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AN APPROXIMATE TECHNIQUE TO DUFFING EQUATION WITH SMALL DAMPING AND SLOWLY VARYING COEFFICIENTS

By

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Abstract

In this article, an approximate technique has been presented for obtaining the analytical approximate solutions of second order strongly nonlinear differential systems with small damping and slowly varying coefficients based on the He's homotopy perturbation and the extended form of the Krylov-Bogoliubov-Mitropolskii methods. An example is given to illustrate the efficiency and implementation of the presented method. The first order analytical approximate solutions obtained by the presented method show a good agreement with the corresponding numerical solutions for the several damping effects.

Keyword and phrases : *doffing equation, damping effect, homotopy perturbation, varying coefficients*

সংক্ষিপ্তসার

এই নিবন্ধে ক্রাইলভ্ - বোগোলিউভব্ - মিট্রোপোল্স্কি (Krylov - Bogoliubov -Mitropolskii) পদ্ধতির বর্ধিত আকার এবং ধীর পরিবর্তী সহগগুলিকে হী'র হোমোটোপী (He's homotopy) বিচলনের উপর ভিত্তি করে ক্ষুদ্র অবমন্দনসহ দ্বিতীয় ক্রমের দৃঢ অরৈখিক অবকলন তন্ত্রের বিশ্লেষণাত্বক আসন্নমানে সমাধান নির্ণয়ের জন্য এক আসন্নমান কৃৎকৌশল পরিবেশন করা হয়েছে। একটি উদাহরণের সাহায্যে উক্ত পরিবেশিত পদ্ধতির কার্যকারী রূপ এবং কর্মক্ষমতাকে দেখানো হয়েছে। উক্ত পদ্ধতির সাহায্যে প্রথম ক্রমের বিশ্লেষণাত্বক আসন্নমানে নির্ণিত সমাধানটি দেখায় যে অনুরূপ বিবিধ অবমন্দনের প্রভাবের জন্য সাংখ্যমানে নির্ণিত সমাধানের সঙ্গেঁ ইহা খুবই সঙ্গঁতিপূর্ণ।

1. Introduction

Most of the physical phenomena and engineering problems occur in nature in the forms of nonlinear differential systems with damping effects and slowly varying coefficients. The common methods for constructing the analytical approximate solutions to the nonlinear oscillator equations are the perturbation methods. Some well known perturbation methods are the Krylov-Bogoliubov- Mitropolskii (KBM) [1-3] method, the Lindstedt-Poincare (LP) method [4-5] and the method of multiple time scales [4]. Almost all perturbation methods are based on an assumption that small parameter must exist in the equations. Lim *et al.* [7] have presented a new analytical approach to the Duffing- harmonic oscillator. In recent years, He [8] has investigated the homotopy perturbation technique. In another paper, He [9] has developed a coupling method of a homotopy perturbation technique and a perturbation technique for strongly nonlinear problems. Recently, He [10] has also presented a new interpretation of homotopy perturbation method for strongly nonlinear differential systems. Belendez *et al.* [11] have presented the application of He's homotopy perturbation method to the Duffing harmonic oscillators. Hu [12] has obtained the solution of a quadratic nonlinear oscillator by the method of harmonic balance. Roy *et al.* [13] have presented the effects of higher approximation of Krylov-Bogoliubov-Mitropolskii solution and matched asymptotic differential systems with slowly varying coefficients and damping near to a turning point for weakly nonlinear systems. Arya and Bojadziev [14] have presented the analytical technique for time depended oscillating systems with damping, slowly varying parameters and delay Alam *et al.* [16] have developed the general Struble technique for weakly nonlinear systems with large damping. Recently Uddin *et al.* [17] have presented an approximate technique for solving strongly nonlinear differential systems with damping effects. The authors [8-12] have studied the nonlinear systems without considering any damping effects. But most of the physical and oscillating systems encounter in presence of small damping in nature and it plays an important role to the systems. In this article, we have extended [17] an approximate technique to obtain the analytical approximate solutions of second order strongly nonlinear differential systems with small damping and slowly varying coefficients [13] based on He's homotopy perturbation [8-11] and the extended form of the KBM [1-3] methods. Figures are provided to compare between the solutions obtained by the presented method with the corresponding numerical (considered to be exact) solutions.

