $f(x) = 0$ for $x < 0$, $f(x) = 1$ for x >0 and $f(0)=0$. Clearly f has a removable quasidiscontinuity at the point 0. But 0 is not a point of removable discontinuity of f.

Remark 4.2. If a function f is quasicontinuous at a point of X, then by Theorem 3.4. f has at worst a removable quasidiscontinuity at that point. But the converse is not true in general as shown by the following example.

Example 4.2. We consider the function f as given in Example 4.1. Clearly f is not quasi-continuous at the point 0 though f has a removable quasidiscontinuity at 0.

However the following theorem is true.

Theorem 4.1. Let f: $X \to Y$ have a closed graph. If f has a removable quasidiscontinuity at that point $x \in X$, f is quasicontinuous at x.

Proof. Let y be the point in Y determined by the definition of removable quasi-discontinuity of f at x.If f is not quasicontinuous at x, $y \neq f(x)$. By Theorem 2.6, there are open sets U and V containing the points x and y respectively such that $V \cap f(U) = \phi$. Since f has a removable quasidiscontinuity at x, there is a semi-open set W containing x such that $f(W\text{-}\{x\}) \subset V$. Since W is a non-empty semi-open set, there exists a non-empty open set G such that $G \subset W \subset Cl\ G$. So $x \in Cl\ G$ and $f(G) \subset V$. Now since $x \in Cl\ G$, we have $G \cap U \neq \phi$. Thus $\phi \neq f(G \cap U) \subset f(G) \cap f(U) \subset V \cap f(U) = \phi$, a contradiction. So f is quasidiscontinuous at x.

Corollary 4.1. Let X be regular and f: $X \to Y$ a closed function with closed point inverses. Then if f has at worst a removable quasidiscontinuity at the point $x \in X$, f is quasi- continuous at x.

5. A set of sufficient conditions for a topological space to be a Baire space

In this section we have found when a topological space will be a Baire space with the aid of point wise quasicontinuity. In this regard we have used a lemma of [17]. Our proof of this lemma follows closely a proposed (but incorrect) proof by Tong [17] is also appealing that it seems useful to record a corrected version.

Lemma 5.1. [17]. If X is an infinite R_0 space with an ascending chain of open sets $O_1, O_2, O_3, \ldots, O_n, \ldots$. such that $O_1 \subsetneq O_2 \subsetneq O_3 \subsetneq \ldots \subsetneq O_n \subsetneq \ldots\ldots$, then there is a countably infinite discrete sets $S = \{ y_1, y_2, y_3, \ldots, y_n\}$ in X such that for each n, there is an open set V_n satisfying $V_n \cap S = y_n$.

Proof. Without loss of generality we may assume that O_1 is non-empty. Let $y1 \in O_1$ be an arbitrary point. Sine X is R_0, $Cl\ \{y_1\} \subset O_1$. Let $V_1 = O_1$. Since $O_1 \subsetneq O_2$, and since X is R_0, we can find a $y_2 \in O_2$ such that $y_2 \notin O_1$ and $Cl\ \{y_2\} \subset O_2$. Let $V_2 = O_2 - Cl\{y_1\}$. Now since $Cl\{y_1\} \subset O_1$ and $y_2 \notin O_1, y_2 \notin Cl\{y_1\}$. It therefore follows that $y_1 \neq y_2,\ y_1 \in V_1,\ y_2 \in V_2,\ y_1 \notin V_2,\ y_2 \notin V_1$.

Again since $O_2 \subset O_3$, there is a $y_3 \in O_3$, such that $y_3 \notin O_2$. Sine X is R_0, $Cl\ \{y_3\} \subset O_3$. We take $V_3 = O_3 - (Cl\{y_1\} \cup Cl\{y_2\})$. Since $Cl\{y_1\} \subset O_1$ and O_1 is a proper subset of O_2 and since $Cl\{y_2\} \subset O_2$, we have $Cl\{y_1\} \cup Cl\{y_2\} \subset O_2$.Now as $y_3 \notin O_2$, $y_3 \notin Cl\{y_1\} \cup Cl\{y_2\}$ which implies $y_3 \neq y_1,\ y_3 \neq y_2,\ y_3 \in V_3$. Again since $O_1 \subsetneq O_2$ (O_1 is a proper subset of O_2), $y_3 \notin O_2 \Rightarrow y_3 \notin O_1$. So $y_3 \notin V_2, V_1$. Also from definition of V_3 it follows that $y_1, y_2 \notin V_3$. Thus we have, $y_1 \in V_1,\ y_2 \in V_2, y_3 \in V_3$; $y_2, y_3 \notin V_1,\ y_3, y_1 \notin V_2,\ y_1, y_2 \notin V_3$.

