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THE EFFECT OF COUPLE STRESS AND GRAVITY ON THE PROPAGATION OF WAVES IN AN ELASTIC LAYER

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Abstract

The object of the present paper is to investigate the joint effect of couple-stress and gravity on the propagation of waves in an elastic layer. It is found that the velocity of propagation of waves in an elastic layer increases due to the presence of couple-stress and the effect of gravity has some effect on the wave velocity when the length of the wave is small compared with the thickness of the layer. It is clear from the phase velocity equation that joint effect of couple-stresses and gravity is superposing effect when this two are acting separately.

Keyword and phrases : *elastic layer, couple-stress, gravity, wave propagation.*

সংক্ষিপ্তসার

বর্ত্মান প্রবন্ধের উদ্দেশ্য হচ্ছে একটি সমতল স্তরে তরঙ্গ প্রবাহের উপর যুগ্মবলজ পীড়ণ এবং অভিকর্ষের যুগ্ম প্রভাবকে অনুসন্ধান করা। এটা দেখা গেছে যে একটি সমতল স্তরে তরঙ্গ প্রবাহের গতিবেগ যুগ্মবলজ পীড়ণের জন্য বৃদ্ধি পায় এবং যখন তরঙ্গের দৈর্ঘ্য স্তরের বেধের তুলনায় ক্ষুদ্র হয় তরঙ্গ-গতিবেগের উপর অভিকর্ষের কিছু প্রভাব লক্ষ্য করা যায়। দশা গতিবেগ সমীকরণ থেকে এটা স্পষ্ট যে যুগ্মবলজ পীড়ণ এবং অভিকর্ষের যুগ্ম প্রভাব একটি উপরিপাত ফল যখন এই দুইটি আলাদাভাবে ক্রিয়া করে।

1. Introduction

In recent years effect of couple-stresses in mechanics of continua is receiving greater attention by many investigators owing to its theoretical and practical importance. The concept of couple-stress in mechanics of continua was originally introduce by Voigt, W [1] and Cosseret brothers [2]. Subsequently Mindlin and Tiersten [8] formulated a linearised theory of couple-stress in elasticity. Using this theory a good many fundamental problems were solved by Sengupta, P.R. and others [6,11]. In the classical theory of elastic waves and vibrations the effect of gravity has not been considered in details. It was Bromwich [4] who first considered the effect of gravity on elastic waves and in particular on an elastic globe. Love [5] presented the influence of gravity on superficial waves and it is shown that rayleigh wave velocity is effected by gravity field.

Boit [3] investigated the influence of gravity on Rayleigh waves assuming the force of gravity to create a type of initial stress of hydrostatic nature and the medium to be incompressible. Subsequently, without assuming the medium to be incompressible Sengupta and Dey [7] investigated a good many fundamental problems of propagation of waves in elastic solids including the effect of gravity as type of initial stress as formulated by Biot in his initial stress theory. These problems were solved in a very general way. Bhattacharyya, P.C. and Senguta, P.R. [9], studied influence of gravity on propagation of waves in a composite elastic layer. Acharya, D.P., Roy, I and Chakraborty, H.S. [10] have investigated the problem on interface waves in second order thermo – visco – elastic solid media under the influence of gravity.

Following the linearised theory of couple-stress of Mindlin and Tiersten [8] and the influence of gravity to create a type of initial stress as suggested by Biot, the author of the present paper has investigated the joint effect of couple-stress and gravity of the propagation of waves in an elastic layer. It is found that effect of couple-stresses increases the wave velocity and the effect of gravity has some effect on the wave velocity when the length of the wave is small in comparison with the thickness of the layer, while in the country case the effect of gravity is insignificant.

2. Statement of the Problem and the Boundary Conditions

From the nature of the problem the non – zero displacements u_1 and u_3, at any point may be written in the form

$$u_1 = \frac{\partial\phi}{\partial x_1} - \frac{\partial\psi}{\partial x_3} \quad , \quad u_3 = \frac{\partial\phi}{\partial x_3} + \frac{\partial\psi}{\partial x_1} \tag{1}$$

Where ϕ and ψ are displacement potentials which are functions of the co-ordinates x_1 , x_3 and time t.

