

MULTIPLE TIME SCALE METHOD FOR OVER-DAMPED PROCESSES IN BIOLOGICAL SYSTEMS

By

¹Md. Abul Kalam Azad, ²M. Ali Akbar and ³M. Abdus Sattar

¹Dept. of Mathematics, Rajshahi University of Engineering & Technology,
Bangladesh

²Dept. of Applied Mathematics, Rajshahi University, Bangladesh

³Dept. of Mathematics, Rajshahi University, Bangladesh

Abstract

An over-damped solution of a nonlinear system has been investigated by multiple time scale method when one of the roots of the unperturbed equation is much smaller than the others. The asymptotic solution shows excellent agreement with the numerical solution. An example is given to biological system.

Keyword and phrases : multiple time scale, over-damped process biological system.

সংক্ষিপ্তসার

বহুলিত - সময় স্কেল পদ্ধতির সাহায্যে অরৈখিক তত্ত্বের একটি অতি-অবমন্দিত সমাধান অনুসন্ধান করা হয়েছে যখন অবিচলিত সমীকরণের একটি বীজ অন্যগুলি থেকে অনেক ক্ষুদ্র হয়। স্পর্শপ্রবণ সমাধানটি সাংখ্যমান সমাধানের সঙ্গে সম্পূর্ণ সংগতিপূর্ণ। জীববিদ্যার ক্ষেত্রে একটি উদাহরণ দেওয়া হয়েছে।

1. Introduction

The mathematical formulation of extensive numbers of physical problems, such as spring mass systems resistor capacitor inductor circuits, bending of beams, chemical reactions, simple and compound pendulums, the biological, the biochemical laws and relations appear in the form of nonlinear differential equations. So, much interest is laid in solving nonlinear differential equation. To solve the nonlinear differential equations, there exist several methods, such as, straight forward expansion method, Renormalization group method, method of averaging, Vander pol's method, Strubles technique, The Krylov-

Bogoliubov-Mitropolskii (KBM) [1, 7] method, the multiple time scale (MTS) method etc. Originally, the KBM method was introduced to obtain periodic solution of a second order nonlinear system with small nonlinearities. Popov [12] extended the KBM method for strong linear damping effects. Following Popov's technique, Murty *et al* [11] have extended the KBM method for over-damped nonlinear systems. Owing to physical importance of damped nonlinear systems Popov's results were rediscovered by Bojadziev [2]. Sattar [13] studied a second order critically damped nonlinear system by making use of the KBM method. But the solution presented in Sattar [13] gives incorrect results for some set of initial conditions. Shamsul [14] has developed a new asymptotic technique for obtaining analytical approximate solutions of second order over-damped and critically damped nonlinear systems.

In MTS method the key time is resolved into several faster and slower times. And as a matter of fact the ordinary differential system changes to a partial differential system. Therefore, it appears the problem has been complicated. This is true, but experiences with this method have shown that the disadvantages of introducing this complication are far outweighed by advantages.

In this article, an asymptotic solution of a biological system whose one of the eigenvalues is very small and the other is large has been found by making use of the MTS method. The results obtained by the presented MTS method show good coincidences with those obtained by numerical method.

Three such biological models are described below:

(i) A modified Lotka-Volterra model: Assuming in presence of predator and a logistic growth for prey the well-known predator-prey [9, 19] model is

$$\dot{N}_1 = N_1(k_{11} + k_{12}N_1 + k_{13}N_2), \quad \dot{N}_2 = N_2(k_{21} + k_{22}N_1 + k_{23}N_2) \quad (1)$$

where N_1 and N_2 are two populations.

(ii) Oscillating chemical reaction: Lefever and Nicolis [8] have considered a set of chemical reactions modeled by the chemical kinetic equations

$$\dot{X} = A + X^2Y - BX - X, \quad \dot{Y} = BX - X^2Y \quad (2)$$

where X and Y are concentrations, and A and B are initial product concentrations. Lefever and Nicolis [8] have studied the phase portrait in the phase plane (X, Y) both analytically and numerically, and shown the existence of a limit cycle.

