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ON $\kappa^{\lambda,\mu,\nu,\beta}$ SUMMABILITY OF A QUADRUPLE FOURIER SERIES

By

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Abstract.

In this paper, Fourier analysis began as an attempt to approximate periodic functions with infinite summations of trigonometric polynomials. For certain functions, these sums, known as Fourier series, converge exactly to the original function. Here extending the result of R. Islam & M. Zaman (1999), a theorem on $k^{\lambda,\mu,\nu,\beta}$ *summability of quadruple Fourier series has been established.*

Keywords and phrases : *Fourier series, approximate periodic function, infinite summation, quadruple Fourier series*

বিমূর্ত সার ***(Bengali version of the Abstract)***

ত্রিকোনোমিতিক বহুপদরাশির অসীম সমষ্টিসহ পর্যাবৃত্ত অপেক্ষকের সন্নিকটস্থ মান নির্ধারণের একটি প্রচেষ্টা হিসাবে ফুরিয়ার বিশ্লেষণ শুরু হয়েছিল। নিশ্চত অপেক্ষকের জন্য ফুরিয়ার শ্রেণি হিসাবে জ্ঞাত এই সমষ্টিগুলি মূল অপেক্ষকে যথার্থ অভিসারী হয়। এখানে আর. ইসলাম (R.Islam) এবং এম. জামান (M.Zaman) - এর ফলাফলকে বর্ধিত করে চতুর্থ ক্রমের ফুরিয়ার শ্রেণির সমস্টি $k^{\lambda,\mu,\nu,\beta}$ *-- এর তত্ত্বকে প্রতিষ্ঠিত করা হয়েছে*

Introduction:

The method k^{λ} was first introduced by Karamata (1935). Lototsky (1963) reintroduced the special case $\lambda = 1$. Only after the paper of Agnew (1957), an intensive study of these and similar methods took place. *Vuĉkoviĉ* (1965) and Kathal (1969) established some interesting results on

k^{λ} summability of Fourier series. Shyam Lal (1997) extended the result of kathal for $k^{\lambda,\mu}$ summability of double Fourier series. R. Islam & M. Zaman (1999) extended the result of Shyam Lal for $k^{\lambda,\mu,\nu}$ summability of triple Fourier series. In this paper, the result of R. Islam & M. Zaman's is extended for quadruple Fourier series.

Some Basic Concept:

1.1 Let $f(w,x,y,z)$ be a periodic function with period 2π in each case which is summable in the quadrate $(-\pi,-\pi,-\pi,-\pi;\pi,\pi,\pi,\pi)$. Then the Fourier series of $f(w,x,y,z)$ is given by

$$f(w,x,y,z)=$$

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{k,l,m,n}\begin{bmatrix} a_{k,l,m,n}\cos kw\cos lx\cos my\cos nz+b_{k,l,m,n}\cos kw\cos lx\cos my\sin nz \\ +c_{k,l,m,n}\cos kw\cos lx\sin my\cos nz+d_{k,l,m,n}\cos kw\sin lx\cos my\cos nz \\ +e_{k,l,m,n}\cos kw\cos lx\sin my\sin nz+f_{k,l,m,n}\cos kw\sin lx\cos my\sin nz \\ +g_{k,l,m,n}\cos kw\sin lx\sin my\cos nz+h_{k,l,m,n}\cos kw\sin lx\sin my\sin nz \\ +i_{k,l,m,n}\sin kw\sin lx\sin my\sin nz+j_{k,l,m,n}\sin kw\sin lx\sin my\cos nz \\ +o_{k,l,m,n}\sin kw\sin lx\cos my\sin nz+p_{k,l,m,n}\sin kw\cos lx\sin my\sin nz \\ +q_{k,l,m,n}\sin kw\sin lx\cos my\cos nz+r_{k,l,m,n}\sin kwcolx\sin my\cos nz \\ +s_{k,l,m,n}\sin kw\cos lx\cos my\sin nz+t_{k,l,m,n}\sin kw\cos lx\cos my\cos nz \end{bmatrix}w$$

here,

$$\lambda_{k,l,m,n}=\begin{cases} \frac{1}{16} & k=l=m=n=0 \\ \frac{1}{8} & k>0,l=m=n=0;\ l>0,k=m=n=0;\ m>0,k=l=n=0;\ n>0,k=l=m=0 \\ \frac{1}{4} & k,l>0,m=n=0;\quad l,m>0,k=n=0;\quad m,n>0,k=l=0;\quad k,n>0,l=m=0 \\ \frac{1}{2} & k,l,m>0,n=0;\quad l,m,n>0,k=0;\quad k,m,n>0,l=0;\quad k,l,n>0,m=0 \\ 1 & k,l,m,n\ge 1 \end{cases} \tag{1}$$

and $a_{k,l,m,n}=\dfrac{1}{\pi^4}\int\int\int\int f(w,x,y,z)\cos kw\cos lx\cos my\cos nzdw\ dx\ dy\ dz$

with similar expressions for other coefficients where c denotes fundamental quadrate.

