ISSN 0973-8975
J.Mech.Cont. & Math. Sci., Vol.-6, No.-1, July (2011) Pages 797-805

A NOTE ON THE GROWTH PROPERTIES OF WRONSKIANS

BY

[1]Sanjib Kumar Dutta & [2]Tanmay Biswas

[1]Department of Mathematics, University of Kalyani, Kalyani, Nadia – 741235, West Bengal, India.

[2]Department of Mathematics, University of North Bengal, Raja Rammohunpur, Darjeeling - 734013, West Bengal, India.

Abstract :

In the paper we study the comparative growth properties of composite entire and meromorphic functions on the basis of $L-(p,q)$ th order ($L-(p,q)$ th lower order) and $L^*-(p,q)$ order ($L^*-(p,q)$ th lower order) where $L \equiv L(r)$ is a slowly changing function and p,q are positive integers and $p > q$.

Keywords and phrases : *composite entire function, composite entire meromorphic function, comparative growth properties*

বিমূর্ত সার *(Bengali version of the Abstract)*

L- (p,q) -তম ক্রমের ভিত্তিতে [L - (p,q) - তম নিম্নক্রম] এবং L- (p,q) তম ক্রমের ভিত্তিতে [L* - (p,q) - তম নিম্নক্রম] যুগ্ম সম্পূর্ণ অপেক্ষক (Entire Function) এবং মেরোমরফিক্ অপেক্ষক (Meromorphic Function) - এর তুলনামূলক বৃদ্ধির ধর্মকে অনুসন্ধান করা হয়েছে যখন $L \equiv L(r)$ ধীরে পরিবর্তনশীল অপেক্ষক এবং p,q ঋণাত্মক পূর্ণ সংখ্যা এবং p>q*

1 Introduction, Definitions and Notations :

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [5] and [2]. In the sequel we use the following notation: $\log^{[k]}x = \log(\log^{[k-1]}x)$ for $k = 1,2,3,\ldots$ and $\log^{[0]}x = x$.

The following definitions are well known.

Definition 1 A meromorphic function $a = a(z)$ is called small with respect to f if $T(r,a) = S(r,f)$.

Definition 2 Let $a_1, a_2, \ldots, a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, \ldots, a_k, f)$ the Wronskian determinant of $a_1, a_2, \ldots, a_k, f$ i.e,

$$L(f) = \begin{vmatrix} a_1 & a_2 & \cdots & a_k & f \\ a_1' & a_2' & \cdots & a_k' & f' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^k & a_2^k & \cdots & a_k^k & f^k \end{vmatrix}.$$

Definition 3 If $a \in \mathbb{C} \cup \{\infty\}$, the quantityType equation here.

$$\delta(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)}$$

$$= \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}$$

is called the Nevanlinna's deficiency of the value $'a'$.

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) \le 2$ {cf. [2], p.43}. If in particular, $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

Definition 4 The order ρ_f and lower order λ_f of a meromorphic function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \quad and \quad \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire then

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad and \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Somasundaram and Thamizharasi [4] introduced the notions of L-order and L-lower order for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant $'a'$. Their definitions are as follows:

Definition 5[4] The L-order ρ_f^L and the L-lower order λ_f^L of an entire function f are defined as follows:

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\,[rL(r)]} \qquad and \qquad \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\,[rL(r)]}.$$

When f is meromorphic, then

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\,[rL(r)]} \qquad and \qquad \lambda_f^L = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log\,[rL(r)]}.$$

Juneja, Kapoor and Bajpai [3] defined the (p,q) th order and (p,q) th lower order of an entire function f respectively as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r} \qquad and \qquad \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r},$$

where p, q are positive integers and $p \geq q$.

When f is meromorphic, one can easily verify that

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r} \qquad and \qquad \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r},$$

where p, q are positive integers and $p \geq q$.

So with the help of the above notion one can easily define the $L-(p,q)$ th order and $L-(p,q)$ th lower order of entire and meromorphic functions.

Definition 6 The $L-(p,q)$ th order $\rho_f^L(p,q)$ and the $L-(p,q)$ th lower order $\lambda_f^L(p,q)$ of an entire function f are defined as

$$\rho_f^L(\mathrm{p},\mathrm{q}) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]}[rL(r)]} \qquad and \qquad \lambda_f^L(\mathrm{p},\mathrm{q}) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]}[rL(r)]},$$

where p, q are positive integers and $p \geq q$.

When f is meromorphic, one can easily verify that

$$\rho_f^L(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}[rL(r)]} \qquad and \qquad \lambda_f^L(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}[rL(r)]},$$

where p, q are positive integers and $p > q$.

