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THE EFFECT OF GRAVITY ON THE PROPAGATION OF WAVES IN AN ELASTIC LAYER IMMERSED IN AN INFINITE LIQUID

By

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Abstract

The object of the present paper is to investigate the propagation of waves in an elastic layer immersed in an infinite liquid and under the influence of gravity. The corresponding velocity equation has been derived. In the limiting case the wave velocity equation so obtained is in good agreement with the corresponding classical problem when gravitational effects are vanishing small .

Keyword and phrases : *elastic layer, propagation of waves, effect of gravity.*

সংক্ষিপ্তসার

এই পত্রের উদ্দেশ্য হচ্ছে অসীম তরলে নিমজ্জিত এবং অভিকর্ষের প্রভাবাধীন স্থিতিস্থাপক সমতল স্তরে তরঙ্গের প্রবাহকে অনুসন্ধান করা। ইহার আনুষঙ্গিক তরঙ্গবেগ সমীকরণ নির্ণয় করা হয়েছে। নির্ণিত তরঙ্গবেগ সমীকরণটি অনুরূপ সুবিদিত সমীকরণের সঙ্গে সঙ্গতিপূর্ণ যখন সীমাস্থ ক্ষেত্র হিসাবে অভিকর্ষের প্রভাবকে না থাকার মত ক্ষুদ্র হিসাবে ধরা হয়।

1. Introduction

The influence of gravity on elastic waves is receiving greater attention by many investigators [5,6] owing to its theoretical and practical interests. The effect of gravity are presented to a limited extent by Ewing, Jardtsky, and Press in their monograph. Firstly Bromwich [3] considered effects of gravity on elastic waves and in particular on an elastic globe. The influence of gravity on superficial waves was investigated by Love [4] and it was shown that the Rayligh wave velocity is effected by the gravity field. In this analysis Love has shown that the effect of gravity increases the Rayligh wave velocity to some significant amount when the wave length is large. Also Biot [2] investigated the influence of gravity on Rayligh waves assuming the force of gravity to create a type of initial stress of hydrostatic nature and the medium to be incompressible. Sengupta, P.R. and De, S.N. [5] investigated influence of gravity on wave propagation in an elastic layer.

As Biot [2] presented in his illustrious work the dynamical equation of motion for a homogeneous isotropic elastic solid medium under initial stress, the author of the present paper has derived the wave equations satisfied by the displacement potentials ϕ, ψ related to this problem. The effect of gravity on the propagation of waves in an elastic layer.immersed in an infinite liquid has been investigated.

2. Statement of the Problem and the Boundary Condition

Let us introduce a Cartesian frame of reference $ox_1x_2x_3$, taking the origine in the middle plane of the elastic layer; the middle plane coincides with the plane ox_1x_2 . Let x_3-axis directed downwards and $x_3 = \pm h$, be the boundary planes of the layer, which is a monochromatic wave propagates with constant velocity c along the x_1 – axis. A plane longitudinal wave in the infinite space would be propagated with velocity c_1 and a transverse one with velocity c_2 . The non – zero displacement components are u_1 and u_3 depending only on the co-ordinates x_1 and x_3 and time t.

If g is the acceleration due to gravity then the components of the body force are X = 0 and Z = g. We shall assume that the initial stress due to gravity is hydrostatic. The state of initial stress are

$$\sigma_{11} = \sigma_{33} = S, \sigma_{13} = 0 \qquad (1)$$

Where S is the function of depth. The equilibrium condition of the initial stress field are

$$\frac{\partial s}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial s}{\partial x_3} + \rho g = 0 \qquad (2)$$

Where ρ is the density of the elastic layer

The dynamical equations of the two dimensional problem under initial stress field are

$$\frac{\partial \sigma_{11}}{\partial x_1}+\frac{\partial \sigma_{33}}{\partial x_3}+\rho g\frac{\partial u_3}{\partial x_1}=\rho\frac{\partial^2 u_1}{\partial t^2} \tag{3}$$

$$\frac{\partial \sigma_{31}}{\partial x_1}+\frac{\partial \sigma_{33}}{\partial x_3}-\rho g\frac{\partial u_1}{\partial x_1}=\rho\frac{\partial^2 u_3}{\partial t^2}$$

where

$$\sigma_{jk} = 2\mu e_{jk} + \lambda e \delta_{jk};$$

$$e_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k}+\frac{\partial u_k}{\partial x_j}\right)$$

and λ and μ are Lami's constants.

