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A PROBLEM OF COUPLED THERMO-ELASTICITY IN A SEMI – INFINITE ELASTIC NON–SIMPLE MEDIUM

By

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Abstract

The object of the present paper is to investigate one – dimensional dynamical problem of coupled thermo- elasticity in a semi- infinite elastic non – simple medium when its surface is under suddenly applied constant pressure . The solution of the problem has been deduced using Laplace transform in Bromwich integral form . The author determined the value of the surface displacements in non – simple medium for small values of time t numerically and presented graphically.

Keyword and phrases : *thermo-elasticity, non-simple medium, surface displacements.*

সংক্ষিপ্তসার

বর্তমান গবেষণা পত্রের উদ্দেশ্য হচ্ছে দ্বন্দ্বযুক্ত তাপ - স্থিতিস্থাপকতার একমাত্রিক গতীয় সমস্যার অনুসন্ধান করা অর্ধ-অসীম স্থিতিস্থাপক অ-সরল মাধ্যমের যখন ইহার তল হঠাৎ প্রযুক্ত ধ্রুবক চাপের অধীন। ল্যাপ্লাস (*Laplace*) রূপান্তরের সাহায্যে ব্রোম্‌উইচ্‌ (*Bromwich*) সমাকলন আকারে উক্ত সমস্যার সমাধান নির্ণয় করা হয়েছে। প্রবন্ধকার সময় t-এর ক্ষুদ্রমানের জন্য অ - সরল মাধ্যমে তল - সরণের মান সাংখ্যমানের সাহায্যে নির্ণয় করেছেন এবং ইহাকে লেখচিত্রের সাহায্যে পরিবেশন করেছেন।

1. Introduction

The subject thermo–elasticity is the peripheral topics in the domain of solid mechanics. The rapid progress in the fields of structural engineering, aircraft engineering, nuclear engineering has given rise to wide range of problems where thermoelasticity played vital role. Well known researchers like Sneddon, Nowacki, Longscope, Forrestal and Warren and many others have investigated the disturbances produced in continuous medium taking into consideration suitable thermal fields.

In the context of solid mechanics, a simple material is defined to be one for which entropy, the internal energy, the stress and the heat–flux at a point, are determined in terms of the history of the deformation gradient. Gurtin and Williams in the year 1966 generalized the definition of a simple material to include a second temperature. They however showed that the satisfaction of Clausius – Duhem inequality requires that two temperatures are identical for simple material Chen & Gurtin(1) in the year 1968 proposed a theory of non–simple rigid materials for which two temperatures are not identical. That theory was further extended in the year 1969 by Chen, Gurtin & Williams (2) to deformable bodies. Assuming isotropy and linearity it was shown that two temperatures are related by

$$\theta = T - a.\nabla^2 T, \qquad a \geq 0$$

where θ is the thermodynamic temperature and T is the conductive temperature. The equation of heat conduction for such materials was shown to contain an additional term involving the time derivative of the Laplacian of the conductive temperature, the equations of motion were also shown to contain additional terms involving space derivatives of the Laplacian of the conductive temperature. Kar,T.K.and Lahiri, A.(8) in the year 2008 have solved some one-dimensional problems of thermo – elasticity in a non – simple medium.

In this paper the author deals with a one–dimensional dynamical problem of thermoelasticity in a semi–infinite non–simple medium when its surface is subjected to a suddenly applied constant pressure.

The exact solution of the problem has been obtained by the application of the Laplace transform in Bromwich integral form. Due to the complicated nature of the integrands, it is not possible to get simpler expressions for the solution. So, we have deduced the approximate values for small values of time only. Finally, we have obtained the temperature field and displacement for small values of time of a non-simple medium. Numerical solutions of surface displacement for small values of time have been shown.

2. The Problem and Fundamental Equations

We consider a semi-infinite elastic non-simple medium occupying the space $x \geq 0$. The plane surface $x = 0$ is subjected to impulsive normal pressure having Heaviside step function time dependent.