The displacement equations of motion are

$$\mu\nabla^2 u_1 + (\lambda+\mu)\frac{\partial^2 u_3}{\partial x_3 \partial x_1} + \eta\nabla^2\left(\frac{\partial^2 u_3}{\partial x_3\partial x_1} - \nabla^2 u_1\right) + \rho g\frac{\partial u_3}{\partial x_1} = \rho\frac{\partial^2 u_1}{\partial t^2} \tag{2.1}$$

$$\mu\nabla^2 u_3 + (\lambda+\mu)\frac{\partial u_1}{\partial x_1 \partial x_3} + \eta\nabla^2\left(\frac{\partial u_1}{\partial x_1\partial x_3} - \nabla^2 u_3\right) - \rho g\frac{\partial u_1}{\partial x_1} = \rho\frac{\partial^2 u_3}{\partial t^2} \tag{2.2}$$

The equation of motion (2.1) and (2.2) in view of (1), yields the following differential equations

$$\nabla^2\phi - \frac{1}{c_1^2}\ddot{\phi} + \frac{g}{c_1^2}\frac{\partial\psi}{\partial x_1} = 0 \tag{3.1}$$

$$\nabla^2\psi - l^2\nabla^4\psi - \frac{1}{c_2^2}\ddot{\psi} - \frac{g}{c_2^2}\frac{\partial\phi}{\partial x_1} = 0 \tag{3.2}$$

where

$$C_1^2 = \frac{\lambda+2\mu}{\rho}, C_2^2 = \frac{\mu}{\rho} \text{ and } l^2 = \frac{\eta}{\mu}$$

η is a constant characterizing the existence of couple stresses and λ, μ are Lami's elastic constants. If l, the parameter of couple stress, be zero the classical result of the corresponding problem follows at once.

Our problem here is to seek solutions of the equations (3.1) and (3.2) subject to the boundary conditions

$$\sigma_{33} = 2\mu e_{33} + \lambda(e_{11} + e_{33}) \tag{4}$$

The stresses σ_{ij} and couple-stress μ_{ij} are given by [1,2,3]

$$\begin{aligned}
&\sigma_{33} = 2\mu e_{33} + \lambda(e_{11} + e_{33}) \\
&\sigma_{11} = 2\mu e_{11} + \lambda(e_{11} + e_{33}) \\
&\sigma_{13} + \sigma_{31} = 4\mu e_{13}\ , \quad \mu_{32} = 4\eta(e_{31,3} - e_{33,1}) \\
&\sigma_{13} - \sigma_{31} = \mu_{12,1} + \mu_{32,3} \qquad \mu_{12} = 4\eta(e_{11,3} - e_{13,1}) \\
&e_{ii} = u_{i,i}\ , \quad e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)
\end{aligned} \tag{5}$$

On substituting for u_1 and u_3 from (1) in (5) we get

$$\begin{aligned}
&\sigma_{11} = 2\mu(\phi,_{11} - \psi,_{13}) + \lambda\nabla^2\phi \\
&\sigma_{33} = 2\mu(\phi,_{33} + \psi,_{13}) + \lambda\nabla^2\phi \\
&\sigma_{13} = \mu\left(2\phi,_{13} - \psi,_{33} + \psi,_{11}\right) - \eta\nabla^4\psi \\
&\sigma_{31} = \mu\left(2\phi,_{31} - \psi,_{33} + \psi,_{11}\right) + \eta\nabla^4\psi \\
&\mu_{32} = -2\eta\nabla^2(\psi,_3) \\
&\mu_{12} = -2\eta\nabla^2(\psi,_1)
\end{aligned} \tag{6}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}$$