(iii) The FitzHugh equations: To investigate the physiological state of nerve membranes, FitzHugh [3] introduce a theoretical model described by

$$\dot{x}_1 = \alpha + x_1 + x_2 - \frac{x_1}{3}, \quad \dot{x}_2 = \rho(\gamma - x_1 - \beta x_2) \quad (3)$$

where it is assume that $\alpha, \gamma \in (-\infty, \infty)$ and $\beta, \rho \in (0, 1)$. For $\alpha = \beta = \gamma = 0$, equation (3) reduces to a Van der Pol equation. This particular model has been studied by Troy [18], Hsu and Kazarinoff [5]. FitzHugh [3] investigated the model quantitatively in the phase plane, while Hsu and Kazarinoff [5] dealt with periodic solutions using the Poincare-Hopf bifurcation theory.

It will be shown that all the modeling equations (i)-(iii) can be presented in the neighborhood of the equilibrium position by a second order differential equation of the type [6]

$$\ddot{x} + 2k\dot{x} + cx = \varepsilon f_1(x, \dot{x}) + \varepsilon^2 f_2(x, \dot{x}) + \dots \quad (4)$$

where ε is a small positive parameter, and the significant damping term is expressed by the linear term $2k\dot{x}$. The damping coefficients k [of order $O(1)$], and also c , are constant. The assumption $k > \sqrt{c}$ ensures that the system is over-damped. When $\varepsilon = 0$, equation (4) has two roots, say λ_1, λ_2 . Therefore, the unperturbed solution of the equation (4) is $x(t) = ae^{\lambda_1 t} + be^{\lambda_2 t}$, which describes a non-oscillatory motion. Here a and b are two arbitrary constants.

We use the multiple time scale perturbation method to obtain quantitative information about the non-oscillatory processes when one of the roots is much

smaller than the other roots. In Section 2, the method is given for equation (4) and in Section 3 the technique obtained in Section 2 is applied to the model (1)

2. The Multiple Time Scale Method

Equation (4) is slightly more general than the equation initially studied by Popov [12], which does not include the term $\varepsilon^2 f_2(x, \dot{x})$. Following the multiple time scale method, the solution of (4) is sought in the form [15]

$$x(t, \varepsilon) = a_1(t) + a_2(t) + \varepsilon u_1(a_1, a_2, t) + \varepsilon^2 \dots \quad (5)$$

Here the solution (5) is not the formal form of the MTS method, rather than a_1 and a_2 are unusual variable. Yet the presentation of this variable is importation for the formulation of the method as well as determination of an approximate solution from the drive formula. Generally, a_1 and a_2 depend on the several time t_0, t_1, t_2, \dots , where $t = t_0 + \varepsilon t_1 + \varepsilon^2 t_2 + \dots$.

Now we can write the equation (4) as

$$(D - \lambda_1)(D - \lambda_2)x = \varepsilon f_1 + \varepsilon^2 f_2 + \dots \quad (6)$$

Substituting equation (5) into equation (6) and equating the coefficients of $\varepsilon^1, \varepsilon^2$, we obtain

$$\begin{aligned} (D_0 - \lambda_2)(D_1 a_1) + (D_0 - \lambda_1)(D_1 a_2) \\ + (D_0 - \lambda_1)(D_0 - \lambda_2)u_1 = f(a_1 + a_2, D_0 a_1 + D_0 a_2) \end{aligned} \quad (7)$$

$$\begin{aligned} (D_0 - \lambda_2)(D_2 a_1) + (D_0 - \lambda_1)(D_2 a_2) \\ \text{and } + [D_1(D_0^1 + c_1) + D_0 D_1]u_1 + D_1^2(a_1 + a_2) \\ + (D_0 - \lambda_1)(D_0 - \lambda_2)u_2 = u_1 f_x(a_1 + a_2, D_0 a_1 + D_0 a_2) \\ + (D_0 u_1 + D_0 a_1 + D_0 a_2) \times f_x(a_1 + a_2, D_0 a_1 + D_0 a_2) + \dots \end{aligned} \quad (8)$$

where c_1, c_2 are the coefficients of the algebraic equation $\prod_{j'=1, j' \neq j}^2 (\lambda_j - \lambda_{j'}) = 0$.