1.2 Let us define the number, $\begin{bmatrix} k \\ s' \end{bmatrix}$ for $k = 1, 2, 3, ...$ and $0 \le s \le 1$,

$$\prod_{v=0}^{k-1} (w+v) = \frac{\Gamma(w+k)}{\Gamma(w)} = \sum_{s'=0}^{k} \begin{bmatrix} k \\ s' \end{bmatrix} w^{s'} \tag{2}$$

$$\prod_{v=0}^{l-1} (x+v) = \frac{\Gamma(x+l)}{\Gamma(x)} = \sum_{p'=0}^{l} \begin{bmatrix} l \\ p' \end{bmatrix} x^{p'} \tag{3}$$

$$\prod_{v=0}^{m-1} (y+v) = \frac{\Gamma(y+m)}{\Gamma(y)} = \sum_{q'=0}^{m} \begin{bmatrix} m \\ q' \end{bmatrix} y^{q'} \tag{4}$$

and $$\prod_{v=0}^{n-1} (z+v) = \frac{\Gamma(z+n)}{\Gamma(z)} = \sum_{r'=0}^{n} \begin{bmatrix} n \\ r' \end{bmatrix} z^{r'} \tag{5}$$

The numbers $\begin{bmatrix} k \\ s' \end{bmatrix}, \begin{bmatrix} l \\ p' \end{bmatrix}, \begin{bmatrix} m \\ q' \end{bmatrix}$ and $\begin{bmatrix} n \\ r' \end{bmatrix}$ are known as the absolute value of Stirling numbers of first kind.

1.3 Let, $\{S_{k,l,m,n}\}$ be the sequence of partial sums of quadruple infinite series $\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \alpha_{k,l,m,n}$ and

$$S_{k,l,m,n} = \frac{\Gamma(\lambda)}{\Gamma(\lambda+k)} \frac{\Gamma(\mu)}{\Gamma(\mu+l)} \frac{\Gamma(\nu)}{\Gamma(\nu+m)} \frac{\Gamma(\beta)}{\Gamma(\beta+n)} \sum_{s'=0}^{k}\sum_{p'=0}^{l}\sum_{q'=0}^{m}\sum_{r'=0}^{n} \begin{bmatrix} k \\ s' \end{bmatrix}\begin{bmatrix} l \\ p' \end{bmatrix}\begin{bmatrix} m \\ q' \end{bmatrix}\begin{bmatrix} n \\ r' \end{bmatrix} \lambda^{s'} \mu^{p'} \nu^{q'} \beta^{r'} \tag{6}$$

to denote (k,l,m,n)th $K^{\lambda,\mu,\nu,\beta}$ mean of order $(\lambda,\mu,\nu,\beta) > 0$ if $S_{k,l,m,n}^{\lambda,\mu,\nu,\beta} \to S$ as $(k,l,m,n) \to \infty$ where S is a fixed finite quantity , then the sequence

$\{S_{k,l,m,n}\}$ is said to be summable by Karamata method $K^{\lambda,\mu,\nu,\beta}$ of order $(\lambda,\mu,\nu,\beta)>0$ to the sum S and we can write $S^{\lambda,\mu,\nu,\beta}_{k,l,m,n} \to S(K^{\lambda,\mu,\nu,\beta})$ as $(k,l,m,n)\to\infty$

(7)

The method $K^{\lambda,\mu,\nu,\beta}$ is regular for $(\lambda,\mu,\nu,\beta)>0$ and we know that $\phi(w,x,y,z)=\int_0^w dH\int_0^x dh\int_0^y ds\int_0^z |\phi(H,h,\underline{s},\underline{t})|dt$ [See details in R. Islam & M. Zaman (1999)]

and
$$K_{k(w)}=\frac{\sum_{s'=0}^{k}\begin{bmatrix}k\\ s'\end{bmatrix}\lambda^{s'}\sin\left(s'+\frac{1}{2}\right)w}{\Gamma(\lambda+k)\sin\frac{w}{2}},$$

$$K_{l(x)}=\frac{\sum_{p'=0}^{l}\begin{bmatrix}l\\ p'\end{bmatrix}\mu^{p'}\sin\left(p'+\frac{1}{2}\right)x}{\Gamma(\mu+l)\sin\frac{x}{2}}$$

$$K_{m(y)}=\frac{\sum_{q'=0}^{m}\begin{bmatrix}m\\ q'\end{bmatrix}\nu^{q'}\sin\left(q'+\frac{1}{2}\right)y}{\Gamma(\nu+m)\sin\frac{y}{2}},$$

$$K_{n(z)}=\frac{\sum_{r'=0}^{n}\begin{bmatrix}n\\ r'\end{bmatrix}\beta^{r'}\sin\left(r'+\frac{1}{2}\right)z}{\Gamma(\beta+n)\sin\frac{z}{2}}$$

1.4 The following lemmas are essential for the proof of our theorem [See for details in *Vuĉkoviĉ* (1965)]

a. Let $\lambda>0$ and $0<w<\frac{\pi}{2}$ then $\frac{\operatorname{Im}\Gamma\left(\left|\lambda e^{iw}+k\right|\right)}{\Gamma\left(\lambda\cos w+k\right)\sin\frac{w}{2}}$

$$=\frac{|\sin(\lambda\log k.\sin w)|}{\sin\frac{w}{2}}+o(1),\quad \text{as}\quad k\to\infty$$

uniformly in w.

b. Let $\mu > 0$ and $0 < x < \frac{\pi}{2}$ then

$$\frac{\operatorname{Im}\Gamma\left(\left|\mu e^{ixl} + l\right|\right)}{\Gamma\left(\mu\cos x + l\right)\sin\frac{x}{2}} = \frac{\left|\sin(\mu\log l.\sin x)\right|}{\sin\frac{x}{2}} + o(1), \text{as } l \to \infty \ ,$$

uniformly in x.

c. Let $\nu > 0$ and $0 < y < \frac{\pi}{2}$

then $$\frac{\operatorname{Im}\Gamma\left(\left|\nu e^{iy} + m\right|\right)}{\Gamma\left(\nu\cos y + m\right)\sin\frac{y}{2}} = \frac{\left|\sin(\nu\log m.\sin y)\right|}{\sin\frac{y}{2}} + o(1), \text{as } m \to \infty,$$

uniformly in y.

d. Let $\beta > 0$ and $0 < z < \frac{\pi}{2}$

then $$\frac{\operatorname{Im}\Gamma\left(\left|\beta e^{iz} + n\right|\right)}{\Gamma\left(\beta\cos z + n\right)\sin\frac{z}{2}} = \frac{\left|\sin(\beta\log n.\sin z)\right|}{\sin\frac{z}{2}} + o(1), \text{as} \quad n \to \infty,$$

uniformly in z.