2. The method

Consider a nonlinear oscillator [13] modeled by the following equation

$$\ddot{x}+e^{-\tau}x+\varepsilon f(x)=0,\quad x(0)=a_0,\qquad \dot{x}(0)=0, \tag{1}$$

where over dots denote differentiation with respect to time t, τ is a slowly varying time, ε is a positive parameter which is not necessarily small, a_0 is a given positive constant and $f(x)$ is a given nonlinear function which satisfies the following condition

$$f(-x)=-f(x). \tag{2}$$

According to the homotopy perturbation [8-11,17] technique, Eq. (1) can be re-written as

$$\ddot{x}+(e^{-\tau}+\lambda)x=\lambda x-\varepsilon f(x) \tag{3}$$

Eq. (3) yields,

$$\ddot{x}+\omega^2 x=\lambda x-\varepsilon f(x), \tag{4}$$

where

$$\omega^2=e^{-\tau}+\lambda. \tag{5}$$

Herein ω is a constant for undamped nonlinear oscillator and known as the frequency in literature and λ is an unknown function which can be determined by eliminating the secular terms. But for a damped nonlinear differential systems ω is a time dependent function and it varies slowly with time t. To handle this situation, we are interested to use the extended KBM [1-2] method by Mitropolskii [3]. According to the He's [8-11,17] homotopy perturbation method, we have constructed the following homotopy

$$\ddot{x}+\omega^2 x = p(\lambda x - \varepsilon x^3), \tag{6}$$

where p is the homotopy parameter and $f(x)=x^3$. When $p=0$, Eq. (6) becomes the linearized equation

$$\ddot{x}+\omega^2 x = 0, \tag{7}$$

and for the case $p=1$, Eq. (6) becomes the original problem. The homotopy parameter p is used to expand the solution $x(t)$ in powers of p in the following form

$$x(t)=x_0(t)+p x_1(t)+p^2 x_2(t)+\cdots \tag{8}$$

Substituting Eq. (8) into Eq. (6) and then equating the coefficients of the like powers of p, we obtain the following linear differential equations

$$\ddot{x}_0+\omega^2 x_0 = 0,\ x_0(0)=a_0,\ \dot{x}_0(0)=0 \tag{9}$$

$$\ddot{x}_1+\omega^2 x_1 = \lambda x_0 - \varepsilon x_0^3,\ x_1(0)=0\ \dot{x}_1(0)=0, \tag{10}$$

where a_0 is a positive constant. The solution of Eq. (9) is then obtained as

$$x_0(t)=a_0\cos\omega t. \tag{11}$$

Substituting Eq. (11) into Eq. (10), we obtain

$$\ddot{x}_1 + \omega^2 x_1 = (\lambda a_0 - \frac{3}{4}\varepsilon a_0^3)\cos\omega t - \frac{1}{4}\varepsilon a_0^3 \cos 3\omega t. \tag{12}$$

The requirement of no secular terms in particular solution of Eq. (12) implies that the coefficient of the $\cos\omega t$ term is zero. Setting this term to zero, we obtain

$$\lambda a_0 - \frac{3}{4}\varepsilon a_0^3 = 0 \tag{13}$$

For the nontrivial solution *i.e.*, $a_0 \neq 0$, Eq. (13) leads to

$$\lambda = \frac{3\varepsilon a_0^2}{4}. \tag{14}$$

By inserting the value of λ from Eq. (14) into Eq. (5), we obtain the following solution for ω as

$$\omega(a_0) = \sqrt{e^{-\tau} + \frac{3\varepsilon a_0^2}{4}}. \tag{15}$$

From Eq. (15), it is seen that the frequency depends on the initial amplitude a_0 and slowly varying time τ. Now using Eq. (13), Eq. (12) can be rewritten in the following form

$$\ddot{x}_1 + \omega^2 x_1 = -\frac{1}{4}\varepsilon a_0^3 \cos 3\omega t, \tag{16}$$

with the initial conditions

$$x_1(0) = 0, \quad \dot{x}_1(0) = 0. \tag{17}$$

The solution of Eq. (16) is then obtained as

$$x_1 = -\frac{1}{32\omega^2}\varepsilon a_0^3(\cos\omega t - \cos 3\omega t). \tag{18}$$

