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NEAR LATTICE OF FINITELY GENERATED PRINCIPAL N-IDEALS WHICH FORMS A NORMAL NEARLATTICE

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Abstract:

In this paper the author generalize several results of normal nearlattices in terms of n-ideals. It has been proved that the nearlattice of finitely generated principal n-ideals $P_n(S)$ is normal if and only if each prime n-ideal contains a unique minimal prime n-ideal. Also it has been shown that $P_n(S)$ is normal if and only if $(<x>_n \cap <y>_n)^* = <x>_n^* \vee <y>_n^*$ for all x, y∈S.

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এই পত্রে প্রবন্ধকার n - আইডিয়ালের (*n-ideal*) পদে স্বাভাবিক নিকট - ল্যাটিসের (*nearlattice*) বহুবিধ ফলাফলকে সামান্যীকরণ করেছেন। ইহা প্রমাণ করা হয়েছে যে নিকট - ল্যাটিসের সসীমভাবে সৃষ্ট মূখ্য n - আইডিয়াল $P_n(S)$ স্বাভাবিক হয় যদি এবং কেবলমাত্র যদি n - আইডিয়ালের একটি অদ্বিতীয় লঘিষ্ঠ মৌলিক n - আইডিয়াল ধারণ করে। এটাও দেখনো হয়েছে যে $P_n(S)$ স্বাভাবিক হয় যদি এবং কেবলমাত্র যদি $(<x>_n \cap <y>_n)^* = <x>_n^* \vee <y>_n^*$) সকল $x, y \in S$.

1. Introduction

Normal lattices have been studied by several author including Cornish [2] and Monterio [5]. A distributive nearlattice S with 0 is *normal* if every prime ideal of S contains a unique minimal prime ideal. Equivalently, S is called *normal* if each prime filter of S is contained in a unique utrafilter (maximal and proper) of S.

In this paper, we have generalized several important results of normal nearlattices in terms of n-ideals by using some results on minimal prime n-ideals. We have proved that $P_n(S)$ is normal if and only if each prime n-ideal contains a unique minimal prime n-ideal.

1.Prelimenaries: For a fixed element $n \in S$, a convex subnearlattice containing n is called an n-ideal. The concept of n-ideals is a kind of generalization of ideals and filters of a nearlattice. For detailed literature on n-ideals we refer the reader to consult [3] and [4].

For a medial element n, an n-ideal P of a nearlattice S is called a *prime* n-*ideal* if $P \neq S$ and $m(x, n, y) \in P$ $(x, y \in S)$ implies either $x \in P$ *or* $y \in P$ where $m(r, n, s) = (r \wedge n) \vee (n \wedge s) \vee (r \wedge s)$.

The set of all n-ideals of a nearlattice S is denoted by $I_n(S)$ which is an algebraic nearlattice. For two n-ideals I and J of a nearlattice S, $I \vee J = \{x: i \wedge j \leq x = \vee^{p}_{r=1} (x \wedge a_r)$ for some positive integer p where $i, j \in I \cup J\}$ while the set theoretic intersection is their infimum. Moreover, $I \cap J = \{m(i, n, j): i \in I, j \in J\}$.

An n-ideal generated by a finite number of elements $a_1, a_2, \ldots, a_n$ is called a *finitely generated* n-*ideal*, denoted by $< a_1, a_2, \ldots, a_n >_n$. By [3], $< a_1, a_2, \ldots, a_m >_n = \{y \in S: a_1 \wedge \ldots \wedge a_m \wedge n \leq y = (y \wedge a_1) \vee \ldots \vee (y \wedge a_m) \vee (y \wedge n)\}$, *provided S is distributive*

$$= < a_1 \wedge a_2 \wedge \ldots \wedge a_m \wedge n, a_1 \vee a_2 \vee \ldots \vee a_m \vee n >_n.$$

The set of finitely generated n-ideals is denoted by $F_n(S)$ which is a nearlattice.

An n-ideal generated by a single element a is called a *principal* n-*ideal*, denoted by $< a >_n$. The set of principal n-ideals is denoted by $P_n(S)$.

Moreover, by [3] we have following result.

Proposition 1.1 For a central elemenl $n \in S$, $P_n(S) \cong (n]^d \times [n)$.

A prime n-ideal P is said to be a *minimal prime* n-*ideal* belonging to n-ideal I if (i) $I \subseteq P$ and (ii) There exists no prime n-ideal Q such that $Q \neq P$ and $I \subseteq Q \subseteq P$

A prime n-ideal P of a nearlattice S is called a ***minimal prime*** n-***ideal*** if there exists no prime n-ideal Q such that $Q \neq P$ and $Q \subseteq P$. Thus a minimal prime n-ideal is a minimal prime n-ideal belonging to $\{n\}$.