Theorem: On $K^{\lambda,\mu,\nu,\beta}$ summability of quadruple Fourier series has been obtained in the following form: If

$$\phi(w,x,y,z) = \int_0^w dH \int_0^x dh \int_0^y ds \int_0^z \left|\phi(H,h,s,t)\right| dt = o\left[\frac{w}{\log\left(\frac{1}{w}\right)} \frac{x}{\log\left(\frac{1}{x}\right)} \frac{y}{\log\left(\frac{1}{y}\right)} \frac{z}{\log\left(\frac{1}{z}\right)}\right], \text{ as } (w,x,y,z) \to +0$$

(8)

then the quadruple Fourier series (1) of the function $f(w,x,y,z)$ is summable $K^{\lambda,\mu,\nu,\beta}$ $(\lambda,\mu,\nu,\beta) > 0$) to the sum $f(s,t,u,v)$ at $w = s,\ x = t,\ y = u,\ z = v$.

Proof: Let $S_{s',p',q',r'}(s,t,u,v)$ denote the partial sum of the series (1) as $w = s, x = t, y = u, z = v$ then we have

$$S_{s',p',q',r'} - f(s,t,u,v) = \frac{1}{16\pi^4}\int_0^\pi\int_0^\pi\int_0^\pi\int_0^\pi \phi(w,x,y,z)\frac{\sin\left(s'+\frac{1}{2}\right)w\sin\left(p'+\frac{1}{2}\right)x\sin\left(q'+\frac{1}{2}\right)y\sin\left(r'+\frac{1}{2}\right)z}{\sin\frac{w}{2}\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}}dwdxdydz$$

Now

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+k)}\cdot\frac{\Gamma(\mu)}{\Gamma(\mu+l)}\cdot\frac{\Gamma(\nu)}{\Gamma(\nu+m)}\cdot\frac{\Gamma(\beta)}{\Gamma(\beta+n)}\cdot \sum_{s'=0}^{k}\sum_{p'=0}^{l}\sum_{q'=0}^{m}\sum_{r''=0}^{n}\begin{bmatrix}k\\ s'\end{bmatrix}\begin{bmatrix}l\\ p'\end{bmatrix}\begin{bmatrix}m\\ q'\end{bmatrix}\begin{bmatrix}n\\ r'\end{bmatrix}\lambda^{s'}\mu^{p'}\nu^{q'}\beta^{r'}.\left\{S_{s',p',q',r'} - f(s,t,u,v)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\mu)\Gamma(\nu)\Gamma(\beta)}{8\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi\int_0^\pi \phi(w,x,y,z)\sum_{s'=0}^{k}\begin{bmatrix}k\\ s'\end{bmatrix}\frac{\lambda^{s'}\sin\left(s'+\frac{1}{2}\right)w}{\Gamma(\lambda+k)\sin\frac{w}{2}}\sum_{p'=0}^{l}\begin{bmatrix}l\\ p'\end{bmatrix}\frac{\mu^{p'}\sin(p'+\frac{1}{2})x}{\Gamma(\mu+l)\sin\frac{x}{2}}.$$

$$\sum_{q'=0}^{m}\begin{bmatrix}m\\ q'\end{bmatrix}\frac{\nu^{q'}\sin\left(q'+\frac{1}{2}\right)y}{\Gamma(\nu+m)\sin\frac{y}{2}}\sum_{r''=0}^{n}\begin{bmatrix}n\\ r'\end{bmatrix}\frac{\beta^{r'}\sin\left(r'+\frac{1}{2}\right)z}{\Gamma(\beta+n)\sin\frac{z}{2}}dwdxdydz$$

$$\therefore S_{s,p,q,r}^{\lambda,\mu,\nu,\beta} - f(s,t,u,v)$$

$$= \frac{\Gamma(\lambda)\Gamma(\mu)\Gamma(\nu)\Gamma(\beta)}{16\pi^4}\int_0^\pi\int_0^\pi\int_0^\pi\int_0^\pi \phi(w,x,y,z)K_k(w)K_l(x)K_m(y)K_n(z)dwdxdydz$$

$$= o\left[\left\{\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} + \int_{\frac{1}{k}}^{\pi}\int_{\frac{1}{l}}^{\pi}\int_{\frac{1}{m}}^{\pi}\int_{1Ln}^{\pi} |\phi(w,x,y,z)||k_k(w)||k_l(x)||k_m(y)||k_n(z)|dwdxdydz\right\}\right]$$

$$= o(I') + o(I'')$$

(9)

Again,

$$\left|K_k(w)\right| = o\left[\frac{\operatorname{Im}\left\{e^{iw/2}\,\dfrac{\Gamma\left(\lambda e^{iw}+k\right)}{\Gamma\left(\lambda e^{iw}\right)}\right\}}{\Gamma(\lambda+k)\sin\dfrac{w}{2}}\right] = o\left[\frac{\operatorname{Im}\Gamma\left(\lambda e^{iw}+k\right)}{\Gamma(\lambda+k)\sin\dfrac{w}{2}}\right] + o\left[\frac{\operatorname{Re}\Gamma\left(\lambda e^{iw}+k\right)}{\Gamma(\lambda+k)}\right]$$

$$= o\left[\frac{\Gamma(\lambda\cos w+k).\operatorname{Im}\Gamma\left(\lambda e^{iw}+k\right)}{\Gamma(\lambda+k).\Gamma(\lambda\cos w+k)\sin\dfrac{w}{2}}\right] + o\left[\frac{\Gamma(\lambda\cos w+k)}{\Gamma(\lambda+k)}\right]$$