Thus we obtain the first order analytical approximate solution of Eq. (1) by setting $p = 1$ in the following form

$$\ddot{x}_1 + \omega^2 x_1 = (\lambda a_0 - \frac{3}{4}\varepsilon a_0^3)\cos\omega t - \frac{1}{4}\varepsilon a_0^3 \cos 3\omega t. \qquad (12)$$

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Thus we obtain the first order analytical approximate solution of Eq. (1) by setting $p = 1$ in the following form

$$x = x_0 + x_1 = \left(\frac{32\omega^2 - \varepsilon a_0^2}{32\omega^2}\right) a_0 \cos\omega t + \frac{\varepsilon a_0^3}{32\omega^2}\cos 3\omega t, \tag{19}$$

where the frequency ω is given by Eq. (15).

But most of the physical and oscillating systems occur in presence of damping in nature with slowly varying coefficients. From our study, it is seen that the most of the authors [7-11] have presented the analytical technique for solving nonlinear oscillators without considering damping effects. So in this article, we are interested to consider a strongly nonlinear oscillator [13] with small damping and slowly varying coefficients in the following form

$$\ddot{x} + 2k(\tau)\dot{x} + e^{-\tau}x = -\varepsilon f(x), \quad k << 1, \tag{20}$$

where $2k$ is the linear damping coefficient which varies slowly with time $t, \tau = kt$ is the slowly varying time.

Eq. (20) leads to Eq. (1) when $k = 0$. Let us assume the following transformation

$$x = y(t)e^{-kt}. \tag{21}$$

Differentiating Eq. (21) twice with respect to time t and substituting $\ddot{x}, \dot{x}$ together with x into Eq. (20) and then simplifying them, we obtain

$$\ddot{y} + (e^{-\tau} - k^2)y = -\varepsilon e^{kt} f(ye^{-kt}). \tag{22}$$

According to the homotopy perturbation method [8-11,17], Eq. (22) can be written as

$$\ddot{y} + \omega^2 y = \lambda y - \varepsilon e^{kt} f(ye^{-kt}), \tag{23}$$

where

$$\omega^2 = e^{-\tau} - k^2 + \lambda. \tag{24}$$

Herein λ is an unknown constant which can be determined by eliminating the secular terms (as it is eliminated for the undamped problem). However, for

a damped nonlinear differential system ω is a time dependent function and it varies slowly with time t. To handle this situation, we can use the extended KBM [1-2] method by Mitropolskii [3]. According to this technique, we are going to choose the analytical approximate solution of Eq. (23) in the following form

$$y = a\cos\varphi, \tag{25}$$

where a and φ vary slowly with time t, a and φ satisfy the following first order differential equations

$$\begin{aligned} \dot{a} &= kA_1(a,\tau) + k^2 A_2(a,\tau) + \cdots, \\ \dot{\varphi} &= \omega(\tau) + kB_1(a,\tau) + k^2 B_2(a,\tau) + \cdots, \end{aligned} \tag{26}$$

where k is a small positive parameter and A_j, B_j ,are unknown functions. It is clear that, this solution is similar to the undamped solution if $k \to 0$ and $a \to a_0$, $\varphi \to \omega t$. Now differentiating Eq. (25) twice with respect to time t, utilizing the relations Eq. (26) and substituting $\ddot{y}$ and y into Eq. (23) and then equating the coefficients of $\sin\varphi$ and $\cos\varphi$, we obtain

$$A_1 = -\omega' a/(2\omega), \ B_1 = 0, \tag{27}$$

where a prime denotes differentiation with respect to τ. Now putting Eq. (25) into Eq. (21) and Eq. (27) into Eq. (26) we obtain the following equations

$$x = a e^{-kt}\cos\varphi, \tag{28}$$

and

$$\begin{aligned} \dot{a} &= -k\,\omega' a/(2\omega), \\ \dot{\varphi} &= \omega(\tau). \end{aligned} \tag{29}$$

Eq. (28) represents the first order analytical approximate solution of Eq (20) by the presented method. Usually, the integration of Eq. (29) is performed by well-known techniques of calculus [4-5], but sometimes they are solved by a numerical procedure [12-17]. Thus the determination of the first order analytical approximate solution of Eq. (20) is completed.