The coupled heat conduction equation, (vide, Nowacki[3]) is given by

$$\nabla^2 T - \frac{1}{K_1}\dot{T} - \eta\dot{e} = -\frac{Q}{K_1}$$

In the case of non-simple medium, the above equation (Vide, Iesan[4]) takes the form

$$\nabla^2 T + \frac{a}{K_1}\nabla^2\dot{T} - \frac{1}{K_1}\dot{T} - \eta\dot{e} + \frac{r}{K_1} = -\frac{Q}{K_1}$$

where T is the conductive temperature, a is the temperature discrepancy factor, r is the heat supplied per unit volume from outside and other symbols have the same meanings as mentioned in Nowacki[3]. The equations of motion (vide, Chen, Gurtin and Williams [2] (1969)), are given by

$$\mu U_{i,kk} + (\lambda+\mu).U_{k,ki} + F_i - \gamma.(T_{,i} - a\nabla^2 T_{,i}) = \rho\,\ddot{U}_i\,.$$

We assume the solid to be mechanically constrained so that the displacement U and temperature T depend on x and t only.

Assuming r = 0, $F_i = 0$ and Q = 0, the basic equations reduce to the forms

$$\left[D^2 + \frac{a}{K_1} D^2 D' - \frac{1}{K_1} D'\right] T - \eta D D' U = 0 \tag{1}$$

$$\left[(\lambda + 2\mu).D^2 - \rho.D'^2\right] U - \gamma.D\left[1 - aD^2\right] T = 0, \tag{2}$$

where D and D' denote respectively partial differentiation operator with respect to x and t.

We derive the non–dimension form of the above equations by means of the transformations (vide, Chakraborty[5]) and are given by

$$(U, T, x, t) \rightarrow \left(\ell_o U, \frac{T}{\alpha_1}, \ell_o x, t_o t\right)$$

where

$$\ell_o = \frac{K_1}{V}, t_o = \frac{\ell_o}{V}, \alpha_1 = \frac{\gamma}{\rho V^2}, V^2 = \frac{(\lambda + 2\mu)}{\rho},$$

so that we obtain the following equations

$$[D^2 + bD^2 D' - D'] T - \varepsilon D D' U = 0 \tag{3}$$

$$[D^2 - D'^2] U - D(1 - bD^2) T = 0 \tag{4}$$

where

$$b = \frac{aV^2}{K_1^2}, \qquad \varepsilon = \frac{\eta K_1 \gamma}{\rho V^2}$$

The equations (3) and (4) are to be solved subject to the following conditions

$$U(x,0) = T(x,0) = \frac{\partial U(x,0)}{\partial t} = 0 \qquad \text{for } x > 0, \tag{5}$$

$$\sigma(0,t) = \sigma_o H(t), T(0,t) = 0 \qquad \text{for } t > 0, \tag{6}$$

$$U, T, \frac{\partial U}{\partial x}, \frac{\partial U}{\partial t} \to 0 \qquad \text{as } x \to \infty \tag{7}$$

where σ_0, T_0 and a are constants, H(t) is the Heviside step function and $\sigma(x, t)$ is given by

$$\sigma(x,t) = \frac{\partial U(x,t)}{\partial x}.$$

3. Solution of The Problem

Applying Laplace Transforms

$$[\overline{U}(x,s), \overline{T}(x,s)] = \int_0^\infty [U(x,t), T(x,t)]e^{-st}dt$$

to equations (3) to (7), we obtain

$$[D^2 + bsD^2 - s]\overline{T} - \varepsilon sD\overline{U} = 0 \tag{8}$$

$$D.(1 - bD^2)\overline{T} - [D^2 - s^2].\overline{U} = 0 \tag{9}$$

with

$$\left.\begin{array}{ll} \text{(i)} & \overline{U}(x,s), \overline{T}(x,s) \to 0 \quad \text{as } x \to \infty \\ \text{(ii)} & \left.\begin{array}{l} \overline{T} = 0 \\ \\ \overline{\sigma} = \dfrac{d\overline{U}}{dx} = \dfrac{\sigma_o}{s} \end{array}\right\} \text{on } x = 0 \\ \text{(iii)} & \end{array}\right\} \tag{10}$$

The solutions of (8) and (9) satisfying the regularity conditions of [10(i)] can be taken as

$$\left.\begin{aligned} \overline{U}(x,s) &= A_1 e^{-p_1 x} + B_1 e^{-p_2 x} \\ \overline{T}(x,s) &= A e^{-p_1 x} + B e^{-p_2 x} \end{aligned}\right\} \quad (11)$$

where p_1^2 and p_2^2 are the roots of the biquadratic equation

$$x^4\{1+bs(1+\varepsilon)\} - s\{s(1+bs)+1+\varepsilon\}x^2 + s^3 = 0 \quad (12)$$