If $0 < w < \frac{1}{k}$, then

$$\frac{\Gamma(\lambda\cos w+k)}{\Gamma(\lambda+k)} = \left[K^{-\lambda(1-\cos w)}\right] = \left[K^{-\lambda(1-\cos w)\log k}\right] = \left[e^{-\frac{\lambda}{2}w^2\log k}\right]$$

since, for $0 < w < \frac{1}{k}$, $0 < 1-\cos w < \frac{w}{2}$

$$\therefore \left|K_k(w)\right| = o\left[e^{-\frac{\lambda}{2}w^2\log k}\,\frac{\operatorname{Im}\Gamma\left(\lambda e^{iw}+k\right)}{\Gamma(\lambda\cos w+k)\sin\dfrac{w}{2}}\right] + o\left[e^{-\frac{\lambda}{2}w^2\log k}\right],$$ for $0 < w < \frac{1}{k}$

Similarly,

$$\left|K_l(x)\right| = o\left[e^{-\frac{\mu}{2}x^2\log l}\,\frac{\operatorname{Im}\Gamma\left(\mu e^{ix}+l\right)}{\Gamma(\mu\cos x+l)\sin\dfrac{x}{2}}\right] + o\left[e^{-\frac{\mu}{2}x^2\log l}\right],\ \text{for } 0 < x < \frac{1}{l}$$

$$\left|K_m(y)\right| = o\left[e^{-\frac{\nu}{2}y^2\log m}\,\frac{\operatorname{Im}\Gamma\left(\nu e^{im}+m\right)}{\Gamma(\nu\cos y+m)\sin\dfrac{y}{2}}\right] + o\left[e^{-\frac{\nu}{2}y^2\log m}\right],\ \text{for } 0 < y < \frac{1}{m}$$

$$and \qquad |K_n(z)| = o\left[e^{-\frac{\beta}{2}z^2\log n}\frac{\operatorname{Im}\Gamma\left(\beta e^{iz}+n\right)}{\Gamma\left(\beta\cos z+n\right)\sin\frac{z}{2}}\right] + o\left[e^{-\frac{\beta}{2}z^2\log n}\right],$$

for $0 < z < \frac{1}{n}$

Now,

$$I' = o\left[\left\{\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}|\Phi(w,x,y,z)||K_k(w)||K_l(x)||K_m(y)||K_n(z)|\right\}dwdxdydz\right] \tag{10}$$

Substitute the values of $|K_k(w)|, |K_l(x)|, |K_m(y)|, |K_n(z)|$ in equation (10).

We will get sixteen terms and consider the terms are

$$I' = I'_{k1,l1,m1,n1} + I'_{k1,l1,m1,n2} + I'_{k1,l1,m2,n1} + I'_{k1,l2,m1,n1} + I'_{k2,l1,m1,n1} + I'_{k1,l1,m2,n2}$$
$$+ I'_{k1,l2,m2,n1} + I'_{k2,l2,m1,n1} + I'_{k1,l2,m1,n2} + I'_{k2,l1,m2,n1} + I'_{k2,l1,m1,n2} + I'_{k1,l2,m2,n2}$$
$$+ I'_{k2,l2,m2,n1} + I'_{k2,l1,m2,n2} + I'_{k2,l2,m1,n2} + I'_{k2,l2,m2,n2}$$

Now consider **1st term**,

$$I'_{k1,l1,m1,n1} = o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left[\begin{array}{l} e^{-\frac{\lambda}{2}w^2\log k}\dfrac{\operatorname{Im}\Gamma\left(\lambda e^{iw}+k\right)}{\Gamma\left(\lambda\cos w+k\right)\sin\frac{w}{2}}e^{-\frac{\mu}{2}x^2\log l}\dfrac{\operatorname{Im}\Gamma\left(\mu e^{ix}+l\right)}{\Gamma\left(\mu\cos x+l\right)\sin\frac{x}{2}} \\ .e^{-\frac{\nu}{2}y^2\log m}\dfrac{\operatorname{Im}\Gamma\left(\nu e^{iy}+m\right)}{\Gamma\left(\nu\cos y+m\right)\sin\frac{y}{2}}e^{-\frac{\beta}{2}z^2\log n}\dfrac{\operatorname{Im}\Gamma\left(\beta e^{iz}+n\right)}{\Gamma\left(\beta\cos z+n\right)\sin\frac{z}{2}}|\phi(w,x,y,z)| \end{array}\right]dwdxdydz\right]$$

$$=o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left[\begin{array}{c} e^{-\frac{\lambda}{2}w^2\log k}\dfrac{|\sin(\lambda\log k.\sin w)|}{\sin\frac{w}{2}}e^{-\frac{\mu}{2}x^2\log l}\dfrac{|\sin(\mu\log l.\sin x)|}{\sin\frac{x}{2}}e^{-\frac{\nu}{2}y^2\log m}\dfrac{|\sin(\nu\log m.\sin y)|}{\sin\frac{y}{2}} \\ .e^{-\frac{\beta}{2}z^2\log n}\dfrac{|\sin(\beta\log n.\sin z)|}{\sin\frac{z}{2}}|\phi(w,x,y,z)| \end{array}\right]dwdxdydz\right]$$