3. Example

As an example of the above procedure, we are going to consider the Duffing equation in the following form [13]

$$\ddot{x}+2k(\tau)\dot{x}+e^{-\tau}x=-\varepsilon x^3, \tag{30}$$

where $f(x)=x^3$. Now using the transformation Eq. (21) into Eq. (30) and then simplifying them, we obtain

$$\ddot{y}+(e^{-\tau}-k^2)y=-\varepsilon e^{-2kt}y^3. \tag{31}$$

According to the homotopy perturbation [8-11,17] method, Eq. (31) can be rewritten as

$$\ddot{y}+\omega^2 y=\lambda y-\varepsilon e^{-2kt}y^3, \tag{32}$$

where

$$\omega^2=e^{-\tau}-k^2+\lambda. \tag{33}$$

According to the extended form of the KBM [1-3] method, the solution of Eq. (32) is given Eq. (25). The requirement of no secular terms in particular solution of Eq. (32) implies that the coefficient of the $\cos\omega t$ term is zero. Setting this term to zero, we obtain

$$\lambda a-\frac{3\varepsilon a^3 e^{-2kt}}{4}=0, \tag{34}$$

For the nontrivial solution *i.e.*, $a\neq 0$, Eq. (34) leads to

$$\lambda=\frac{3\varepsilon a^2 e^{-2kt}}{4}. \tag{35}$$

Inserting the value of λ from Eq. (35) into Eq. (33), it yields

$$\omega^2 = e^{-\tau} - k^2 + \frac{3\varepsilon a^2 e^{-2kt}}{4}. \tag{36}$$

This is a time dependent frequency equation of the given nonlinear system. As $t \to 0$, Eq. (36) yields

$$\omega_0 = \omega(0) = \sqrt{1 - k^2 + \frac{3\varepsilon a_0^2}{4}}. \tag{37}$$

Integrating the first equation of Eq. (29), we get

$$a = a_0 \sqrt{\frac{\omega_0}{\omega}}, \tag{38}$$

where a_0 is a constant of integration which represents the initial amplitude of the nonlinear systems. Now putting Eq. (38) into Eq. (36), we obtain the following equation

$$\omega^3 + q\omega + r = 0, \tag{39}$$

where

$$q = -(e^{-\tau} - k^2), \quad r = -\frac{3\varepsilon a_0^2 \omega_0 e^{-2kt}}{4}. \tag{40}$$

Eq. (39) is a cubic equation in ω. It has an analytical solution for every real value of $e^{-\tau}$. When $e^{-\tau} > k$, then the solution of Eq. (39) becomes (see also [6] for details)

$$\omega = \left(-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} \right)^{1/3} + \left(-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} \right)^{1/3}. \tag{41}$$

Now substituting $r = -2R, q = -3Q$ into Eq. (41), then it becomes

$$\omega = \left(R + \sqrt{R^2 - Q^3} \right)^{1/3} + \left(R - \sqrt{R^2 - Q^3} \right)^{1/3}, \quad R > Q, \tag{42}$$

or,

$$\omega = \left(R + i\sqrt{Q^3 - R^2}\right)^{1/3} + \left(R - i\sqrt{Q^3 - R^2}\right)^{1/3}, \quad R < Q. \tag{43}$$

Herein the relations among Q, R, ν, k, ω_0 and a_0 are obtained as

$$Q = \frac{(e^{-\tau} - k^2)}{3}, \; R = \frac{3\varepsilon\omega_0 a_0^2 e^{-2kt}}{8}. \tag{44}$$

According to [6], the real form of Eq. (43) is obtained as

$$\omega = 2\sqrt{Q}\cos\left(\frac{\pi + \tan^{-1} V/R}{3}\right) = 2\sqrt{Q}\cos\left(\frac{\tan^{-1} V/R}{3}\right), \tag{45}$$

where

$$V = \sqrt{Q^3 - R^2}. \tag{46}$$

The solution of the second equation of Eq. (29) becomes

$$\varphi = \varphi_0 + \int_0^t \omega(t)dt, \tag{47}$$

where φ_0 is the initial phase and ω is given by Eqs. (42) or (45). Therefore, the first order analytical approximate solution of Eq (30) is obtained by Eq. (28) and the amplitude a and the phase φ are calculated from Eq. (38) and Eq. (47) respectively. Thus the determination of the first order analytical approximate solution of Eq. (30) is completed by the presented analytical technique by coupling He's homotopy technique and the KBM method.