For any n-ideals J of a distributive nearlattice S, we define

$J^* = \{x \in S: m(x, n, j) = n$ for all $j \in J\}$ where n is a medial element. Obviously, J^* is an n-ideal and $J \cap J^* = \{n\}$. In fact, this the largest n-ideal which annihilates J and so it is the pseudocomplement of J in $I_n(S)$. Moreover, for a distributive nearlattice S, $I_n(S)$ is a distributive algebraic nearlattice and so it is pseudocomplemented.

Recently [1] have established the following results on prime n-ideal of a nearlattice. These results will be needed in establishing the other results of the paper.

Lemma 1.2 *If* n is *a medial element and* P *is a prime* n-*ideal of a nearlattice* S. *Then* P contains *either* (n] or [n), *but not both.* □

Lemma 1.3 *Let* n be *a medial element of a nearlattice* S. *Then every prime* n-*ideal* P of S is *either an ideal or a filter. If it is an ideal, then it is also a prime ideal. If it is a filter, then it is a prime filter.* □

Lemma 1.4 Let I be an ideal and D be a convex sub-nearlattice of a distributive nearlattice S with $I \cap D = \phi$. Then there exists a prime ideal $P \supseteq I$ such that $P \cap D = \phi$. □

Lemma 1.5 *Let* n *be a medial element of a distributive nearlattice* S. *Then every*

n-*ideal* I *of* S *is the intersection of all prime* n-*ideals containing it.* □

Theorem 1.6 *Let* S be *a distributive nearlattice with an upper element* n *and let* I, J *be two* n-*ideals of* S. *Then for any* $x \in I \vee J$, $x \vee n = i \vee j$ *and* $x \wedge n = i' \wedge j'$ *for some* $i, i' \in I, j, j' \in J$ *with* $i, j \geq n$ *and* $i', j' \leq n$.

Proof. It is very easy.So we omit this proof.

2. Nearlattices whose $P_n(S)$ are normal nearlattices.

We start with the following results due to [2, Lemma 1.4].

Lemma 2.1 *If* S_1 *is a subnearlattice of a distributive nearlattice* S *and* P_1 *is a prime ideal (filter) in* S_1, *then there exists a prime ideal (filter)* P *in* S *such that* $P_1 = P \cap S_1$. □

Lemma 2.2 *Let* S *be a distributive nearlattice and* $n \in S$ *be a medial element. Then for any* $I, J \in I_n(S)$, $(I \cap J)^* \cap I = J^* \cap I$.

Proof. Since $I \cap J \subseteq J$. so R.H.S. $\subseteq$ L.H.S.

To prove the reverse inclusion, let $x \in$ L.H.S. Then $x \in I$ and $m(x, n, t) = n$ for all $t \in I \cap J$. Since $x \in I$, so $m(x, n, j) \in I \cap J$. Thus, $m(x, n, m(x, n, j)) = n$. But it can be easily seen that $m(x, n, m(x, n, j)) = m(x, n, j)$. This implies $m(x, n, j) = n$ for all $j \in J$. Hence $x \in$ R.H.S, and so L.H.S. $\subseteq$ R.H.S. Thus $(I \cap J)^* \cap I = J^* \cap I$. □

Lemma 2.3 *Suppose* n *is a medial element of a nearlattice* S. *If* $I \subseteq J$, $I, J \in I_n(S)$, *then* (i) $I^+ = I^* \cap J$ and (ii) $I^{++} = I^{**} \cap J$.

Proof. (i) is trivial. For (ii), using (i), we have,

$I^{++} = (I^+)^* \cap J = (I^* \cap J)^* \cap J$. Thus by Lemma 2.1, $I^{++} = I^{**} \cap J$. □

Theorem 2.4 *Suppose* S *be a distributive nearlattice and* $n \in S$. *Let* $x, y \in S$ *with*

$<x>_n \cap <y>_n = \{n\}$. *Then the following conditions are equivalent.*

(i) $<x>_n^* \vee <y>_n^* = S$.

(ii) *For any* $t \in S$, $<m(x, n, t)>_n^+ \vee <m(y, n, t)>_n^+ = <t>_n$ *where* $<m(x, n, t)>_n^+$ *denote the relative pseudo-complement of* $<m(x, n, t)>_n$ *in* $[\{n\}, <t>_n]$.

Proof. It is obvious by using Lemmas 2.2 and 2.3.

Following result is due to [2]

Theorem 2.5 *Let* S *be a distributive nearlattice with* 0. *Then the following conditions are equivalent.*

(i) S *is normal.*

(ii) *Every prime ideal contains a unique minimal prime ideal.*

(iii) *For all* $x, y \in S$, $x \wedge y = 0$ *implies* $(x]^* \vee (y]^* = S$.