3. Solution and Phase Velocity Equation.

To solve the equations (3.1) and (3.2) we assume

$$\phi(x_1, x_3, t) = \phi^*(x_3)e^{i(\omega t - \alpha x_1)} \tag{7}$$

$$\psi(x_1,x_3,t)=\psi^*(x_3)e^{i(\omega t-\alpha x_1)}$$

Introducing equation (7) into equations (3.1) and (3.2) we arrive at two ordinary differential equations

$$\left(\frac{\partial^2}{\partial x_3^2}-\nu_1^2\right)\phi^*-\frac{i\alpha g}{c_1^2}\psi^*=0 \tag{8.1}$$

$$\left[l^2\left(\frac{\partial^4}{\partial x_3^2}-\alpha^2\right)-\left(\frac{\partial^4}{\partial x_3^2}-\nu_2^2\right)\right]\psi^*-\frac{i\alpha g}{c_2^2}\phi^*=0 \tag{8.2}$$

where

$$\nu_1^2=\alpha^2-\frac{\omega^2}{c_1^2}\quad\text{and}\quad\nu_2^2=\alpha^2+l^2\alpha^4-\frac{\omega^2}{c_2^2}$$

Eliminating $\phi^* and \psi^*$ from equations (8.1) and (8.2) we have

$$\left[\left\{l^2\left(\frac{\partial^2}{\partial x_3^2}-\alpha^2\right)^2-\left(\frac{\partial^2}{\partial x_3^2}-\nu_2^2\right)\right\}\left\{\frac{\partial^2}{\partial x_3^2}-\nu_1^2\right\}+\frac{\alpha^2g^2}{c_1^2c_2^2}\right](\phi^*,\psi^*)=0 \tag{9}$$

where

The solution of the differential equation (9) are as follows

$$\begin{aligned}\phi^*&=A\sinh\lambda_1x_3+B\cosh\lambda_1x_3+C\sinh\lambda_2x_3+D\cosh\lambda_2x_3\\&\quad+E\sinh\lambda_3x_3+F\cosh\lambda_3x_3\\\psi^*&=A'\sinh\lambda_1x_3+B'\cosh\lambda_1x_3+C'\sinh\lambda_2x_3+D'\cosh\lambda_2x_3\\&\quad+E'\sinh\lambda_3x_3+F'\cosh\lambda_3x_3\end{aligned} \tag{10}$$

where,

$$\lambda_1^2+\lambda_2^2+\lambda_3^2=\frac{l^2\nu_1^2+(1+2\alpha^2l^2)}{l^2}$$

$$\lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2=\frac{(l^2\alpha^4+\nu_2^2)+\nu_1^2(1+2\alpha^2l^2)}{l^2} \tag{11}$$

$$\lambda_1^2\lambda_2^2\lambda_3^2 = \frac{c_1^2c_2^2\nu_1^2(l^2\alpha^4+\nu_2^2)-\alpha^2g^2}{c_1^2c_2^2l^2}$$

Putting the values of $\phi^* and \psi^*$ in equation (8.1) we get

$$A' = -im_1A, \quad B' = -im_1B, \quad C' = -im_2C, \quad D' = -im_2D,$$
$$E' = -im_3E, \quad F' = -im_3F \tag{12}$$

where,

$$m_j = \frac{c_1^2(\lambda_j^2-\nu_1^2)}{\alpha g}, \quad j = 1,\ 2, 3$$

and ultimately we get

$$\phi = (A\sinh\lambda_1x_3 + B\cosh\lambda_1x_3 + C\sinh\lambda_2x_3 + D\cosh\lambda_2x_3$$
$$+E\sinh\lambda_3x_3 + F\cosh\lambda_3x_3)e^{i(\alpha t-\alpha x_1)}$$

$$\psi = -i(Am_1\sinh\lambda_1x_3 + Bm_1\cosh\lambda_1x_3 + Cm_2\sinh\lambda_2x_3 + Dm_2\cosh\lambda_2x_3$$
$$+Em_3\sinh\lambda_3x_3 + Fm_3\cosh\lambda_3x_3)e^{i(\alpha t-\alpha x_1)} \tag{13}$$