[using Lemma ***a, b, c, d***]

$$=o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left[\begin{array}{c}\dfrac{|\sin(\lambda\log k.\sin w)|}{\sin\frac{w}{2}}.\dfrac{|\sin(\mu\log l.\sin x)|}{\sin\frac{x}{2}}\dfrac{|\sin(\nu\log m.\sin y)|}{\sin\frac{y}{2}}\dfrac{|\sin(\beta\log n.\sin z)|}{\sin\frac{z}{2}}. \\ .|\phi(w,x,y,z)|\end{array}\right]dwdxdydz\right]$$

$$=o[(\lambda\log k).(\mu\log l).(\nu\log m).(\beta\log n)].\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}|\phi(w,x,y,z)|dwdxdydz$$

$$=o\left[\frac{(\lambda\log k).(\mu\log l).(\nu\log m).(\beta\log n)}{(k\log k).(l\log l).(m\log m).(n\log n)}\right] \quad \text{by using equation (8).}$$

$$=o\left[\frac{\lambda.\mu.\nu.\beta}{k.l.m.n}\right]=o(1) \quad \text{as} \quad (k.l.m.n)\to\infty.$$

2nd term,

$$I'_{k1,l1,m1,n2}=o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left[\begin{array}{c}e^{-\frac{\lambda}{2}w^2\log k}\dfrac{|\sin(\lambda\log k\sin w)|}{\sin\frac{w}{2}}.e^{-\frac{\mu}{2}x^2\log l}\dfrac{|\sin(\mu\log l\sin x)|}{\sin\frac{x}{2}}.e^{-\frac{\nu}{2}y^2\log m} \\ .\dfrac{|\nu\log m\sin y|}{\sin\frac{y}{2}}.e^{-\frac{\beta}{2}z^2\log n}|\phi(w,x,y,z)|\end{array}\right].dwdxdydz\right]$$

$$= o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} \frac{|\sin(\lambda\log k\sin w)|}{\sin\frac{w}{2}}\frac{|\sin(\mu\log l\sin x)|}{\sin\frac{x}{2}}\frac{|\sin(\nu\log m\sin y)|}{\sin\frac{y}{2}}|\phi(w,x,y,z)dwdxdydz|\right]$$

$$= o\left[\lambda\log k \quad \mu\log l \quad \nu\log m\right]\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}|\phi(w,x,y,z)|\,dwdxdydz$$

$$= o\left[\frac{\lambda\log k. \quad \mu\log l. \quad \nu\log m}{k\log k. \quad l\log l. \quad m\log m. \quad n\log n}\right] = o\left[\frac{\lambda\mu\nu}{klmn\log n}\right] = o(1) \quad \text{as} \quad (k,l,m,n)\to\infty.$$

3rd term,

$$I'_{k1,l1,m2,n1} = o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left[\begin{array}{l} e^{-\frac{\lambda}{2}w^2\log k}\frac{|\sin(\lambda\log k\sin w)|}{\sin\frac{w}{2}}.e^{-\frac{\mu}{2}x^2\log l}\frac{|\sin(\mu\log l\sin x)|}{\sin\frac{x}{2}}. \\ e^{-\frac{\nu}{2}y^2\log m}.e^{-\frac{\beta}{2}z^2\log n}.\frac{|\sin(\beta\log n\sin z)|}{\sin\frac{z}{2}}|\phi(w,x,y,z)| \end{array}\right]dwdxdydz\right]$$

$$= o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left[\frac{|\sin(\lambda\log k\sin w)|}{\sin\frac{w}{2}}\frac{|\sin(\mu\log l\sin x)|}{\sin\frac{x}{2}}\frac{|\sin(\beta\log n\sin z)|}{\sin\frac{z}{2}}.|\phi(w,x,y,z)z|\right]dwdxdyd\right]$$

$$= o\left[\lambda\log k\ \mu\log l\ \beta\log n\right]\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}|\phi(w,x,y,z)|dwdxdydz$$

$$= o\left[\frac{\lambda\log k.\mu\log l.\beta\log n}{k\log k.l\log l.m\log m.n\log n}\right] = o\left[\frac{\lambda\mu\beta}{klmn\log m}\right] = o(1) \ \ as \ \ (k,l,m,n)\to\infty.$$

4th term,

$$I'_{k1,l2,m1,n1} = o\left[\begin{array}{l}\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} e^{-\frac{\lambda}{2}w^2\log k}\frac{|\sin(\lambda\log k\sin w)|}{\sin\frac{w}{2}}.e^{-\frac{\mu}{2}x^2\log l}.e^{-\frac{\nu}{2}y^2\log m}\frac{|\nu\log m\sin y|}{\sin\frac{y}{2}} \\ \qquad .e^{-\frac{\beta}{2}z^2\log n}\frac{|\sin(\beta\log m\sin z)|}{\sin\frac{z}{2}}.|\phi(w,x,y,z)|dwdxdydz\end{array}\right]$$

$$= o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\frac{|\sin(\lambda\log k\sin w)|}{\sin\frac{w}{2}}\frac{|\sin(\nu\log m\sin y)|}{\sin\frac{y}{2}}\frac{|\sin(\beta\log n\sin z)|}{\sin\frac{z}{2}}.|\phi(w,x,y,z)dwdxdydz|\right]$$

$$= o[\lambda\log k\ \beta\log n\ \nu\log m]\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}|\phi(w,x,y,z)|dwdxdydz$$

$$= o\left[\frac{\lambda\log k.\beta\log n.\nu\log m}{k\log k.l\log l.m\log m.n\log n}\right] = o\left[\frac{\mu\beta\nu}{klmn\log l}\right] = o(1)\ \ as\ (k,l,m,n)\to\infty$$