(iv) $(x \wedge y]^* = (x]^* \vee (y]^*$.

(v) *For any two minimal prime ideals* P *and* Q *of* S, $P \vee Q = S$. □

Theorem 2.6 *Let* S *be a distributive nearlattice and* n *be a central element of* S. *The following conditions are equivalent.*

(i) $P_n(S)$ *is normal.*

(ii) *Every prime* n-*ideal of* S *contains a unique minimal prime* n-*ideal.*

(iii) *For any two minimal prime* n-*ideals* P *and* Q *of* S, $P \vee Q = S$.

Proof. **(i)⇒(ii).** Let $P_n(S)$ be normal, since $P_n(S) \cong (n]^d \times [n)$, so both $(n]^d$ and $[n)$ are normal. Suppose P is any prime n-ideal of S. Then by Lemma 1.2, either $P \supseteq (n]$ or $P \supseteq [n)$. Without loss of generality, suppose $P \supseteq (n]$. Then by Lemma 1.3, P is prime ideal of S. Hence by Lemma 2.1, $P_1 = P \cap [n)$ is a prime ideal of $[n)$. Since $[n)$ is normal, so by [2, Th. 2.4] P_1 contains a unique minimal prime ideal R_1 of $[n)$. Therefore, P contains a unique minimal prime ideal R of S where $R_1 = R \cap [n)$. Since $n \in R_1$ so $n \in R$ and hence R is a minimal prime n-ideal of S. Thus (ii) holds.

(ii)⇒(i). Suppose (ii) holds. Let P_1 be a prime ideal in $[n)$. Then by Lemma 2.1, $P_1 = P \cap [n)$ for some prime ideal P of S. Since $n \in P_1 \subseteq P$, so P is prime n-ideal. Therefore, P contains a unique minimal prime n-ideal R of S. Thus by Lemma 2.1 P_1 contains the unique minimal prime ideal $R_1 = R$

$\cap$ [n) of [n). Hence by definition [n) is normal. Similarly, we can prove that $(n]^d$ is also normal. Since $P_n(S) \cong (n]^d \times [n)$, so $P_n(S)$ is normal.

(ii)⇔(iii) is trivial. □

For a prime ideal P of a distributive nearlattice S with 0, Cornish in [2] has defined $0(P) = \{x \in S: x \wedge y = 0$ for some $y \in S - P\}$. Clearly, 0(P) is an ideal and $0(P) \subseteq P$. Cornish in [2] has shown that 0(P) is the intersection of all the minimal prime ideals of S which are contained in P.

For a prime n-ideal P of a distributive nearlattice S, we write $n(P) = \{y \in S: m(y, n, x) = n$ for some $x \in S - P\}$. Clearly, n(P) is an n-ideal and $n(P) \subseteq P$.

Lemma 2.7 *Let* n *be a medial element of a distributive nearlattice* S *and* P *be a prime* n*-ideal in* S. *Then each minimal prime* n*-ideal belonging to* n(P) *is contained in* P.

Proof. Let Q be a minimal prime n-ideal belonging to n(P). If $Q \not\subset P$, then choose $y \in Q - P$. Since Q is a prime n-ideal, so by Lemma 1.3, we know that Q is either an ideal or a filter. Without loss of generality, suppose Q is an ideal. Now let

$T = \{t \in S: m(y, n, t) \in n(P)\}$. We shall show that $T \not\subset Q$.

If not, let $D = (S - Q) \vee [y)$. Then $n(P) \cap D = \phi$.

For otherwise, $y \wedge r \in n(P)$ for some $r \in S - Q$. Then by convexity,

$y \wedge r \leq m(y, n, r) \leq (y \wedge r) \vee n$ implies $m(y,n,r) \in n(P)$.

Hence $r \in T \subseteq Q$,which is a contradiction. Thus by Stone's separation theorem for

n-ideals, there exists a prime n-ideal R containing n(P) disjoint to D.Then $R \subseteq Q$.

Moreover, $R \neq Q$ as $y \notin R$, this shows that Q is not a minimal prime n-ideal belonging to n(P), which is a contradiction. Therefore, $T \not\subset Q$. Hence there exists

$z \notin Q$ such that $m(y, n, z) \in n(P)$. Thus $m(m(y, n, z), n, x) = n$ for some $x \in S - P$. It is easy to see that $m(m(y, n, z), n, x) = m(m(y, n, x), n, z)$. Hence $m(m(y, n, x), n, z) = n$. Since P is prime and $y, x \notin P$ so $m(y, n, x) \notin P$. Therefore, $z \in n(P) \subseteq Q$, which is a contradiction. Hence $Q \subseteq P$. □

Proposition 2.8 *If* n *is a medial element of a distributive nearlattice* S *and* P *is a prime* n*-ideal in* S, *then* n(P) *is the intersection of all minimal prime* n*-ideals contained in* P.