Hence, we get

$$p_1^2 + p_2^2 = \frac{s\{s(1+bs)+1+\varepsilon\}}{\{1+bs(1+\varepsilon)\}},$$

$$p_1^2 p_2^2 = \frac{s^3}{\{1+bs(1+\varepsilon)\}},$$

and

$$(p_1, p_2) = \frac{\sqrt{s}(\alpha_o \pm \beta_o)}{2\sqrt{1+bs(1+\varepsilon)}},$$

where

$$\alpha_o = \left[1+\varepsilon+s(1+bs)+2\sqrt{s}.\sqrt{1+bs(1+\varepsilon)}\right]^{1/2}$$

$$\beta_o = \left[1+\varepsilon+s(1+bs)-2\sqrt{s}.\sqrt{1+bs(1+\varepsilon)}\right]^{1/2}.$$

Now the boundary conditions (10(ii)) and (10(iii)) with the aid of (11) give

$$\left.\begin{aligned} &B=-A \\ &B_1=\left(\frac{\sigma_o}{p_2 s}+\frac{p_1}{p_2}A_1\right) \end{aligned}\right\} \tag{13}$$

Replacing B_1 and B by the above relations, we obtain from (11) with the aid of (13), we get

$$\left.\begin{aligned} &\overline{U}=A_1.e^{-p_1x}-\left(\frac{p_1}{p_2}A_1+\frac{\sigma_o}{sp_2}\right).e^{-p_2x}, \\ &\overline{T}=A\left(e^{-p_1x}-e^{-p_2x}\right) \end{aligned}\right\} \tag{14}$$

Now substituting the solution (14) in the differential equation (9), we obtain

$$e^{-p_1x}\left[-p_1A+bp_1^3A\right]+e^{-p_2x}\left[p_2A-bp_2^3A\right]=e^{-p_1x}\left[p_1^2A_1-s^2A_1\right]$$

$$+e^{-p_2x}\left[A_1\frac{s^2p_1}{p_2}-p_1p_2A_1-\frac{\sigma_o p_2}{s}+\frac{\sigma_o s^2}{p_2}\right]$$

This will be satisfied if

$$-p_1A+bp_1^3A=p_1^2A_1-s^2A_1$$

$$p_2A-bAp_2^3=-p_1p_2A-\frac{\sigma_o}{s}p_2+\frac{p_1}{p_2}s^2A_1+\frac{\sigma_o}{p_2}s$$

Solving for A and A_1 we get

$$A=\frac{\sigma_o(p_1^2-s^2)(s^2-p_2^2)}{s\left[p_2^2(p_1^2-s^2)(1-bp_2^2)+p_1^2(p_2^2-s^2)(bp_1^2-1)\right]}$$

and

$$A_1 = \frac{\sigma_o(s^2 - p_2^2)(bp_1^2 - 1)p_1}{s\left[p_2^2(p_1^2 - s^2)(1 - bp_2^2) + p_1^2(p_2^2 - s^2)(bp_1^2 - 1)\right]}$$

Hence we obtain,

$$\overline{T}(x,s) = \frac{\sigma_o(p_1^2 - s^2)(s^2 - p_2^2)\{e^{-p_1x} - e^{-p_2x})}{s\left[p_2^2(p_1^2 - s^2)(1 - bp_2^2) + p_1^2(p_2^2 - s^2)(bp_1^2 - 1)\right]} \qquad (15)$$

and

$$\overline{U}(x,s) = \frac{\sigma_o(s^2 - p_2^2)(bp_1^2 - 1)p_1\{e^{-p_1x} - \frac{p_1}{p_2}e^{-p_2x}\}}{s\left[p_2^2(p_1^2 - s^2)(1 - bp_2^2) + p_1^2(p_2^2 - s^2)(bp_1^2 - 1)\right]} - \frac{\sigma_o}{sp_2}e^{-p_2x} \qquad (16)$$