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Now if y_{n-1} is chosen and $V_{n-1} = O_{n-1} - (\bigcup_{i=1}^{n-2} Cl\{y_i\})$ is defined, then since $O_{n-1} \subsetneq O_n$ and since X is R_0, we may choose $y_n \in O_n$ such that $y_n \notin O_{n-1}$ and $Cl\ \{y_n\} \subset O_n$. Clearly $y_n \notin \bigcup_{i=1}^{n-2} Cl\{y_i\}$. Let $V_n = O_n - (\bigcup_{i=1}^{n-1} Cl\{y_i\})$. Then $y_n \in V_n$. Thus we have a countably infinite set of distinct points $S = \{y_1, y_2, ..., y_n, ...\}$ and a countable infinite distinct open sets $V_1, V_2, ..., V_n, ...$ such that $y_n \in V_n$ for $n = 1, 2, 3, ...$

We now show that $y_i \notin V_n$ for $i = 1, 2, 3, ..., n-1, n+1, n+2, ...$, so that $V_n \cap S = \{y_n\}$. Since $V_n = O_n - (\bigcup_{i=1}^{n-1} Cl\{y_i\})$, we have,

$$y_i \notin V_n \quad \text{for } i = 1, 2, 3, ..., n-1. \tag{1}$$

Also by construction, $y_{n+1} \in O_{n+1}$ but $y_{n+1} \notin O_n$. So $y_{n+1} \notin V_n$. Again for $m > 1$, $y_{n+m} \in O_{n+m}$ but $y_{n+m+1} \notin O_{n+m-1}$, As $O_n \subsetneq O_{n+m-1}$, $y_{n+m} \notin O_n$ for $m > 1$ and so $y_{n+m} \notin V_n$ for $m > 1$. Hence it follows that

$$y_{n+m} \notin V_n \text{ for } m \geq 1. \tag{2}$$

Thus combining (1) and (2) we have, $V_n \cap S = \{y_n\}$, which implies that S is a discrete set of countably infinite distinct points. This proves the lemma.

Theorem 5.1. Let Y be an infinite R_0 space with an ascending chain of open sets. If X is a topological space such that every mapping $f: X \to Y$ is quasicontinuous on a dense subset of X, then X is a Baire space.

Proof. If possible, let X be not a Baire space. Then there exists a sequence of dense open sets $D_1, D_2, D_3, ...$ such that $\bigcap_{i=1}^{\infty} D_i$ is not dense in X and so by Lemma 2.2 there exists a non-empty semi-open set say U of X such that $U \subset X - \bigcap_{i=1}^{\infty} D_i = \bigcup_{i=1}^{\infty} (X - D_i)$.

Now for each i, $SInt\ SCl\ (X-D_i)=\ SInt\ (X-D_i)=\phi$, By Lemma 2.2 and so by Lemma 2.3, $(X-D_i)$ is nowhere dense in X.

Let $U_i = U - D_i$. Then $U = \bigcup_{i=1}^{\infty} U_i$.

Without loss of generality we may assume that we may assume that these U_i are pair wise disjoint (and non-empty), for otherwise we may instead choose $U_1' = U_1$, $U_n' = U_n - \bigcup_{i=1}^{\infty} U_i$, $n \geq 2$. Now by Lemma 5.1there exists a countably infinite discrete subspace $S = \{y_1, y_2, ..., y_n, ...\}$ of Y. We then consider the mapping f: $X \rightarrow Y$ defined by

$$
\begin{aligned}
f(x) &= y_{n+1},\ if\ x \in U_n\ for\ some\ n \\
&= y_1,\ otherwise
\end{aligned}
$$

It is readily seen that f is a well defined mapping. Therefore by hypothesis of the theorem, this mapping f: $X \rightarrow Y$ is quasicontinuous on a dense subset $D(f)$ of X. Now if U is non-empty and semi-open, we must have $U \cap D(f) \neq \phi$. Then since $x_0 \in U_m$ for some U_m we have $f(x_0) = y_{m+1}$.

Since X is a discrete subspace of Y, there exists an open neighbourhood V_{m+1} of y_{m+1} such that $V_{m+1} \cap S = \{y_{m+1}\}$.

Then since $U_m \subset X - D_m$ and U_m is nowhere dense, for any open set V containing $f(x_0)$ such that $V \subset V_{m+1}$, $SInt\ SCl\ f^{-1}(V) = \phi$, by Lemma 2.3. Hence $SInt\, f^{-1}(V) = \phi$. But $f(x_0) \in V\ (= Int\ \ V\ )$ and since $SInt\, f^{-1}(V) = \phi$, $x_0 \notin SInt\, f^{-1}(V)$. So by Theorem 3.1 (c) f is not quasicontinuous at $x_0 \in D(f)$, a contradiction. Hence X is a Baire space.