For the second approximate solution of equation (4), we have to use the formula (7) and (8). To avoid the secular terms in solution Eq. (5), it has been proposed in [15] that u_1 and u_2 exclude the terms $a_1^{i_1} a_2^{i_2}$ when $i_1 = i_2 \pm 1$. But this assumption is not suitable when one root will be multiple of the other roots

and when one root will be very smaller than the other roots. So, for this problem we assume that, the correction terms u_1, u_2, \dots do not contain the terms $a_1^{i_1} a_2^{i_2}$ when $i_1 \leq 1$ if $\lambda_1 > \lambda_2$ (see [16]). Under this assumption the equations (7) and (8) can be separated as:

$$(D_0 - \lambda_2)(D_1 a_1) + (D_0 - \lambda_1)(D_1 a_2) = F_1(a_1^{i_1} a_2^{i_2}) \quad \text{where } i_1 \leq 1 \quad (9)$$

$$(D_0 - \lambda_1)(D_0 - \lambda_2)u_1 = F_2(a_1^{i_1} a_2^{i_2}) \quad \text{where } i_1 > 1 \quad (10)$$

And

$$(D_0 - \lambda_2)(D_2 a_1) + (D_0 - \lambda_1)(D_2 a_2) = G_1(a_1^{i_1} a_2^{i_2}) \quad \text{where } i_1 \leq 1 \quad (11)$$

$$(D_0 - \lambda_1)(D_0 - \lambda_2)u_2 = G_2(a_1^{i_1} a_2^{i_2}) \quad \text{where } i_1 > 1 \quad (12)$$

Now the equation (9) can be separated for $D_1 a_1$ and $D_1 a_2$ subject to the condition that the coefficients of $D_1 a_1$ and $D_1 a_2$ do not become large and solving equation (9) we obtain $D_1 a_1$, $D_1 a_2$ and solving equation (10) we obtain the value of u_1 . Substituting $D_1 a_1$, $D_1 a_2$ and u_1 into (11) and (12) and using the same assumption, we shall be able to separate the equation (11) and solving equation (11) we get $D_2 a_1$, $D_2 a_2$ and solving equation (12) we get u_2 . This completes the determination of the second approximate solution of the equation (4).

3. A Modified Lotk - Volterra Model

A special case (over-damped) of the model (1) has been discussed by Goh [4]. Goh used

$$\dot{N}_1 = N_1(5.6 - 0.5N_1 - 0.6N_2), \quad \dot{N}_2 = N_2(-1 + N_1 + N_2). \quad (13)$$

where N_1 is the prey density and N_2 is the predator density. There exists a single steady state solution $N_1^* = 10$, $N_2^* = 1$ of (13), obtained from the equilibrium equations $\dot{N}_1 = 0$, $\dot{N}_2 = 0$. Here we see that, one of the roots is much smaller than the other. Goh [4] showed that the equilibrium of the model (13) is locally stable. If the solution initially starts for $N_1^* = 11$, $N_2^* = 3$ it tends rapidly

to $N_1^* = 0$ and $N_2^* = \infty$, although the steady state solution $N_1^* = 10$, $N_2^* = 1$ is not very far from $N_1^* = 11$, $N_2^* = 3$. That is why we are interested to investigate quantitative solutions in the neighborhood of the steady state solution. The solution in the neighborhood of the steady state is presented by x and y , where

$$N_1(t) = N_1^* + \varepsilon x(t), \quad N_2(t) = N_2^* + \varepsilon y(t) \quad (14)$$

Using (13) and (14), we obtain

$$\begin{aligned} \dot{x} &= -(10 + \varepsilon x)(0.5x + 0.6y), \\ \dot{y} &= x + y + \varepsilon(xy + y^2). \end{aligned} \quad (15)$$

Eliminating y from two equations (15) leads to a second order nonlinear differential equation for x

$$\ddot{x} + 4\dot{x} + x = \varepsilon \left(\frac{11}{15}x^2 - \frac{7}{6}x\dot{x} - \frac{1}{15}\dot{x}^2 \right) + O(\varepsilon^2) \quad (16)$$

Here the unperturbed equation, *i. e.* $\ddot{x} + 4\dot{x} + x = 0$ has the roots $-2+\sqrt{3}$ and $-2-\sqrt{3}$. It is clear that the ratio of the roots is 12.34. *i. e.* one of the roots is much smaller than the other.