The more generalised concept of $L-(p,q)$ th order and $L-(p,q)$ th lower order of entire and meromorphic functions are $L^*(p,q)$ th order and $L^*(p,q)$ th lower order respectively. In order to prove our results we require the following definitions:

Definition 7 The $L^*(p,q)$ -order $\rho_f^{L^*}$ and the $L^*(p,q)$ -lower order $\lambda_f^{L^*}$ of a meromorphic function f are defined by

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[r e^{L(r)}\right]} \qquad and \qquad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log\left[r e^{L(r)}\right]}.$$

When f is entire, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[r e^{L(r)}\right]} \qquad and \qquad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[r e^{L(r)}\right]}.$$

Definition 8 The $L^*(p,q)$ th order $\rho_f^{L^*}(p,q)$ and the $L^*(p,q)$ th lower order $\lambda_f^{L^*}(p,q)$ of an entire function f are defined as

$$\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]}\left[r e^{L(r)}\right]} \qquad and \qquad \lambda_f^{L^*}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]}\left[r e^{L(r)}\right]},$$

where p, q are positive integers and $p > q$.

When f is meromorphic, one can easily verify that

$$\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}\left[r e^{L(r)}\right]} \qquad and \qquad \lambda_f^{L^*}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}\left[r e^{L(r)}\right]}.$$

where p, q are positive integers and $p > q$.

Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the Wronskians. In the paper we study the comparative growth properties of composite entire and meromorphic functions on the basis of $L-(p,q)$ th order ($-(p,q)$ th lower order) and $L^*(p,q)$ order ($L^*(p,q)$ th lower order) where $L \equiv L(r)$ is a slowly changing function.

2 Lemmas.

In this section we present a lemma which will be needed in the sequel.

Lemma 1 [1] If f be a transcendental meromorphic function having the maximum deficiency sum. Then the $L^* - (p,q)$ th order (the $L^* - (p,q)$ th lower order) of $L(f)$ and that of f are same where p, q are positive integers and $p > q$.

Lemma 2 [1] Let f be a transcendental meromorphic function having the maximum deficiency sum. Then the $L - (p,q)$ th order (the $L - (p,q)$ th lower order) of $L(f)$ and that of f are same where p, q are positive integers and $p > q$.

3 Theorems.

In this section we present the main results of the paper.

Theorem 1 Let f be meromorphic and g be a transcendental entire function with the maximum deficiency sum such that $0 < \lambda_{f\circ g}^{L^*}(p,q) \leq \rho_{f\circ g}^{L^*}(p,q) < \infty$ and $0 < \lambda_{g}^{L^*}(m,q) \leq \rho_{g}^{L^*}(m,q) < \infty$. Then

$$\frac{\lambda_{f\circ g}^{L^*}(p,q)}{\rho_{g}^{L^*}(m,q)} \leq \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[m-1]}(r, L(g))} \leq \min\left\{\frac{\lambda_{f\circ g}^{L^*}(p,q)}{\lambda_{g}^{L^*}(m,q)}, \frac{\rho_{f\circ g}^{L^*}(p,q)}{\rho_{g}^{L^*}(m,q)}\right\}$$

$$\leq \max\left\{\frac{\lambda_{f\circ g}^{L^*}(p,q)}{\lambda_{g}^{L^*}(m,q)}, \frac{\rho_{f\circ g}^{L^*}(p,q)}{\rho_{g}^{L^*}(m,q)}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[m-1]}(r, L(g))} \leq \frac{\rho_{f\circ g}^{L^*}(p,q)}{\lambda_{g}^{L^*}(m,q)}$$

where p, m and q are positive integers such that $p > q$ and $m > q$ i.e., $min\{p,m\} > q$.

Proof From the definition of $L^*(p,q)$ th order and $L^*(p,q)$ th lower order of L(g) where g is a transcendental entire function we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[m-1]}(r, L(g)) < (\rho_{L(g)}^{L^*}(m,q) + \varepsilon) \log^{[q]}[r e^{L(r)}] \qquad (1)$$

and

$$\log^{[m-1]}(r, L(g)) \geq (\lambda^{L^*}_{L(g)}(m,q) - \varepsilon)\log^{[q]}[re^{L(r)}]. \tag{2}$$

Now in view of Lemma 1 we get from (1) for all sufficiently large values of r that

$$\log^{[m-1]}(r, L(g)) \leq (\rho^{L^*}_{g}(m,q) + \varepsilon)\log^{[q]}[re^{L(r)}]. \tag{3}$$