5th term,

$$I'_{k2,l1,m1,n1} = o\left[\begin{array}{l}\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} e^{-\frac{\lambda}{2}w^2\log k}.e^{-\frac{\mu}{2}x^2\log l}\frac{|\sin(\mu\log l\sin x)|}{\sin\frac{x}{2}}.e^{-\frac{\nu}{2}y^2\log m}\frac{|\nu\log m\sin y|}{\sin\frac{y}{2}}\\ e^{-\frac{\beta}{2}z^2\log n}\frac{|\sin(\beta\log n\sin z)|}{\sin\frac{z}{2}}|\phi(w,x,y,z)|dwdxdydz\end{array}\right]$$

$$= o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\frac{|\sin(\beta\log n\sin z)|}{\sin\frac{z}{2}}\frac{|\sin(\mu\log l\sin x)|}{\sin\frac{x}{2}}\frac{|\sin(\nu\log m\sin y)|}{\sin\frac{y}{2}}.|\phi(w,x,y,z)|dwdxdydz\right]$$

$$= o[\beta\log n\ \mu\log l\ \nu\log m]\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}|\phi(w,x,y,z)|dwdxdydz$$

$$= o\left[\frac{\mu\log l.\beta\log n.\nu\log m}{k\log k.l\log l.m\log m.n\log n}\right] = o\left[\frac{\beta\mu\nu}{klmn\log k}\right] = o(1)\qquad as\ (k,l,m,n)\to\infty.$$

6th term,

$$I'_{k1,l1,m2,n2} = o\left[\begin{array}{l}\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} e^{-\frac{\lambda}{2}w^2\log k}\ \frac{|\sin(\lambda\log k\sin w)|}{\sin\frac{w}{2}} e^{-\frac{\mu}{2}x^2\log l}\ \frac{|\sin(\mu\log l\sin x)|}{\sin\frac{x}{2}} \\ e^{-\frac{\nu}{2}y^2\log m} e^{-\frac{\beta}{2}z^2\log n}\ |\phi(w,x,y,z)|dwdxdydz\end{array}\right]$$

$$= o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\frac{|\sin(\lambda\log k\sin w)|}{\sin\frac{w}{2}}\frac{|\sin(\mu\log l\sin x)|}{\sin\frac{x}{2}}.|\phi(w,x,y,z)|dwdxdydz\right]$$

$$= o[\lambda\log k\ \mu\log l\ ]\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}|\phi(w,x,y,z)|dwdxdydz$$

$$= o\left[\frac{\lambda\log k.\ \ \mu\log l.}{k\log k.\ \ l\log l.\ \ m\log m.\ \ n\log n}\right] = o\left[\frac{\lambda\mu}{klmnl\log m\log n}\right] = o(1) \quad \text{as} \quad (k,l,m,n)\to\infty$$

7th term,

$$I'_{k1,l2,m2,n1} = o\left[\begin{array}{l}\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} e^{-\frac{\lambda}{2}w^2\log k}\ \frac{|\sin(\lambda\log k\sin w)|}{\sin\frac{w}{2}} e^{-\frac{\mu}{2}x^2\log l} e^{-\frac{\nu}{2}y^2\log m} e^{-\frac{\beta}{2}z^2\log n} \\ \frac{|\sin(\beta\log n\sin z)|}{\sin\frac{z}{2}}.\ |\phi(w,x,y,z)|dwdxdydz\end{array}\right]$$

$$= o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} \frac{|\sin(\lambda \log k \sin w)|}{\sin\frac{w}{2}} \frac{|\sin(\beta \log n \sin z)|}{\sin\frac{z}{2}} .|\phi(w,x,y,z)| dwdxdydz\right]$$

$$= o[\lambda \log k \ \beta \log n]\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} |\phi(w,x,y,z)| dwdxdydz$$

$$= o\left[\frac{\lambda \log k. \ \ \beta \log n}{k\log k. \ \ l\log l. \ \ m\log m. \ \ n\log n}\right] = o\left[\frac{\lambda\beta}{klmn\log l \ \ \log m}\right] = o(1) \qquad \text{as} \quad (k,l,m,n)\to\infty.$$

8th term,

$$I'_{k2,l2,m1,n1} = o\left[\begin{array}{l}\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} e^{-\frac{\lambda}{2}w^2 \log k} .e^{-\frac{\mu}{2}x^2 \log l} .e^{-\frac{\nu}{2}y^2 \log m} \frac{|\nu \log m \sin y|}{\sin\frac{y}{2}}. \\ e^{-\frac{\beta}{2}z^2 \log n} \frac{|\sin(\beta \log n \sin z)|}{\sin\frac{z}{2}} |\phi(w,x,y,z)| dwdxdydz\end{array}\right]$$

$$= o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} \frac{|\sin(\beta \log n \sin z)|}{\sin\frac{z}{2}} \frac{|\sin(\nu \log m \sin y)|}{\sin\frac{y}{2}} .|\phi(w,x,y,z)| dwdxdydz\right]$$

$$= o[\beta \log n \ \ .\nu \log m]\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} |\phi(w,x,y,z)| dwdxdydz = o\left[\frac{\beta\log n. \ \ \nu\log m}{k\log k. \ \ l\log l. \ \ m\log m. \ \ n\log n}\right]$$

$$= o\left[\frac{\beta\nu}{klmn\log l\log k}\right] = o(1) \qquad \text{as} \quad (k,l,m,n)\to\infty.$$