Introducing the value of $\phi^* and \psi^*$ from equation (13) into equation (4) and also using boundary condition we get

$$A(\xi_1p_1-\eta_1q_1)+B(\xi_1q_1-\eta_1p_1)+C(\xi_2p_2-\eta_2q_2)+D(\xi_2q_2-\eta_2p_2)$$
$$+E(\xi_3p_3-\eta_3q_3)+F(\xi_3q_3-\eta_3p_3)=0 \tag{14.1}$$

$$-A(\xi_1p_1+\eta_1q_1)+B(\xi_1q_1+\eta_1p_1)-C(\xi_2p_2+\eta_2q_2)+D(\xi_2q_2+\eta_2p_2)$$
$$-E(\xi_3p_3+\eta_3q_3)+F(\xi_3q_3+\eta_3p_3)=0 \tag{14.2}$$

$$A(n_1p_1-l_1q_1)+B(n_1q_1-l_1p_1)+C(n_2p_2-l_2q_2)+D(n_2q_2-\eta_2p_2)$$
$$+E(n_3p_3-l_3q_3)+F(n_3q_3-l_3p_3)=0 \tag{14.3}$$

$$-A(n_1p_1+l_1q_1)+B(n_1q_1+l_1p_1)-C(n_2p_2+l_2q_2)+D(n_2q_2+\eta_2p_2)$$
$$-E(n_3p_3+l_3q_3)+F(n_3q_3+l_3p_3)=0 \tag{14.4}$$

$$A\, r_1\, q_1 + B\, r_1\, p_1 + C\, r_2\, q_2 + D\, r_2\, p_2 + E\, r_3\, q_3 + F\, r_3\, p_3 = 0 \tag{14.5}$$

$$A r_1 q_1 - B r_1 p_1 + C r_2 q_2 - D r_2 p_2 + E r_3 q_3 - F r_3 p_3 = 0 \quad (14.6)$$

Eliminating A, B, C, D, E, F from equations (14.1) to (14.6) we get

$$\begin{vmatrix} \xi_1 p_1 & \eta_1 p_1 & \xi_2 p_2 & \eta_2 p_2 & \xi_3 p_3 & \eta_3 p_3 \\ n_1 p_1 & l_1 p_1 & n_2 p_2 & l_2 p_2 & n_3 p_3 & l_3 p_3 \\ 0 & r_1 p_1 & 0 & r_2 p_2 & 0 & r_3 p_3 \\ \eta_1 q_1 & \xi_1 q_1 & \eta_2 q_2 & \xi_2 q_2 & \eta_3 q_3 & \xi_3 q_3 \\ l_1 q_1 & n_1 q_1 & l_2 q_2 & n_2 q_2 & l_3 q_3 & n_3 q_3 \\ r_1 q_1 & 0 & r_2 q_2 & 0 & r_3 q_3 & 0 \end{vmatrix} = 0 \quad (15)$$

when,

$$\xi_j = c_1^2\left(\lambda_j^2 - \alpha^2\right) + 2c_2^2\alpha^2$$

$$\eta_j = 2c_2^2\alpha\lambda_j m_j \quad (16)$$

$$l_j = 2\alpha\lambda_j$$

$$n_j = m_j\left(\lambda_j^2 + \alpha^2\right) - \frac{\eta}{\mu} m_j\left(\lambda_j^4 + \alpha^4\right)$$

$$r_j = 2i\eta m_j\left(\lambda_j^2 - \alpha^2\right), \quad p_j = Sinh\lambda_j h, \quad q_j = Cosh\lambda_j h, \quad j = 1, 2, 3$$

4. Discussion.

The transcendental equation (15) is in the determinant form. It represent the wave velocity equation of wave propagated in the elastic solid layer under the influence of couple-stress and gravity. It is easy to note that this wave propagation velocity essentially depends upon the couple-stress and gravity field.

If the length of the wave is very small with respect to its thickness 2h of the layer then the quantities $\lambda_j h$, j = 1, 2, 3 are large and the approximations.

$$\frac{p_j}{q_j} = \tanh \lambda_j h \simeq 1, \quad j = 1, 2, 3 \tag{17}$$

may be made. Now introducing equation (17) in equation (15) and then using the properties of the determinant for simplification it is natural to expect Rayligh wave velocity equation in layered elastic solid medium under the influence of coupl-stress and gravity.