Therefore, for modeling equation (13), equation (7) becomes

$$\begin{aligned} &(D_0 - \lambda_2)(D_1 a_1) + (D_0 - \lambda_1)(D_1 a_2) \\ &+ (D_0 - \lambda_1)(D_0 - \lambda_2)u_1 = \frac{a_1^2}{30}(22 - 35\lambda_1 - 2\lambda_1^2) \\ &+ \frac{a_2^2}{30}(22 - 35\lambda_2 - 2\lambda_2^2) + \frac{a_1 a_2}{30}(44 - 35\lambda_1 - 35\lambda_2 - 4\lambda_1 \lambda_2) \end{aligned} \quad (17)$$

Using the condition u_1 does not contains the terms $a_1^{i_1} a_2^{i_2}$ where $i_1 \leq 1$ and $\lambda_1 > \lambda_2$, we obtain

$$(D_0 - \lambda_2)(D_1 a_1) = \frac{a_1 a_2}{30}(44 - 35\lambda_1 - 35\lambda_2 - 4\lambda_1 \lambda_2) \quad (18)$$

$$(D_0 - \lambda_1)(D_1 a_2) = \frac{a_2^2}{30}(22 - 35\lambda_2 - 2\lambda_2^2) \quad (19)$$

$$(D_0 - \lambda_1)(D_0 - \lambda_2)u_1 = \frac{a_1^2}{30}(22 - 35\lambda_1 - 2\lambda_1^2) \quad (20)$$

Solving equations (18)-(20), we obtain

$$D_1 a_1 = l_1 a_1 a_2 \quad (21)$$

$$D_1 a_2 = m_1 a_2^2 \quad (22)$$

$$u_1 = n_1 a_1^2 \quad (23)$$

where

$$l_1 = \frac{1}{30\lambda_1}(44 - 35\lambda_1 - 35\lambda_2 - 4\lambda_1\lambda_2), \quad m_1 = \frac{1}{30(2\lambda_2 - \lambda_1)}(22 - 35\lambda_2 - 2\lambda_2^2),$$

$$n_1 = \frac{1}{30\lambda_1(2\lambda_1 - \lambda_2)}(22 - 35\lambda_1 - 2\lambda_1^2)$$

Substituting the values of u_1 from equation (23) into equation (8), utilizing the equations (21)-(22) and imposing the restriction that u_2 excludes the terms $a_1^{i_1} a_2^{i_2}$ where $i_1 \leq 1$ and $\lambda_1 > \lambda_2$, equations for $D_2 a_1$, $D_2 a_2$ and u_2 can be separated into three parts as:

$$(D_0 - \lambda_2)(D_2 a_1) = (-l_1^2 - l_1 m_1 - 7(l_1 + m_1)/6 - 2(m_1 \lambda_1 + l_1 \lambda_2)/15) a_1 a_2^2 \quad (24)$$

$$(D_0 - \lambda_1)(D_2 a_2) = (-2m_1^2 - 7m_1/6 - 2m_1 \lambda_2/15) a_2^3 \quad (25)$$

$$(D - \lambda)(D - \lambda)u_2 = (22n_1/15 - 7n_1 \lambda_1/2 - 4n_1 \lambda_1^2/15) a_1^3 + (22n_1/15 - 7(l_1 + n_1 \lambda_2 + 2n_1 \lambda_1)/6 - (2l_1 \lambda_1 + 4n_1 \lambda_1 \lambda_2) - 6l_1 n_1 \lambda_1) a_1^2 a_2 \quad (26)$$