Similarly in view of Lemma 1 we get from (2) for all sufficiently large values of r

$$\log^{[m-1]}(r, L(g)) \geq (\lambda^{L^*}_{g}(m,q) - \varepsilon)\log^{[q]}[re^{L(r)}]. \tag{4}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[m-1]}(r, L(g)) \leq (\lambda^{L^*}_{L(g)}(m,q) + \varepsilon)\log^{[q]}[re^{L(r)}] \tag{5}$$

and

$$\log^{[m-1]}(r, L(g)) \geq (\rho^{L^*}_{L(g)}(m,q) + \varepsilon)\log^{[q]}[re^{L(r)}]. \tag{6}$$

So in view of Lemma 1 we get from (5) for a sequence of values of r tending to infinity that

$$\log^{[m-1]}(r, L(g)) \leq (\lambda^{L^*}_{g}(m,q) + \varepsilon)\log^{[q]}[re^{L(r)}]. \tag{7}$$

Also in view of Lemma 1 we get from (6) for a sequence of values of r tending to infinity that

$$\log^{[m-1]}(r, L(g)) \geq (\rho^{L^*}_{g}(m,q) - \varepsilon)\log^{[q]}[re^{L(r)}]. \tag{8}$$

Now again from the definition of $L^* - (p,q)$ th order and $L^* - (p,q)$ th lower order of the composite function $f \circ g$ we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[p-1]}(r, f \circ g) \leq (\rho^{L^*}_{f \circ g}(p,q) + \varepsilon)\log^{[q]}[re^{L(r)}] \tag{9}$$

and

$$\log^{[p-1]}(r, f \circ g) \geq (\lambda^{L^*}_{f \circ g}(p,q) - \varepsilon)\log^{[q]}[re^{L(r)}]. \tag{10}$$

Again for a sequence of values of r tending to infinity

$$\log^{[p-1]}(r, f \circ g) \leq (\lambda^{L^*}_{f \circ g}(p,q) + \varepsilon)\log^{[q]}[re^{L(r)}] \tag{11}$$

and

$$\log^{[p-1]}(r, f\circ g) \geq (\rho_{f\circ g}^{L^*}(p,q) - \varepsilon)\log^{[q]}[re^{L(r)}]. \tag{12}$$

Now from (3) and (10) it follows for all sufficiently large values of r,

$$\frac{\log^{[p-1]}T(r,f\circ g)}{\log^{[m-1]}(r,L(g))} \geq \frac{\lambda_{f\circ g}^{L^*}(p,q) - \varepsilon}{\rho_g^{L^*}(m,q) + \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}T(r,f\circ g)}{\log^{[m-1]}(r,L(g))} \geq \frac{\lambda_{f\circ g}^{L^*}(p,q)}{\rho_g^{L^*}(m,q)}. \tag{13}$$

Again combining (4) and (11) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p-1]}T(r,f\circ g)}{\log^{[m-1]}(r,L(g))} \leq \frac{\lambda_{f\circ g}^{L^*}(p,q) + \varepsilon}{\lambda_g^{L^*}(m,q) - \varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary,it follows that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}T(r,f\circ g)}{\log^{[m-1]}(r,L(g))} \leq \frac{\lambda_{f\circ g}^{L^*}(p,q)}{\lambda_g^{L^*}(m,q)}. \tag{14}$$

Similarly from (8) and (9) it follows for a sequence of values of r tending to infinity that,

$$\frac{\log^{[p-1]}T(r,f\circ g)}{\log^{[m-1]}(r,L(g))} \leq \frac{\rho_{f\circ g}^{L^*}(p,q) + \varepsilon}{\rho_g^{L^*}(m,q) - \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}T(r,f\circ g)}{\log^{[m-1]}(r,L(g))} \leq \frac{\rho_{f\circ g}^{L^*}(p,q)}{\rho_g^{L^*}(m,q)}. \tag{15}$$

Now combining (13) ,(14) and (15) we get that

$$\frac{\lambda_{f\circ g}^{L^*}(p,q)}{\rho_g^{L^*}(m,q)} \leq \liminf_{r\to\infty}\frac{\log^{[p-1]}T(r,f\circ g)}{\log^{[m-1]}(r,L(g))} \leq \min\left\{\frac{\lambda_{f\circ g}^{L^*}(p,q)}{\lambda_g^{L^*}(m,q)}, \frac{\rho_{f\circ g}^{L^*}(p,q)}{\rho_g^{L^*}(m,q)}\right\}. \tag{16}$$

Now from (7) and (10) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p-1]}T(r,f\circ g)}{\log^{[m-1]}(r,L(g))} \geq \frac{\lambda_{f\circ g}^{L^*}(p,q) - \varepsilon}{\lambda_g^{L^*}(m,q) + \varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[m-1]}(r, L(g))} \ge \frac{\lambda^{L^*}_{f\circ g}(p,q)}{\lambda^{L^*}_{g}(m,q)} . \tag{17}$$