Assuming that the displacement u_1 and u_3 are derivable from the displacement potentials $\varphi(x_1,x_3,t)$ and $\psi(x_1,x_3,t)$ by the relations

$$u_1=\frac{\partial \phi}{\partial x_1}-\frac{\partial \psi}{\partial x_3} \quad \text{and} \quad u_3=\frac{\partial \phi}{\partial x_3}+\frac{\partial \psi}{\partial x_1} \tag{4}$$

we obtain the wave equation

$$\nabla^2\phi-\frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2}+\frac{g}{c_1^2}\frac{\partial \psi}{\partial x_1}=0 \tag{5.1}$$

$$\nabla^2\psi-\frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2}-\frac{g}{c_2^2}\frac{\partial \phi}{\partial x_1}=0 \tag{5.2}$$

where

$$c_1^2=\frac{\lambda+2\mu}{\rho}, \quad c_2^2=\frac{\mu}{\rho}, \quad \nabla^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_3^2} \tag{6}$$

The following boundary conditions should be satisfied on the edge (at solid-liquid interface) of the layer $x_3 = \pm h$,

$$\sigma_{33} = \sigma'_{33}\ , \quad \sigma_{13} = 0 \quad \text{and} \quad u_3 = u'_3 \tag{7}$$

The unprimed and primed quantities are related to the solid and liquid respectively.

Due to the participation of the liquid, let us introduce the potentials φ_0, φ_2 respectively for the liquid above and below the layer

Then

$$u'_1 = \phi_{0,1} \quad ; \qquad u'_3 = \phi_{0,3} \tag{8}$$

and

$$u'_1 = \phi_{2,1} \quad ; \qquad u'_3 = \phi_{2,3}$$

are respectively the displacements of liquids above and below the layer.

The appropriate solutions of the equations of motion for the liquid are

$$\phi_0 = A_0 e^{v_0 x_3} + i(\omega t - \alpha x_1) \tag{9}$$

$$\phi_2 = A_2 e^{-v_0 x_3} + i(\omega t - \alpha x_1)$$

where,

$$v_0 = (\alpha^2 - K_{\alpha_0}^2)^{\frac{1}{2}}, \quad K_{\alpha_0}^2 = \frac{\omega^2}{\alpha_0^2} \quad \text{and} \quad \alpha_0^2 = \frac{\lambda_0}{\rho_0} \tag{10}$$

$\lambda_0, \rho_0, \alpha_0$ are the constants characterizing the properties of the liquid.

Now, the normal stress σ'_{33} for the liquid is

$$\sigma'_{33} = -\rho_0\omega^2\phi_0 \quad \text{or} \quad -\rho_0\omega^2\phi_2 \tag{11}$$

according as the liquid is above or below the layer.

The force stresses σ_{ij} may be expressed for a layer in terms of the displacement potentials, we have

$$\left.\begin{aligned}\sigma_{33} &= 2\mu(\phi_{,33}+\psi_{,13})+\lambda\nabla^2\phi \\ \sigma_{31} &= \mu(2\phi_{,13}-\psi_{,33}+\psi_{,11})\end{aligned}\right\} \tag{12}$$

Now, we shall find out the solution of the equations (5.1) and (5.2) subject to the boundary conditions (7).

3. Solution and Phase Velocity Equation.

Let us assume,

$$\left.\begin{aligned}\phi &= \phi^*(x_3)e^{i(\omega t-\alpha x_1)} \\ \psi &= \psi^*(x_3)e^{i(\omega t-\alpha x_1)}\end{aligned}\right\} \tag{13}$$

Introducing equations (13) into equations (5.1) and (5.2) we get,

$$\left(\frac{\partial^2}{\partial x_3^2}-\nu_1^2\right)\phi^*-\frac{i\alpha g}{c_1^2}\psi^*=0 \tag{14.1}$$

$$\left(\frac{\partial^2}{\partial x_3^2}-\nu_2^2\right)\psi^*+\frac{i\alpha g}{c_2^2}\phi^*=0 \tag{14.2}$$

where

$$\nu_1^2=\alpha^2-\frac{\omega^2}{c_1^2} \quad \text{and} \quad \nu_2^2=\alpha^2-\frac{\omega^2}{c_2^2}$$

Eleminating ϕ^* or ψ^* from the equation (14.1) and (14.2) we obtain the wave velocity equation

$$\left(\frac{\partial^2}{\partial x_3^2}-\lambda_1^2\right)\left(\frac{\partial^2}{\partial x_3^2}-\lambda_2^2\right)(\phi^*,\psi^*)=0 \tag{15}$$

where

$$\left.\begin{aligned} \lambda_1^2+\lambda_2^2 &= \nu_1^2+\nu_2^2 \\ \lambda_1^2\lambda_2^2 &= \nu_1^2\nu_2^2-\nu_3^2 \\ \nu_3^2 &= \frac{\alpha^2 g^2}{c_1^2 c_2^2} \end{aligned}\right\} \tag{16}$$

obviously the solution of the equations (15) are

$$\phi^* = A\sinh\lambda_1 x_3 + BCosh\lambda_1 x_3 + C\sinh\lambda_2 x_3 + D\cosh\lambda_2 x_3 \tag{17}$$

$$\psi^* = A'\sinh\lambda_1 x_3 + B'Cosh\lambda_1 x_3 + C'\sinh\lambda_2 x_3 + D'\cosh\lambda_2 x_3$$

where A', B', C', D' are related respectively to A, B, C, D by means of equations (14.1) and (14.2). Equating the coefficients of

