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SENSITIVITY AND ACCUARACY OF EIGENVALUES RELATIVE TO THEIR PERTURBATION

By

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Abstract

The main objective of this paper is to study the sensitivity of eigenvalues in their computational domain under perturbations, and to provide a solid intuition with some numerical example as well as to represent them in graphically. The sensitivity of eigenvalues, estimated by the condition number of the matrix of eigenvectors has been discussed with some numerical example. Here, we have also demonstrated, other approaches imposing some structures on the complex eigenvalues, how this structure affects the perturbed eigenvalues as well as what kind of paths do they follow in the complex plane.

Keywords and phrases *: sensitivity, eigenvalues, perturbations, complex eigenvalues.*

বিমূর্ত সার *(Bengali version of the Abstract)*

বিচলনাধীন গণনা অঞ্চলে আইগেন্ মান (Eigen Value)- এর সুগ্রাহিতাকে অনুসন্ধান করা এবং কিছু সাংখ্য উদাহরণের সাহায্যে দৃঢ় স্বত্তাকে প্রদান করা হয় এবং ইহাকে লেখচিত্রের সাহায্যে ও পরিবেশন করা এই পত্রের মূখ্য উদ্দেশ্য। শত্রসূচক ম্যাট্রিক্সের আইগেন ভেক্টরের সাহায্যে আইগেন মানের সুগ্রাহিতাকে গণনা করা হয়েছে এবং কিছু সাংখ্য উদাহরণের সাহায্যে আলোচনা করা হয়েছে। জটিল আইগেন মানে কিছু কাঠামো আরোপ করে অন্য ধরণের অভিগমনের ক্ষেত্রে এই কাঠামো বিচলিত আইগেন মানে কি ভাবে প্রভাবিত করে এবং জটিল-তলে কোন ধরণের পথ অনুসরণ তা প্রদর্শন করা হয়েছে।

1. INTRODUCTION :

Mathematical models for the description of the physical behavior of a system typically contain measurement errors or modeling errors in different parameters. This type of models are treated numerically, discretization and rounding errors are introduced. Moreover, usually a given model is applicable only for values of its parameters within certain bounds. For parameter values out of these bounds the model is not correct and the solution of the corresponding computational problem may not exist or may have no physical meaning. It is essential for understanding the problems and estimating the accuracy of the computed results. Indeed, the mathematical models that are used to solve application problems are typically subject to modelling uncertainties (due to simplifications), and measurement errors in the data. Furthermore, the solution of the problem is usually carried out with numerical methods that may include approximation errors due to truncation of infinite series or discretization of continuous processes. In addition to, the final result is contaminated by rounding errors due to the implementation of computational algorithms in finite precision arithmetic. The influence of the above uncertainties and errors on the computed result depends on the sensitivity of the problem. Thus, without a detailed perturbation analysis, it is not possible to assess the quality of the computed results. The main goal of this paper is to study the sensitivity of eigenvalues in computational domain, mathematical models under perturbations with matrix condition number and to illustrate them with some numerical example as well as to represent them in graphically. Here, we discuss how the solution changes when the data of the problem are changed. In a more restricted framework the objective of sensitivity analysis of eigenvalues is, to provide computable bounds for the perturbation in the solution of a given problem as a function of the perturbation in the data. At present, sensitivity analysis techniques of eigenvalues under perturbation are important issues in numerical analysis and also in all areas of science and engineering. There is a huge literature on this topic that covers existence and uniqueness of solutions, numerical methods and also, more recently, perturbation analysis for special classes of such equations. In summary, we have seen that there are at least some important reasons to study the sensitivity of eigenvalue problems relative to their perturbations in the data from a given class. First, sensitivity analysis of eigenvalues may provide an independent and deep insight in the very nature of the problem, being therefore of independent theoretical interest. Perturbation bounds provide a realistic framework for most problems in mathematical modelling of objects and processes. Indeed, in practice there are inevitable measurement and other parametric or structural uncertainties. This means that we have to deal with a family of models rather than with a single model. Having a model with given parameters and estimates for their values, the only thing that we can rigorously claim is that the model will behave within a framework predicted by perturbation analysis. Lastly, when a numerically stable algorithm is applied to solve a problem, then the solution, computed in finite precision arithmetic, will be close to the solution of a near problem. Having tight perturbation bounds and a knowledge about the equivalent perturbation for the computed solution, it is possible to derive condition and accuracy estimation. Without such estimation a computational algorithm cannot be recognized as reliable from the viewpoint of modern computing standards.