Taking the inverse Transform of (15) and (16), we obtain

$$T(x,t) = \frac{\sigma_o}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{\sigma_o(p_1^2 - s^2)(s^2 - p_2^2)\{e^{-p_1x} - e^{-p_2x})}{s\left[p_2^2(p_1^2 - s^2)(1 - bp_2^2) + p_1^2(p_2^2 - s^2)(bp_1^2 - 1)\right]}\right]e^{st}\,ds \qquad (17)$$

$$U(x,t) = \frac{\sigma_o}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{\sigma_o(s^2 - p_2^2)(bp_1^2 - 1)p_1\{e^{-p_1x} - \frac{p_1}{p_2}e^{-p_2x}\}}{s\left[p_2^2(p_1^2 - s^2)(1 - bp_2^2) + p_1^2(p_2^2 - s^2)(bp_1^2 - 1)\right]} - \frac{\sigma_o}{sp_2}e^{-p_2x}\right]e^{st}\,ds \qquad (18)$$

where all the singularities lie to the loft of the Bromwich contour Re s = C.

It is observed from equation (17) and (18) that since the integrands are too much complicated, the exact solution for non–simple material cannot be evaluated easily. Therefore, we have obtained approximate solutions for small values of time to get an idea about the initial nature of the disturbances in the medium.

4. Approximate Solutions

Now, we calculate the thermo-elastic deformation and temperature field for small values of time. In this case the Laplace parameter s is large and the expressions will be expanded in inverse powers of s, neglecting squares and higher powers of s. We note that

$$p_1 \simeq \frac{s}{\sqrt{1+\varepsilon}} + N_1 \quad ; \; p_2 \simeq \frac{1}{\sqrt{b}}, \quad \text{as } s \to \infty$$

where

$N_1 = \frac{\varepsilon}{2b(1+\varepsilon)^{3/2}}$, ε being the thermo–elastic coupling factor.

Therefore from equations (16) and (15) we obtain

$$\frac{\bar{U}(x,s)}{\sigma_0} = -\left[\frac{\sqrt{1+\varepsilon}}{s^2} - \frac{N_1(1+\varepsilon)}{s^3}\right]\left[e^{-\left(\frac{s}{\sqrt{1+\varepsilon}}+N_1\right)x} - \left(\frac{s\sqrt{b}}{\sqrt{1+\varepsilon}} + N_1\sqrt{b}\right)e^{-x/\sqrt{b}}\right] - \frac{\sqrt{b}}{s}e^{-x/\sqrt{b}} \tag{19}$$

and

$$\frac{\bar{T}(x,s)}{\sigma_0} = -\frac{1}{b}\left[\frac{2N_1(1+\varepsilon)^{3/2}(1+2\varepsilon)}{s} - \frac{\varepsilon(1+\varepsilon)}{s^3}\right]\left[e^{-\left(\frac{s}{\sqrt{1+\varepsilon}}+N_1\right)x} - e^{-x/\sqrt{b}}\right] \tag{20}$$

The inversions of (19) and (20), obtained by using standard results [vide, Erdelyi[6]] are

$$-\frac{U(x,t)}{\sigma_0} = \sqrt{1+\varepsilon}.e^{-N_1 x}\left\{t - \frac{x}{\sqrt{1+\varepsilon}}\right\}.H\left\{t - \frac{x}{\sqrt{1+\varepsilon}}\right\}$$

$$-N_1(1+\varepsilon)e^{-N_1 x}\left\{\frac{\left\{t-\frac{x}{\sqrt{1+\varepsilon}}\right\}^2}{2}\right\}.H\left\{t-\frac{x}{\sqrt{1+\varepsilon}}\right\}$$

$$+N_1^2\sqrt{b}(1+\varepsilon)e^{-x/\sqrt{b}}.\frac{t^2}{2}H(t) \tag{21}$$

and

$$-\frac{T(x,t)}{\sigma_o}=\frac{1}{b}\left[2N_1(1+\varepsilon)^{3/2}(1+2\varepsilon)e^{-N_1x}H\left\{t-\frac{x}{\sqrt{1+\varepsilon}}\right\}\right.$$

$$-\varepsilon(1+\varepsilon)e^{-N_1x}\left\{\frac{t-\dfrac{x}{\sqrt{1+\varepsilon}}}{2}\right\}^2.H\left\{t-\frac{x}{\sqrt{1+\varepsilon}}\right\}$$

$$\left.-2N_1(1+\varepsilon)^{5/2}e^{-x/\sqrt{b}}H(t)+\varepsilon(1+\varepsilon)e^{-x/\sqrt{b}}\frac{t^2}{2}H(t)\right] \qquad (22)$$