$\sinh\lambda_1 x_3$, $Cosh\lambda_1 x_3$, $\sinh\lambda_2 x_3$, $Cosh\lambda_2 x_3$

to zero, we obtain from the equations (14.1) and (14.2).

$$A' = -im_1 A, B' = -im_1 B, C' = -im_2 CandD' = -im_2 D \tag{18}$$

where,

$$m_j = c_1^2(\lambda_i^2 - \nu_1^2)/\alpha g, i = 1,2$$

Using the relations (18) into equations (17)we get

$$\left.\begin{aligned} \phi &= (A\sinh\lambda_1 x_3 + BCosh\lambda_1 x_3 + C\sinh\lambda_2 x_3 + D\cosh\lambda_2 x_3)e^{i(\omega t-\alpha x_1)} \\ \psi &= -i(m_1 A\sinh\lambda_1 x_3 + m_1 BCosh\lambda_1 x_3 + m_2 C\sinh\lambda_2 x_3 + m_2 D\cosh\lambda_2 x_3)e^{i(\omega t-\alpha x_1)} \end{aligned}\right\} \tag{19}$$

Now, introducing the value of ϕ and ψ from equation (19) into equation (7), (8), (11) and using the boundary conditions we arrive at the following equations :

$$(\xi_1 p_1 - \eta_1 q_1)A + (\xi_1 q_1 - \eta_1 p_1)B + (\xi_2 p_2 - \eta_1 q_1)C + (\xi_2 q_2 - \eta_2 p_2)D + f_1 e^{i\omega h} A_0 = 0 \tag{20.1}$$

$$-(\xi_1 p_1 + \eta_1 q_1)A + (\xi_1 q_1 + \eta_1 p_1)B - (\xi_2 p_2 + \eta_1 q_1)C + (\xi_2 q_2 + \eta_2 p_2)D + f_1 e^{i\omega h} A_0 = 0 \tag{20.2}$$

$$(n_1 p_1 - l_1 q_1)A + (n_1 q_1 - l_1 p_1)B + (n_2 p_2 - l_1 q_1)C + (n_2 q_2 - l_2 p_2)D = 0 \tag{20.3}$$

$$-(n_1 p_1 + l_1 q_1)A + (n_1 q_1 + l_1 p_1)B - (n_2 p_2 + l_1 q_1)C + (n_2 q_2 + l_2 p_2)D = 0 \tag{20.4}$$

$$(s_1 p_1 - \lambda_1 q_1)A + (s_1 q_1 - \lambda_1 p_1)B + (s_2 p_2 - \lambda_2 q_2)C + (s_2 q_2 - \lambda_2 p_2)D + \nu_0 e^{i\omega h} A_0 = 0 \tag{20.5}$$

$$(s_1 p_1 + \lambda_1 q_1)A - (s_1 q_1 + \lambda_1 p_1)B + (s_2 p_2 + \lambda_2 q_2)C - (s_2 q_2 + \lambda_2 p_2)D + \nu_0 e^{i\omega h} A_2 = 0 \tag{20.6}$$

Eliminating A,B,C,D, A_0 and A_2 from equations (20.1) to (20.6) we obtain the wave velocity equation.

$$\Delta = \begin{vmatrix} \xi_1 p_1 & \eta_1 p_1 & \xi_2 p_2 & \eta_2 p_2 & -f_1 & 0 \\ n_1 p_1 & l_1 p_1 & n_2 p_2 & l_2 p_2 & 0 & 0 \\ s_1 p_1 & \lambda_1 p_1 & s_2 p_2 & \lambda_2 p_2 & 0 & \nu_0 \\ \eta_1 q_1 & \xi_1 q_1 & \eta_2 q_2 & \xi_2 q_2 & 0 & -f_1 \\ l_1 q_1 & n_1 q_1 & l_2 q_2 & n_2 q_2 & 0 & 0 \\ \lambda_1 q_1 & s_1 q_1 & \lambda_2 q_2 & s_2 q_2 & \nu_0 & 0 \end{vmatrix} = 0 \tag{21}$$

when,

$$\left.\begin{aligned} \xi_j &= c_1^2(\lambda_1^2 - \alpha^2) + 2c_2^2\alpha^2 \\ \eta_j &= 2c_2^2 \alpha \lambda_j m_j \\ n_j &= m_j(\lambda_j^2 + \alpha^2) \\ l_j &= 2\alpha\lambda_j \\ s_j &= \alpha m_j \\ f_1 &= \frac{\rho_0 \omega^2}{\rho} \end{aligned}\right\} \qquad (21)$$

4. Discussion

The transcendental equation is in the determinantal form. It represents the wave velocity equation of wave propagated in the elastic solid layer immersed in an infinite liquid under the influence of gravity only.