Solving equations (24)-(26), we obtain

$$D_2 a_1 = l_2 a_1 a_2^2 \quad (27)$$

$$D_2 a_2 = m_2 a_2^3 \quad (28)$$

$$u_2 = n_2 a_1^2 a_2 + n_3 a_1^3 \quad (29)$$

where

$$l_2 = (-l_1^2 - l_1 m_1 - 7(l_1 + m_1)/6 - 2(m_1 \lambda_1 + l_1 \lambda_2)/15)/(\lambda_1 + \lambda_2)$$

$$m_2 = (-2m_1^2 - 7m_1/6 - 2m_1 \lambda_2/15)/(3\lambda_2 - \lambda_1)$$

$$n_2 = (22n_1/15 - 7(l_1 + n_1\lambda_2 + 2n_1\lambda_1)/6 - (2l_1\lambda_1 + 4n_1\lambda_1\lambda_2)/15 - 6l_1n_1\lambda_1)/(2\lambda_1(\lambda_1 + \lambda_2))$$

$$n_3 = (22n_1/15 - 7n_1\lambda_1/2 - 4n_1\lambda_1^2/15)/(2\lambda_1(3\lambda_1 - \lambda_2))$$

All these results obtained from equations (21)-(23) and (27)-(29) give the second approximate solution of equation (16). Now we write the variational equations as follows:

$$\begin{aligned}\dot{a}_1 &= Da_1 = (D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \dots) a_1 = \lambda_1 a_1 + \varepsilon l_1 a_1 a_2 + \varepsilon^2 l_2 a_1 a_2^2 + \dots \\ \dot{a}_2 &= Da_2 = (D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \dots) a_2 = \lambda_2 a_2 + \varepsilon m_1 a_1^2 + \varepsilon^2 m_2 a_1 a_2^3 + \dots\end{aligned}\quad (30)$$

Thus the second approximate solution of the equation (16) is

$$x = a_1 + a_2 + \varepsilon u_1 + \varepsilon^2 u_2 \dots \quad (31)$$

4. Initial Conditions for MTS Method

For the second order system considered in equation (15), we have the initial conditions

$$\begin{aligned}x(0) &= a_{1,0} + a_{2,0} + \varepsilon n_1 a_{1,0}^2 + \varepsilon^2 (n_2 a_{1,0}^2 a_{2,0} + n_3 a_{1,0}^3) \\ \dot{x}(0) &= \lambda_1 a_{1,0} + \lambda_2 a_{2,0} + \varepsilon (l_1 a_{1,0} a_{2,0} + m_1 a_{2,0}^2 + 2n_1 \lambda_1 a_{1,0}^2) \\ &\quad + \varepsilon^2 (l_2 a_{1,0} a_{2,0}^2 + 2l_1 n_1 a_{1,0}^2 a_{2,0} + 2n_2 \lambda_1 a_{1,0}^2 a_{2,0} + n_2 \lambda_2 a_{1,0}^2 a_{2,0} + 3n_3 \lambda_1 a_{1,0}^3 + m_2 a_{2,0}^3)\end{aligned}\quad (32)$$

Usually, in a problem, the initial conditions $[x(0), \dot{x}(0)]$ are specified. Then one has to solve nonlinear algebraic equations in order to determine the two arbitrary constants $a_{1,0}$ and $a_{2,0}$ that appear in the solution (32). Here Newton-Raphson method is used to solve (32) for $x(0) = 0.4$ and $\dot{x}(0) = -2.04$, and we obtained $a_{1,0} = 0.529798$ and $a_{2,0} = -0.153345$.

5. Results and Discussion

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we compare the approximate results to the numerical results. With regard to such a comparison concerning the presented MTS method of this article, we refer to the work of Murty and Deekshatulu

[10] and Shamsul *et al.* [17]. In this article, we have compared the approximate results obtained by (32) (when $k = 2$, $c = 1$ and $\varepsilon = 0.5$) to those obtained by a fourth order Runge-Kutta method.