9th term,

$$I'_{k1,l2,m1,n2} = o\left[\begin{array}{l}\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} e^{-\frac{\lambda}{2}w^2\log k}\dfrac{\left|\sin(\lambda\log k\sin w)\right|}{\sin\frac{w}{2}}.e^{-\frac{\mu}{2}x^2\log l}. \\ e^{-\frac{\nu}{2}y^2\log m}\dfrac{\left|\nu\log m\sin y\right|}{\sin\frac{y}{2}}.e^{-\frac{\beta}{2}z^2\log n}\left|\phi(w,x,y,z)\right|dwdxdydz\end{array}\right]$$

$$= o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\frac{\left|\sin(\lambda\log k\sin w)\right|}{\sin\frac{w}{2}}.\frac{\left|\sin(\nu\log m\sin y)\right|}{\sin\frac{y}{2}}.\left|\phi(w,x,y,z)\right|dwdxdydz\right]$$

$$= o\left[\lambda\log k.\nu\log m\right]\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left|\phi(w,x,y,z)\right|dwdxdydz$$

$$= o\left[\frac{\lambda\log k.\ \nu\log m}{k\log k.\ l\log l.\ m\log m.\ n\log n}\right] = o\left[\frac{\lambda\nu}{klmn\log l\log n}\right] = o(1)\quad \text{as}\quad (k,l,m,n)\to\infty.$$

10th term,

$$I'_{k2,l1,m2,n1} = o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left[\begin{array}{l}e^{-\frac{\lambda}{2}w^2\log k}.e^{-\frac{\mu}{2}x^2\log l}\dfrac{\left|\sin(\mu\log l\sin x)\right|}{\sin\frac{x}{2}}.e^{-\frac{\nu}{2}y^2\log m}. \\ .e^{-\frac{\beta}{2}z^2\log n}\dfrac{\left|\sin(\beta\log n\sin z)\right|}{\sin\frac{z}{2}}\left|\phi(w,x,y,z)\right|\end{array}\right]dwdxdydz\right]$$

$$= o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\frac{\left|\sin(\beta\log n\sin z)\right|}{\sin\frac{z}{2}}\frac{\left|\sin(\mu\log l\sin x)\right|}{\sin\frac{x}{2}}.\left|\phi(w,x,y,z)dwdxdydz\right|\right]$$

$$= o\left[\beta \log n.\mu \log l\ \right]\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} |\phi(w,x,y,z)|\,dwdxdydz$$

$$= o\left[\frac{\beta \log n.\ \ \mu \log l}{k\log k.\ \ l\log l.\ \ m\log m.\ \ n\log n}\right] = o\left[\frac{\beta\mu}{klmn \log k \log m}\right] = o(1) \qquad \text{as} \quad (k,l,m,n)\to\infty.$$

11^th^term,

$$I'_{k2,l1,m1,n2} = o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left[\begin{array}{l} e^{-\frac{\lambda}{2}w^2\log k}\, e^{-\frac{\mu}{2}x^2\log l}\,\dfrac{|\sin(\mu \log l \sin x)|}{\sin\frac{x}{2}} \\ e^{-\frac{\nu}{2}y^2\log m}\,\dfrac{|\nu \log m \sin y|}{\sin\frac{y}{2}}\, e^{-\frac{\beta}{2}z^2\log n}\,|\phi(w,x,y,z)| \end{array}\right] dwdxdydz\right]$$

$$= o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} \frac{|\sin(\mu\log l \sin x)|}{\sin\frac{x}{2}}.\frac{|\nu \log m \sin y|}{\sin\frac{y}{2}}.\ |\phi(w,x,y,z)|\,dwdxdydz\right]$$

$$= o\left[\mu \log l\, \nu \log m\right]\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} |\phi(w,x,y,z)|\,dwdxdydz = o\left[\frac{\mu\log l.\nu\log m}{k\log k.l\log l.m\log m.n\log n}\right]$$

$$= o\left[\frac{\mu\nu}{klmn\log k \log n}\right] = o(1) \qquad as\ (k,l,m,n)\to\infty.$$

12^th^term,

$$I'_{k1,l2,m2,n2} = o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left[\begin{array}{l} e^{-\frac{\lambda}{2}w^2\log k}\,.\dfrac{|\sin(\lambda \log k \sin w)|}{\sin\frac{w}{2}}.e^{-\frac{\mu}{2}x^2\log l} \\ .e^{-\frac{\nu}{2}y^2\log m}\,.e^{-\frac{\beta}{2}z^2\log n}.\ |\phi(w,x,y,z)| \end{array}\right] dwdxdydz\right]$$

$$= o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} \frac{|\sin(\lambda \log k \sin w)|}{\sin\frac{w}{2}}.|\phi(w,x,y,z)dwdxdydz|\right] = o[\lambda \log k]\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}|\phi(w,x,y,z)|dwdxdydz$$

$$= o\left[\frac{\lambda \log k}{k \log k . l \log l . m \log m . n \log n}\right] = o\left[\frac{\lambda}{klmn \log l . \log m . \log n}\right] = o(1) \quad as \ (k,l,m,n) \to \infty.$$

13th term,

$$I'_{k2,l2,m2,n1} = o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left[\begin{array}{l} e^{-\frac{\lambda}{2}w^2 \log k} . e^{-\frac{\mu}{2}x^2 \log l} . e^{-\frac{\nu}{2}y^2 \log m} . \\ e^{-\frac{\beta}{2}z^2 \log n} \dfrac{|\sin(\beta \log n \sin z)|}{\sin \frac{z}{2}}|\phi(w,x,y,z)| \end{array}\right] dwdxdydz\right]$$

$$= o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} \frac{|\sin(\beta \log n \sin z)|}{\sin\frac{z}{2}}.|\phi(w,x,y,z)|dwdxdydz\right] = o[\beta \log n]\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}|\phi(w,x,y,z)|dwdxdydz$$

$$= o\left[\frac{\beta \log n}{k \log k . l \log l . m \log m . n \log n}\right] = o\left[\frac{\beta}{klmn \log k . \log l . \log m}\right] = o(1) \quad as \ (k,l,m,n) \to \infty.$$