If the length of the wave is very small with respect to its thickness 2h of the layer, the quantities $\lambda_j h$, j = 1,2 are large and the approximations.

$$\frac{p_j}{q_j} = \tanh \lambda_j h \cong 1, \quad j = 1, 2 \tag{23}$$

Now, we introduce equation (23) into equation (21) and use the properties of determinants for simplification. We have obtained Rayligh wave velocity equation in layered elastic solid medium immersed in an infinite liquid under the influence of gravity only.

Thus we get,

$$\Delta = \Delta_1 . \Delta_2 = 0 \tag{24}$$

when,

$$\Delta_1 = \begin{vmatrix} \xi_1 + \eta_1 & \xi_2 + \eta_2 & -f_1 + 0 \\ n_1 + l_1 & n_2 + l_2 & 0 + 0 \\ s_1 + \lambda_1 & s_2 + \lambda_2 & 0 + \nu_0 \end{vmatrix} \tag{25}$$

and

$$\Delta_2 = \begin{vmatrix} \xi_1 - \eta_1 & \xi_2 - \eta_2 & -f_1 - 0 \\ n_1 - l_1 & n_2 - l_2 & 0 - 0 \\ s_1 - \lambda_1 & s_2 - \lambda_2 & 0 - \nu_0 \end{vmatrix} \tag{26}$$

From equation (24) we have

$$\text{Either} \quad \Delta_1 = 0 \quad \text{or} \quad \Delta_2 = 0 \tag{27}$$

From equation (27) it is clear that $\Delta_1 = 0$, represents the wave velocity equation corresponding to Rayligh waves propagating in a semi infinite medium under the influence of gravity having the plane horizontal boundary in the upper most part of the elastic solid while the equation $\Delta_2 = 0$ is the wave velocity equation for a semi-infinite medium under the influence of gravity having its horizontal plane boundary in the lower most part of the solid, though the thickness of the layer is assumed to be finite and large, in case of Rayligh wave we shall consider that it is a semi-infinite medium with a plane boundary existing at the upper most part or at the lower most of the semi-infinite medium and that is why Rayligh wave velocity equation occurs twice according as the free plane boundary is at the upper most side or lower most side of the medium.

If the plane boundary be at the upper most side, the medium is extended to infinity at the lower and the wave velocity equation $\Delta_1 = 0$ represents the type of wave propagating in the vicinity of the free plane upper boundary. In a similar manner the wave velocity equation $\Delta_2 = 0$ represents the type of Rayligh wave propagating in the vicinity of the lower plane boundary treating the medium to be extended towards infinity at the upper side.

It is obvious from the mathematical form of the equations $\Delta_1 = 0$ and $\Delta_2 = 0$, that they are interchangeable simply by changing $\lambda_j by - \lambda_j$ (j=1,2) i.e. by changing the direction of the x_3- axis. It explains the existence of 2-wave velocity equations.

Let us first study the equation $\Delta_1 = 0$ and the equation $\Delta_2 = 0$, may be treated similarly.

Irrespective of the thickness of the layer and in absence of liquid when we consider that the effect of gravity on the propagation of wave in an infinite solid medium we take $f_1 = 0$ in the equation $\Delta_1 = 0$ of equation (25), we get the wave velocity equation of Rayligh waves under the influence of gravity only.

$$(\xi_1+\eta_1)(n_2+l_2)=(\xi_2+\eta_2)(n_1+l_1) \tag{28}$$

This is in good agreement with the paper of De, S.N.and Sengupta P.R.[5].

Let us consider the length of the wave is large compared with the thickness of the layer, the quantities $\lambda_1 h, \lambda_2 h$ and αh can be regarded as small and hyperbolic tangents are replaced by their arguments putting the values of ξ_j, η_j, n_j, l_j when j=1,2 and neglecting the effect of gravity in equation (29) we obtain

$$c^2\, c_1^2 = 4\, c_2^2\, (c_1^2 - c_2^2) \tag{29}$$

which is classical equation and also determines the wave velocity in the elastic layer.

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