Thus we get,

$$\Delta = \Delta_1 . \Delta_2 = 0 \tag{18}$$

where,

$$\Delta_1 = \begin{vmatrix} \xi_1 + \eta_1 & \xi_2 + \eta_2 & \xi_3 + \eta_3 \\ n_1 + l_1 & n_2 + l_2 & n_3 + l_3 \\ 0 + r_1 & 0 + r_2 & 0 + r_3 \end{vmatrix} \tag{19}$$

and

$$\Delta_2 = \begin{vmatrix} \xi_1 - \eta_1 & \xi_2 - \eta_2 & \xi_3 - \eta_3 \\ n_1 - l_1 & n_2 - l_2 & n_3 - l_3 \\ 0 - r_1 & 0 - r_2 & 0 - r_3 \end{vmatrix} \tag{20}$$

From equation (18) we have

Either $\Delta_1 = 0$ or $\Delta_2 = 0$ (21)

It is clear that $\Delta_1 = 0$, represents the wave velocity equation corresponding to Rayligh waves propagating in a semi-infinite medium under the influence of couple-stress and gravity having the plane horizontal boundary in the upper most part of the elastic solid while the equation $\Delta_2 = 0$ is the wave velocity equation for a semi-infinite medium under the influence of couple-stress and gravity having its horizontal plane boundary in the lower most part of the solid, though the thickness of the layer is assumed to be finite and large, in the case of Rayligh wave we shall consider that it is a semi-infinite medium with a plane boundary existing at the upper most part or at the lower most of the semi-infinite medium and that is why Rayligh wave velocity equation occurs twice according as the free plane boundary is at the upper most side or lower most side of the medium.

If this plane boundary be at the upper most side, the medium is extended to infinity at the lower and the wave velocity equation $\Delta_1 = 0$ represent the type of wave propagating in the vicinity of the free plane upper boundary. In a similar manner the wave velocity equation $\Delta_2 = 0$ represents the type of Rayligh wave propagating in the vicinity of lower plane boundary treating the medium to be extended towards infinity at the upper side.

It is obvious from the mathematical form of the equations $\Delta_1 = 0$ and $\Delta_2 = 0$, that they are interchangeable simply by changing λj by $-\lambda j$ (j = 1,2,3) i.e. by changing the direction of the x_3 axis. It explains the existence of 2 – wave velocity equations.

Let us first study the equation $\Delta_1 = 0$ and the equation $\Delta_2 = 0$ may be treated similarly.

J.Mech.Cont.& Math. Sci., Vol.-5, No.-2, January (2011) Pages 664 – 675

When we consider that the effect of couple-stress on the propagation of the waves in an elastic layer is absent irrespective of the thickness of the layer take, $l=0$ and so $\lambda_3=0$ and the value of λ_1 and λ_2 may be obtained by the relation given bellow.

$$\lambda_1^2+\lambda_2^2=\nu_1^2+\nu_2^2$$

$$\lambda_1^2\lambda_1^2=\nu_1^2\nu_1^2-\nu_3^2$$

$$\nu_3^2=\frac{\alpha^2 g^2}{c_1^2 c_2^2} \tag{22}$$

Neglecting the effect of couple-stress in $\Delta_1=0$ of the equation (18) we obtain the wave velocity equation for Rayligh wave under the influence of gravity only

$$(\xi_1+\eta_1)(l_2+n_2)=(\xi_2+\eta_2)(l_1+n_1) \tag{23}$$

This is an good agreement with the paper of De and Sengupta [7]

Let us consider the length of the wave is large compared with the thickness of the layer, the quantities $\lambda_1 h,\lambda_2 h,\alpha h$ can be regarded as small and hyperbolic tangents are replaced by their arguments putting the value of ξ_j,η_j,n_j,l_j where j = 1,2 and neglecting the effect of gravity in equation (23) we obtain

$$c^2c_1^2=4c_2^2(c_1^2-c_2^2) \tag{24}$$

which is classical equation and also this determines the wave velocity in the elastic layer.

From the above discussion we can conclude that joint effect of couple stress and gravity is superposing effect when this are acting separately.

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