Corollary 5.1. If Y be an arbitrary infinite Hausdorff space and X is a topological space such that every mapping f: $X \to Y$ is quasicontinuous on a dense subset of X, then X is a Baire space.

Proof. The proof follows at once from Theorem 5.1 in view of Lemma2.4.

Remark 5.1. In Example 5.1 (below) we have justified that the assumption of Theorem 5.1 and Corollary 5.1, viz., '*every mapping is quasicontinuous on a dense subset*' is consistent.

Example 5.1. Let $\mathbb{N}$ be the set of all positive integers with the topology $\tau^{**} = \{\phi, \mathbb{N}, \{1\}, \{2\}, \{1,2\}\}$. Then $(\mathbb{N}, \tau^{**})$ is a Baire space. Consider any mapping f: $(\mathbb{N}, \tau^{**}) \to (Y, \sigma)$, where (Y, σ) is any topological space. It is easy to verify that f is continuous (and hence quasicontinuous) on the dense subset $\{1,2\}$ of $\mathbb{N}$.

Remark 5.2. From the Example 5.1 it is trivial that the restriction on the range space in Theorem 5.1 and Corollary 5.1 is not necessary. Furthermore, the hypothesis '*every mapping is quasicontinuous on a dense subset*' in Theorem 5.1 and in Corollary 5.1 is only a sufficient condition as shown by the following example.

Example 5.2. Let $\mathbb{N}$ be the set of all positive integers with the topology τ^*, where τ^*-open sets are: $\phi, \mathbb{N}$, $\{1,2\} \subset \{1,2,3\} \subset \{1,2,3,4\} \subset \ldots\ldots..$ Let f: $(\mathbb{N}, \tau^*) \to (\Re, \mathfrak{U})$ be defined by $f(3) = 1$, and $f(n) = n$ otherwise. It can be verified that f is not quasicontinuous on any dense subset of $(\mathbb{N}, \tau^*)$ because each dense subset of $\mathbb{N}$ must contain 1 or 2 or both; but $(\mathbb{N}, \tau^*)$ is a Baire space.

The same conclusion of Theorem 5.1 under independent conditions without any assumption on separation axiom is provided in the following theorem.

Theorem 5.2. Let Y be an infinite space with an ascending chain of regular open sets. If X is a topological space such that every mapping f: $X \to Y$ is quasicontinuous on a dense subset of X, then X is a Baire space.

Proof. Using Lemma 2.6, the proof is all the same as Theorem 5.1.

Remark 5.3. Corollary 5.1 also follows from Theorem 5.2 in view of Lemma 2.5.

Remark 5.4. Similar results of Theorem 5.1 and Corollary 5.1 and Theorem 5.2 can be found in [17], [12], [18] respectively, where instead of taking the function f quasicontinuous they consider f as almost-continuous. But the notion of almost-continuous function at a point and quasicontinuous function at a point are independent (even they are independent on a dense subset) as shown in the following example and hence our study using quasicontinuity is justified.

Example 5.3. We consider Example 5.2 where it was shown that f is not quasicontinuous on any dense subset of $(\mathbb{N}, \tau^*)$. Furthermore, every mapping ψ (and hence f) from $(\mathbb{N}, \tau^*)$ to $(\Re, \mathfrak{U})$ is almost-continuous on the dense subset $\{1\}$ of $\mathbb{N}$ because for any open set V containing the functional value of 1, $\psi^{-1}(V)$ contains 1 and so $Int\ Cl\ \psi^{-1}(V) = Int\ \mathbb{N} = \mathbb{N}$.

Example 5.3. Let $\mathbb{N}$ be the set of all positive integers with the topology τ, where $\tau = \{\phi, \mathbb{N}, \{1\}, \{3\}, \{1,3\}, \{3,4,5,.......\}, \{1,3,4,5,.....\}\}$.

Let f: $(\mathbb{N}, \tau) \to (\Re, \mathfrak{U})$ be defined by

$$f(n) = 1,\ for\ n = 1,2$$
$$= n+1,\ n \geq 3.$$

We $D = \{1,2,3\} \subset \mathbb{N}$. Clearly D is dense in $\mathbb{N}$.

It is easy to verify that f is quasicontinuous on D but f is not almost-continuous on D, because it can be verified that f is not almost-continuous at 2.

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