Here we have considered the equation (13) [special case of modeling equation (1)] in which $N_1^* = 10$, $N_2^* = 1$ (in lakh, 1 lakh=1,00,000). Let us assume that 20 thousands prey have been added to this population. For that we have chosen $y(0) = 0.0$ and $\varepsilon = 0.5$.

First of all, $x(t, \varepsilon)$ has been computed by our asymptotic solution (32) with initial condition $x(0) = 0.4$ and $\dot{x}(0) = -2.04$. Then $N_1(t) = N_1^* + \varepsilon x(t)$ has also been computed. To verify the results, corresponding numerical solutions of $N_1(t)$ has been computed by fourth order Runge-Kutta method. All the results are shown in Fig. 1(a). From Fig. 1(a) it is clear that the asymptotic solution (32) shows excellent agreement with the numerical solution.

To compute $N_2(t)$ or $y(t)$, we have to compute $\dot{x}(t)$. Differentiating $x(t)$ from (32) and then substituting the values of $\dot{x}(t)$ and $x(t)$ into the first equation of (15) and simplifying, we have computed $y(t)$ and then $N_2(t) = N_2^* + \varepsilon y(t)$. Corresponding numerical results of $N_2(t)$ have also been computed and both the results are shown in Fig. 1(b). From Fig. 1 (b), we see that the perturbation results of $N_2(t)$ also agree with the numerical results nicely.