4. Initial conditions

The initial conditions of $\ddot{x} + 2k(\tau)\dot{x} + e^{-\tau}x = -\varepsilon x^3$ are obtained as

$$\begin{aligned} x(0) &= a_0\cos\varphi_0, \\ \dot{x}(0) &= \left(\frac{ka_0(4 + 3\varepsilon a_0^2)}{8(3\omega_0^2 + k^2 - 1)} - ka_0\right)\cos\varphi_0 - a_0\,\omega_0\sin\varphi_0. \end{aligned} \tag{48}$$

In general, the initial conditions $[x(0), \dot{x}(0)]$ are specified. Then one has to solve nonlinear algebraic equation in order to determine the initial amplitude a_0 and the initial phase φ_0 that appear in the solutions, from the initial conditions equation (48).

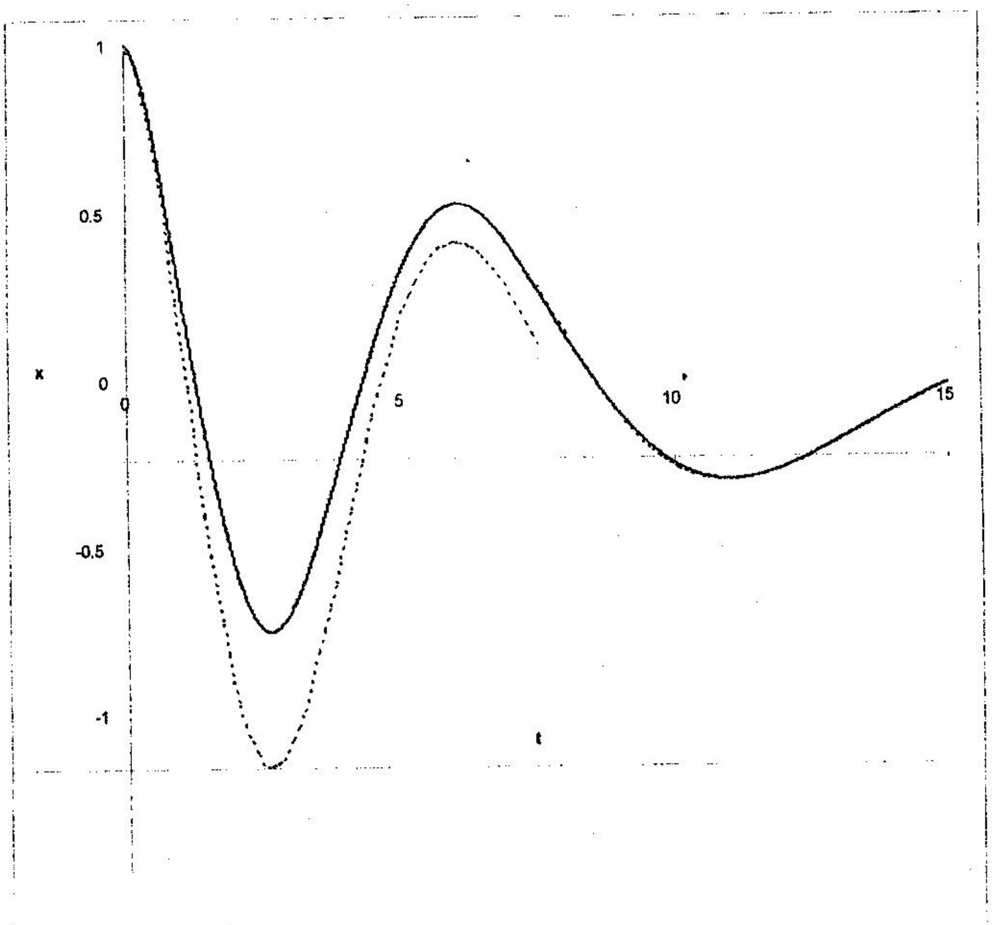

Fig. 1 (a) First approximate solution of Eq. (30) is denoted by $-\bullet-$ (dashed lines) by the presented analytical technique with the initial conditions $a_0 = 1.0,\ \varphi_0 = 0$ or $[x(0) = 1.0,\ \dot{x}(0) = -0.118879]$ with $k = 0.15,\ \varepsilon = 1.0$ and $f = x^3$. Corresponding numerical solution is denoted by - (solid line).