14th term,

$$I'_{k2,l1,m2,n2} = o\left[\begin{array}{l}\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} e^{-\frac{\lambda}{2}w^2\log k}.e^{-\frac{\mu}{2}x^2\log l}\dfrac{\left|\sin(\mu\log l\sin x)\right|}{\sin\frac{x}{2}}\\ .e^{-\frac{\nu}{2}y^2\log m}.e^{-\frac{\beta}{2}z^2\log n}\left|\phi(w,x,y,z)\right|dwdxdydz\end{array}\right]$$

$$= o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\frac{\left|\sin(\mu\log l\sin x)\right|}{\sin\frac{x}{2}}.\ \left|\phi(w,x,y,z)\right|dwdxdydz\right]$$

$$= o[\mu\log l\,]\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left|\phi(w,x,y,z)\right|dwdxdydz = o\left[\frac{\mu\log l}{k\log k.l\log l.m\log m.n\log n}\right]$$

$$= o\left[\frac{\mu}{klmn\ \log k.\log m.\log n}\right] = o(1)\quad as\ (k,l,m,n)\to\infty.$$

15th term,

$$I'_{k2,l2,m1,n2} = o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left[\begin{array}{l}e^{-\frac{\lambda}{2}w^2\log k}.e^{-\frac{\mu}{2}x^2\log l}.e^{-\frac{\nu}{2}y^2\log m.\frac{\left|\nu\log m\sin y\right|}{\sin\frac{y}{2}}}\\ .e^{-\frac{\beta}{2}z^2\log n}.\ \left|\phi(w,x,y,z)\right|\end{array}\right]dwdxdydz\right]$$

$$= o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\frac{\left|\nu\log m\sin y\right|}{\sin\frac{y}{2}}.\ \left|\phi(w,x,y,z)\right|dwdxdydz\right] = o[\nu\log m\,]\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left|\phi(w,x,y,z)\right|dwdxdydz$$

$$= o\left[\frac{\nu\log m}{k\log k.l\log l.m\log m.n\log n}\right] = o\left[\frac{\lambda\mu\nu}{klmn\log l.\log k.\log n}\right] = o(1)\ as\ (k,l,m,n)\to\infty.$$

16th term,

$$I'_{k2,l2,m2,n2} = o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} e^{-\frac{\lambda}{2}w^2\log k}.e^{-\frac{\mu}{2}x^2\log l}.e^{-\frac{\nu}{2}y^2\log m}.e^{-\frac{\beta}{2}z^2\log n}\left|\phi(w,x,y,z)\right|dwdxdydz\right]$$

$$= o\left[\int_0^{\frac{1}{k}}\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left|\phi(w,x,y,z)\right|dwdxdydz\right] = o\left[\frac{1}{k\log k.l\log l.m\log m.n\log n}\right]$$

$$= o\left[\frac{1}{klmn\log k.\log l.\log m.\log n}\right] = o(1) \qquad as\ (k,l,m,n)\to\infty.$$

Thus we get that $I' = o(1)$ as $(k.l.m.n)\to\infty$

(11)

For $\frac{1}{k} \le \partial_1 < w < \pi$, then $\left|K_k(w)\right| = o\left[\frac{l^{-\lambda(1-\cos\delta_1)}}{(\sin\frac{\delta_1}{2})}\right] = o[1]$ as $k \to \infty$

Similarly for, $\frac{1}{l} \le \partial_2 < x < \pi, \quad \left|K_l(x)\right| = o(1)$ as $l\to\infty$

$\frac{1}{m} \le \partial_2 < y < \pi, \quad \left|K_m(y)\right| = o(1)$ as $m\to\infty$

and $\frac{1}{n} \le \partial_2 < z < \pi, \quad \left|K_n(z)\right| = o(1)$ as $n\to\infty$

$\therefore \left|K_k(w)\right|\left|K_l(x)\right|\left|K_m(y)\right|\left|K_n(z)\right| = o(1)$ as $(k,l,m,n)\to\infty$.

Lastly, when, $\frac{1}{k} < w < \pi, \frac{1}{l} < x < \pi, \frac{1}{m} < y < \pi, \frac{1}{n} < z < \pi$ then $\phi(w,x,y,z)$ is bounded. Then we have

$$I'' = o\left[\int_{\frac{1}{k}}^{\pi}\int_{\frac{1}{l}}^{\pi}\int_{\frac{1}{m}}^{\pi}\int_{\frac{1}{n}}^{\pi}\left|\phi(w,x,y,z)\right|\left|K_k(w)\right|\left|K_l(x)\right|\left|K_m(y)\right|\left|K_n(z)\right|dwdxdydz\right]$$

$$= o(1)\int_{\frac{1}{k}}^{\pi}\int_{\frac{1}{l}}^{\pi}\int_{\frac{1}{m}}^{\pi}\int_{\frac{1}{n}}^{\pi}\left|\phi(w,x,y,z)\right|dwdxdydz = o(1) \text{ as } \quad (k,l,m,n)\to\infty.$$

(12)

From (9), (10) & (11) we get,

$$S_{s',p',q',r'}^{\lambda,\mu,\nu,\beta} - f(s,t,u,v) = o(1) + o(1) = o(1) \;\; as\,(k,l,m,n,) \to \infty$$

This completes the proof of the theorem.

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