Proof. Clearly, n(P) is contained in any prime n-ideal which is contained in P. Hence n(P) is contained in the intersection of all minimal prime n-ideals contained in P. Since S is distributive, so by Lemma 1.5, n(P) is the intersection of all minimal prime n-ideals belonging to it. Since each prime n-ideal contains a minimal prime n-ideal, above remarks and Lemma 2.7 establish the proposition. □

Theorem 2.9 *Let* S *be a distributive nearlattice and let* n *be central element in* S. *Then the following conditions are equivalent.*

(i) $P_n(S)$ *is normal.*

(ii) *Every prime* n*-ideal contains a unique minimal prime* n*-ideal.*

(iii) *For each prime* n*-ideal* P, n(P) *is prime* n*-ideal.*

(iv) *For all* $x, y \in S$, $<x>_n \cap <y>_n = \{n\}$ *implies* $<x>_n^* \vee <y>_n^* = S$.

(v) *For all* $x, y \in S$, $(<x>_n \cap <y>_n)^* = <x>_n^* \vee <y>_n^*$.

Proof. **(i)⇒(ii)** holds by Theorem 2.6

(ii)⇒(iii) is a direct consequence of Lemma 2.7.

(iii)⇒(iv). Suppose (iii) holds.

Consider $x, y \in S$ with $<x>_n \cap <y>_n = \{n\}$.

If $<x>_n^* \vee <y>_n^* \neq S$, then by Lemma 1.4, there exists a prime n-ideal P such that $<x>_n^* \vee <y>_n^* \subseteq P$, then $<x>_n^* \subseteq P$ and $<y>_n^* \subseteq P$ imply

$x \notin n(P)$ and $y \notin n(P)$. But $n(P)$ is prime and so $m(x, n, y) = n \in n(P)$ is contradictory. Therefore, $<x>_n^* \vee <y>_n^* = S$.

(iv)⇒(v). Obviously, $<x>_n^* \vee <y>_n^* \subseteq (<x>_n \cap <y>_n)^*$.

Conversely, let $w \in (<x>_n \cap <y>_n)^*$. Then, $<w>_n \cap <x>_n \cap <y>_n = \{n\}$

or,

$<m(w, n, x)>_n \cap <y>_n = \{n\}$

So by (iv), $<m(w, n, x)>_n^* \vee <y>_n^* = S$.

So, $w \in <m(w, n, x)>_n^* \vee <y>_n^*$.

Therefore, $w \wedge n$, $w \vee n \in <m(w, n, x)>_n^* \vee <y>_n^*$. Here $w \vee n$ exists as n is an upper element. Then by Theorem 1.6, $w \vee n = r \vee s$ for some $r \in <m(w, n, x)>_n^*$ and $s \in <y>_n^*$ with $r, s \geq n$.

Now $r \in <m(w, n, x)>_n^*$ implies

$r \wedge [(w \wedge n) \vee (w \wedge x) \vee (x \wedge n)] \vee (r \wedge n) \vee [(w \wedge n) \vee (x \wedge n) \vee (w \wedge x)] \wedge n = n$. Observe that above left hand expression exists as S is medial. That is, $(r \wedge w \wedge n) \vee (r \wedge w \wedge x) \vee (r \wedge x \wedge n) \vee (r \wedge n) \vee (w \wedge n) \vee (x \wedge n) = n$, and so $(r \wedge w \wedge x) \vee n = n$. This implies $(r \vee n) \wedge (w \vee n) \wedge (x \vee n) = n$, so $(r \vee n) \wedge (x \vee n) = n$ as

$r \vee n \leq w \vee n$. Thus, $(r \wedge x) \vee n = n$. Hence $(r \wedge x) \vee (x \wedge n) \vee (r \wedge n) = n$, which implies $r \in <x>_n^*$.

Therefore, $w \vee n \in <x>_n^* \vee <y>_n^*$.

A dual proof of above shows that $w \wedge n \in <x>_n^* \vee <y>_n^*$. So by convexity,

$w \in <x>_n^* \vee <y>_n^*$. Therefore, $(<x>_n \cap <y>_n)^* \subseteq <x>_n^* \vee <y>_n^*$, and so

$(<x>_n \cap <y>_n)^* = <x>_n^* \vee <y>_n^*$, which is (v).

(v)⇒(iv). Let $<x>_n \cap <y>_n = \{n\}$, for some $x, y \in S$.