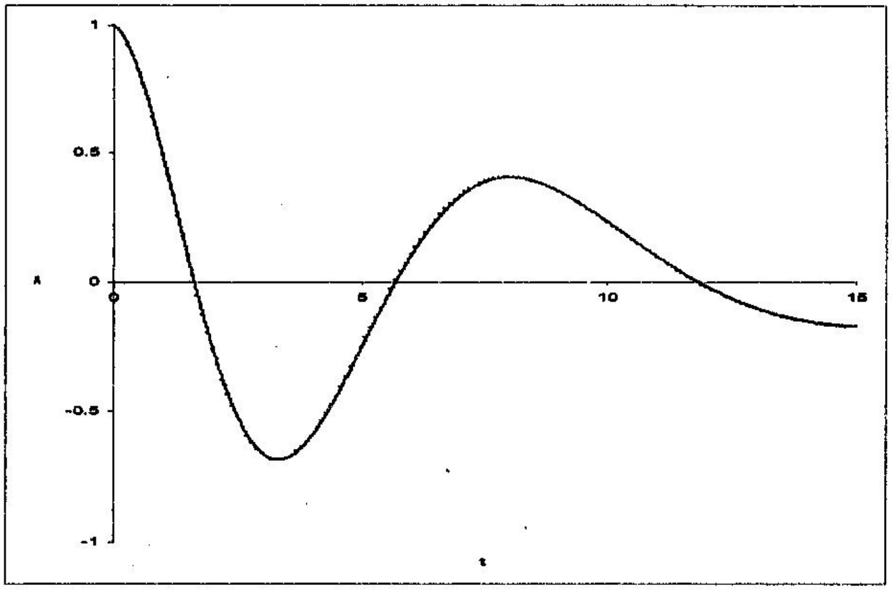

Fig. 1 (b) First approximate solution of Eq. (30) is denoted by $-\bullet-$ (dashed lines) by the presented analytical technique with the initial conditions $a_0 = 1.0$, $\varphi_0 = 0$ or $[x(0) = 1.0,\ \dot{x}(0) = -0.11302]$ with $k = 0.15$, $\varepsilon = 0.1$ and $f = x^3$. Corresponding numerical solution is denoted by - (solid line).

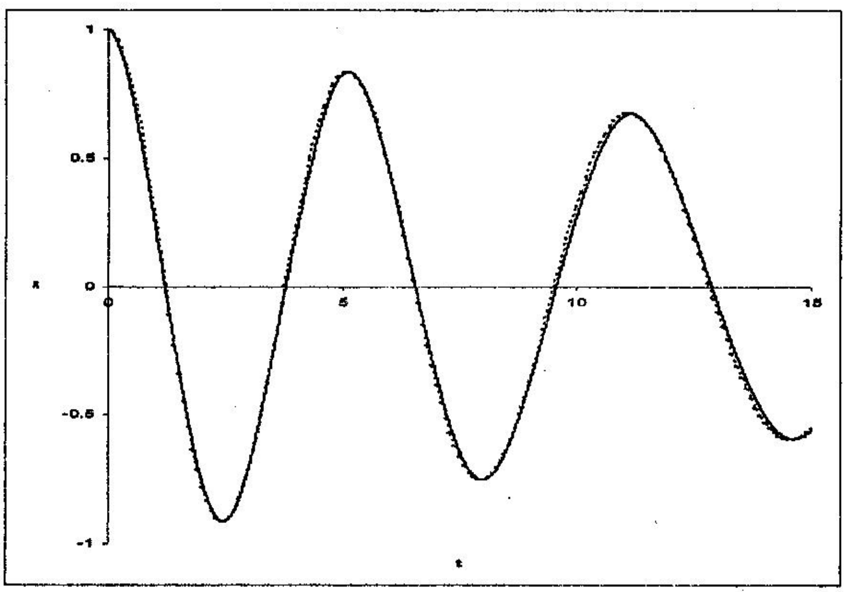

Fig. 2 (a) First approximate solution of Eq. (30) is denoted by $-\bullet-$ (dashed lines) by the presented analytical technique with the initial conditions $a_0 = 1.0$, $\varphi_0 = 0$ or $[x(0) = 1.0,\ \dot{x}(0) = -0.03969]$ with $k = 0.05$, $\varepsilon = 1.0$ and $f = x^3$. Corresponding numerical solution is denoted by - (solid line).