For the displacement on the surface x = 0, the expression (21) becomes

$$-\frac{U(0,t)}{\sigma_o}=\sqrt{1+\varepsilon}\,t\,H(t)-\frac{1}{2}N_1(1+\varepsilon)t^2H(t)+N_1^2\sqrt{b}(1+\varepsilon)\frac{t^2}{2}\dot{H}(t) \qquad (23)$$

5. Numerical Calculations for Surface Displacements

Considering the material of the half-space to be copper, the values of the surface displacements U(0, t) in non-simple medium for small values of time t have been computed and are presented is table as well as in Figure. The values of the material constants are taken as follows:

$\varepsilon = 0.0168$, $\lambda = 1.38 \times 10^{12}$,

$\mu = 0.477 \times 10^{12}$, $\rho = 8.94$.

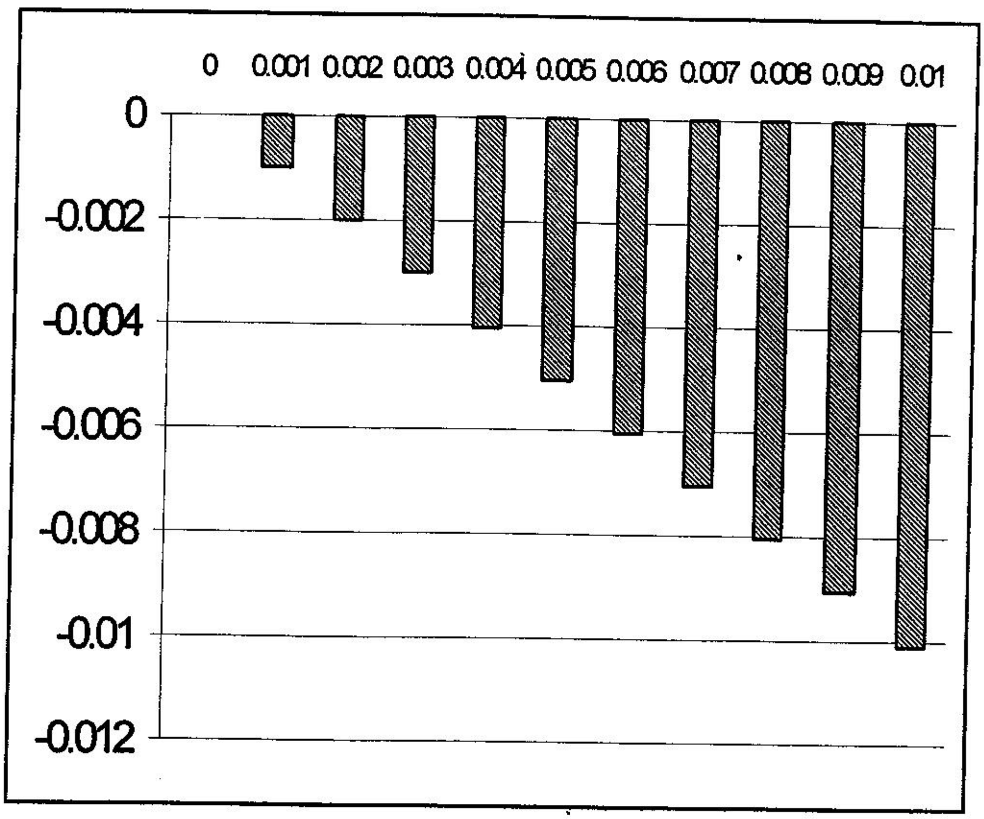
TABLE

Surface displacement for small values of time in a non-simple medium

t	0	0.001	0.002	0.003	0.004	0.005
$\frac{U(0,t)}{\sigma_o}$	0	-0.001008	-0.002016	-0.003024	-0.004032	-0.005040

t	0.006	0.007	0.008	0.009	0.01
$\frac{U(0,t)}{\sigma_o}$	-0.006048	-0.007056	-0.008064	-0.009072	-0.010080

FIGURE



Conclusion

From the graphical presentation we observe that for a small value of time in a non-simple medium the surface displacement increases at t increases.

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