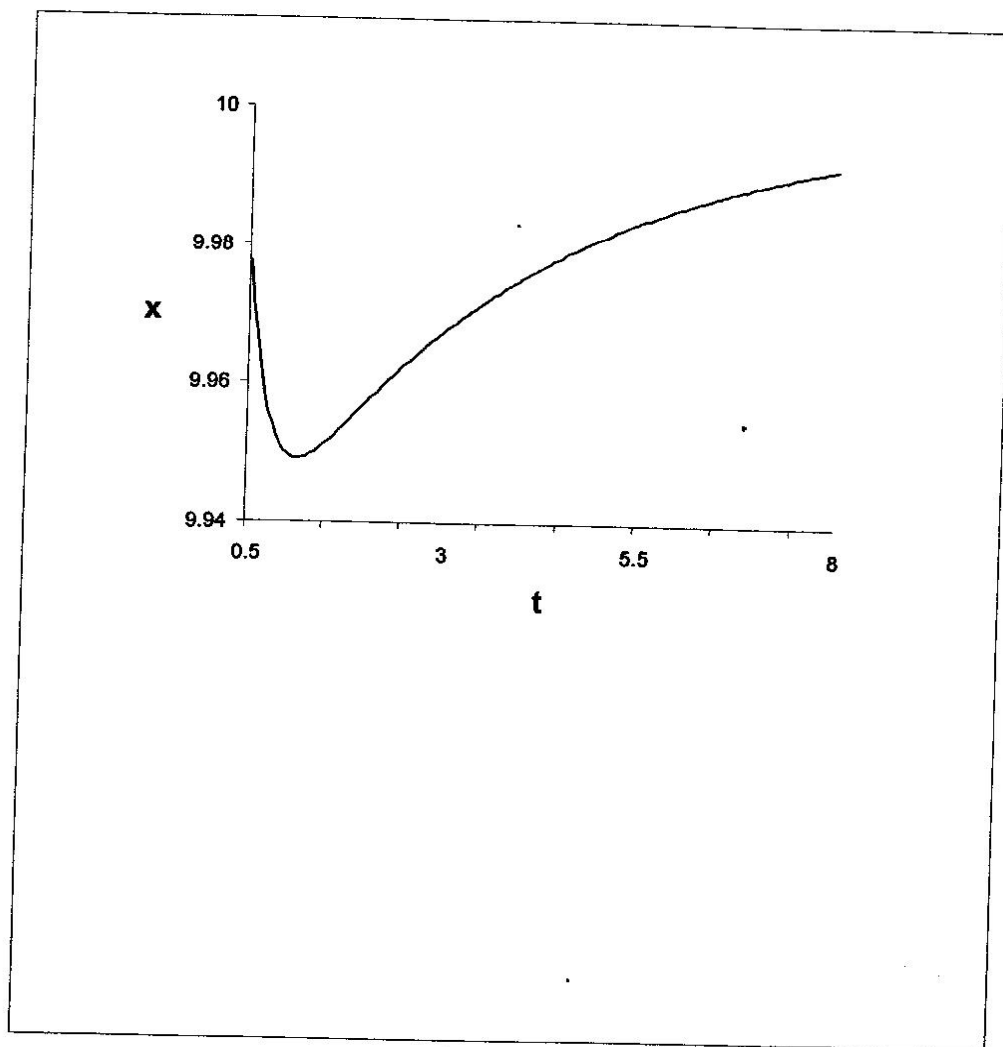


Fig. 1(a): Perturbation solutions (solid line) and numerical solutions (dotted line) of N_1 are computed when $N_1(0)=10.2$ and $N_2(0)=1.0$ [or $N_1=10.2$ and $N_2=1.0$]. In this case, $x(0)=0.4$, $\dot{x}(0)=-2.04$ and $\varepsilon=0.5$ or $a_{1,0}=0.529798$ and $a_{1,2}=-.153345$.

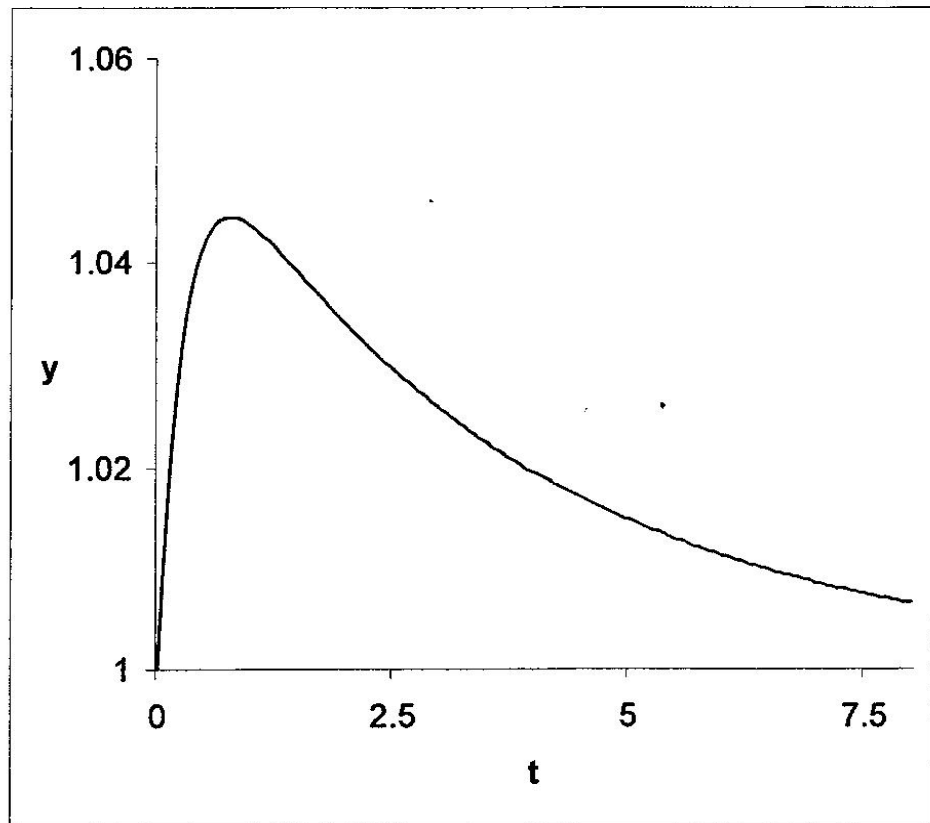


Fig. 1(b): Perturbation solutions (solid line) and numerical solutions (dotted line) of n_2 are computed with the same initial conditions as in Fig. 1(b).

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