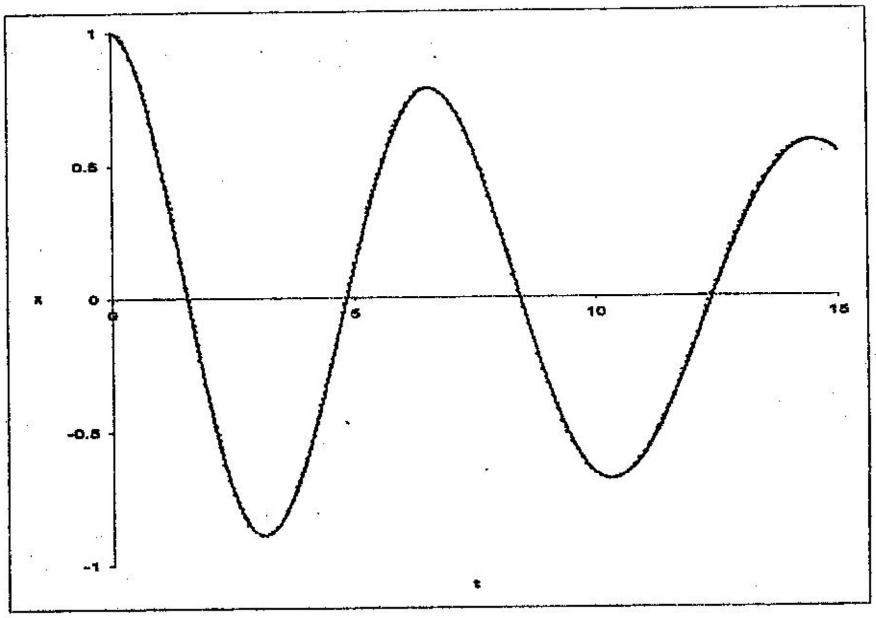

Fig. 2 (b). First approximate solution of Eq. (30) is denoted by $-\bullet-$ (dashed lines) by the presented analytical technique with the initial conditions $a_0 = 1.0,\ \varphi_0 = 0$ or $[x(0) = 1.0,\ \dot{x}(0) = -0.03789]$ with $k = 0.05,\ \varepsilon = 0.1$ and $f = x^3$. Corresponding numerical solution is denoted by - (solid line).

5. Results and discussion

In this article, He's homotopy perturbation technique has been extended based on the extended form of the KBM [1-3] method to strongly nonlinear systems with damping and slowly varying coefficients. From our results, it is seen that the first order analytical approximate solutions show a good agreement with the corresponding numerical solutions for the several damping effects. The analytical approximate solutions of Eq. (30) is computed by Eq. (28) with small damping and slowly varying coefficients and the corresponding numerical solutions are obtained by using fourth order ***Runge-Kutta*** method. This method can also be used to solve the second order strongly nonlinear differential systems without damping (as $k \to 0$). The presented method is very simple in its principle, and is very easy to be applied to the nonlinear systems. The variational equations of the amplitude and phase variables appear in a set of first order nonlinear ordinary differential equations. The integrations of these variational equations are obtained by well-known techniques of calculus [4-5]. In lack of analytical solutions, they are solved by numerical procedure [4,12-17]. The amplitude and phase variables change slowly with time t. The behavior of amplitude and phase variables characterizes the oscillating processes. Moreover, the variational equations of amplitude and phase variables are used to investigate the stability of the nonlinear differential equations. He's homotopy perturbation is valid only for conservative nonlinear systems But the presented method is valid for both conservative and non-conservative nonlinear systems. The presented method can also overcome some limitations of the classical perturbation techniques; it does not require a small parameter $(i.e, \varepsilon = 1)$ in the equations. The advantage of the presented method is that the first order analytical approximate solutions show a good agreement with the corresponding numerical solutions. The method has been successfully implemented to solve for both strongly and weakly cubic nonlinear oscillators with small damping and slowly varying coefficients. Comparison is made between the solutions obtained by the presented coupling analytical technique and those obtained by the numerical solutions in figures graphically.

6. Conclusion

The great achievement of this article is that the presented analytical technique is capable to handle both strongly and weakly nonlinear differential systems with damping and slowly varying coefficients.

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