Again from (4) and (9) it follows for all sufficiently large values of r,

$$\frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[m-1]}(r, L(g))} \le \frac{\rho^{L^*}_{f\circ g}(p,q)+\varepsilon}{\lambda^{L^*}_{g}(m,q)-\varepsilon} .$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[m-1]}(r, L(g))} \le \frac{\rho^{L^*}_{f\circ g}(p,q)}{\lambda^{L^*}_{g}(m,q)} . \tag{18}$$

Similarly combining (3) and (12) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[m-1]}(r, L(g))} \le \frac{\rho^{L^*}_{f\circ g}(p,q)-\varepsilon}{\rho^{L^*}_{g}(m,q)+\varepsilon} .$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[m-1]}(r, L(g))} \ge \frac{\rho^{L^*}_{f\circ g}(p,q)}{\rho^{L^*}_{g}(m,q)} . \tag{19}$$

Therefore combining (17) ,(18) and (19) we get that

$$\max\left\{\frac{\lambda^{L^*}_{f\circ g}(p,q)}{\lambda^{L^*}_{g}(m,q)}, \frac{\rho^{L^*}_{f\circ g}(p,q)}{\rho^{L^*}_{g}(m,q)}\right\} \le \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[m-1]}(r, L(g))} \le \frac{\rho^{L^*}_{f\circ g}(p,q)}{\lambda^{L^*}_{g}(m,q)} . \tag{20}$$

Thus the theorem follows from (16) and (20).

Remark 1 Considering f to be a transcendental meromorphic function with the maximum deficiency sum and g be an entire function if we take $0 < \lambda^{L^*}_{f}(m,q) \le \rho^{L^*}_{f}(m,q) < \infty$ instead of $0 < \lambda^{L^*}_{g}(m,q) \le \rho^{L^*}_{g}(m,q) < \infty$ and the other conditions remain the same then also Theorem 1 holds with g replaced by f in the denominator.

In the line of Theorem 1 and in view of Lemma 2 we may state the following theorem without proof.

Theorem 2 Let f be meromorphic and g be a transcendental entire function with the maximum deficiency sum such that $0 < \lambda^{L}_{f\circ g}(p,q) \leq \rho^{L}_{f\circ g}(p,q) < \infty$ and $0 < \lambda^{L}_{g}(m,q) \leq \rho^{L}_{g}(m,q) < \infty$. Then

$$\frac{\lambda^{L}_{f\circ g}(p,q)}{\rho^{L}_{g}(m,q)} \leq \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[m-1]}(r, L(g))} \leq \min\left\{\frac{\lambda^{L}_{f\circ g}(p,q)}{\lambda^{L}_{g}(m,q)}, \frac{\rho^{L}_{f\circ g}(p,q)}{\rho^{L}_{g}(m,q)}\right\}$$

$$\leq \max\left\{\frac{\lambda^{L}_{f\circ g}(p,q)}{\lambda^{L}_{g}(m,q)}, \frac{\rho^{L}_{f\circ g}(p,q)}{\rho^{L}_{g}(m,q)}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[m-1]}(r, L(g))} \leq \frac{\rho^{L}_{f\circ g}(p,q)}{\lambda^{L}_{g}(m,q)}$$

where p, m and q are positive integers such that $p \geq q$ and $m \geq q$ i.e., $min\{p, m\} \geq q$.

Remark 2 Considering f to be a transcendental meromorphic function with the maximum deficiency sum and g be an entire function if we take $0 < \lambda^{L}_{f}(m,q) \leq \rho^{L}_{f}(m,q) < \infty$ instead of $0 < \lambda^{L}_{g}(m,q) \leq \rho^{L}_{g}(m,q) < \infty$ and the other conditions remain the same then also Theorem 2 holds with g replaced by f in the denominator.

References

1) Datta, S. K. and Biswas, T.: On the $L-(p,q)$ th order of Wronskians, Int. J. Pure Appl. Math., Vol.50, No.3 (2009), pp.373-378.
2) Hayman, W.K.: Meromorphic Functions, The Clarendon Press, Oxford (1964).
3) Juneja, O.P.: Kapoor, G.P. and Bajpai, S.K.: On the $(p,q)-$order and lower $(p,q)-$order of an entire function, J.Reine Angew. Math., 282(1976), pp.53-67.
4) Somasundaram, D and Thamizharasi, R.: A note on the entire functions of L-bounded index and L-type, Indian J. Pure Appl. Math. , Vol. 19, No. 3 (1988), pp. 284-293.
5) Valiron, G.: Lectures on the general theory of integral functions, Chelsea Publishing Company, 1949.