The first general perturbation bounds for eigenvalue were given by Ostrowksi 12), in 1957. Elsner 3), in 1985, showed that an optimal bound of eigenvalues are possible. In these intervening years, a lot of work has taken place in this subject, which attracted the attention of several mathematicians. The most prominent conjecture on eigenvalue perturbation was that, the inequality $d(eigA, eigB) \le \|A - B\|$ would be true for all normal matrices A and B. Another inequality $d(eigA, eigB) \le C\|A - B\|$ is also true for all n-by-n normal matrices A and B with *1.018<c<2.904* published by J. Holbrook 5), in 1992.

These inequalities have been significantly improved by the factor n, and replaced the previous inequalities by $d(eigA, eigB) \le 4(2M)^{1-1/n} \|A-B\|^{1/n}$, where A and B are any two n-by-n matrices. This work is done by D. Phillips and by R. Bhatia, L. Elsner, and G. Krause 2) in 1990. Another consequence result shown by J. G. Sun 16) in 1996 is that, if A is normal and B arbitrary, the last inequality can be improved to $d(eigA, eigB) \le n\|A-B\|$. A lot of work has been done in this field but we try to present it in more rigorous way. In 8), the author presented the classical perturbation theory for Hermitian matrix eigenvalue and singular value problems that provides bounds on the absolute differences between approximate eigenvalues (singular values) and the true eigenvalues (singular values) of a matrix. The sensitivity of Lyapunov equations are discussed in 1), 7), 9) which we encountered in generalized state-spaced systems of the form $E\dot{x} = Ax$ where E is nonsingular, and the system stable**.** Methods are presented in 14) for performing a rigorous sensitivity analysis for general systems of linear and nonlinear equations with respect to weighted perturbations in the input data, but there is no graphical representation. M. Konstanitinov et al. in 10) have applied the theory of condition developed by Rice to define condition numbers of the continuous-time algebraic Riccati equation and the discrete-time algebraic Riccati equation in the Frobenius norm and derive explicit expressions of the condition numbers in a uniform manner. In 11) author presented a detail discussion on the sensitivity of eigenvalues with example but the related theorems are not discussed sufficiently. In this paper we have tried to present an overall discussion of the sensitivity of eigenvalus and to make a solid intuition behind by presenting graphically to show their geometrical behavior.

2. Paper outline :

The rest of this paper is organized as follows. The notations used throughout the rest of this paper are given in section III. We discuss the relative perturbations theorems for eigenvalue in section IV. Eigenvalue sensitivity is presented in section V. Concluding remarks are presented in section VI.

3. Background Notations :

In order to facilitate discussion in subsequent sections we introduce relevant notation first. We shall adopt the following convention: capital letters denote unperturbed matrices and capital letter with *tildes* denote their perturbed matrices. Throughout the paper, capital letters are for matrices, lowercase Latin letters for column vectors or scalars and lowercase Greek letters for scalars.

4. Relative perturbation theorems for eigenvalue problems :

Matrix Condition Number: It is convenient to have some number which defines the condition of a matrix with respect to a computing problem and we can say such a number is 'condition number'. Generally, it should provide some 'overall assessment' of the rate of change of the solution with respect to changes in the coefficients and should therefore be in some way proportional to this rate of change. If we have the eigenvalues which are very sensitive then the condition number would have to be very large, even if some other eigenvalues are very insensitive.

Let x be a vector and A be a matrix. The multiplication of x by A results Ax, can have a very different norm from x. This change in norm is directly related to the sensitivity. The range of the possible change can be expressed by two numbers

$$M = \max \frac{\|Ax\|}{\|x\|}$$

$$m = \min \frac{\|Ax\|}{\|x\|}$$

The maximum and minimum are taken over all non-zero vectors x. If A is singular then m=0.

The ratio $\frac{M}{m} = \psi(X) = \frac{\max \frac{\|Ax\|}{\|x\|}}{\min \frac{\|Ax\|}{\|x\|}}$

is called the condition number of A. Consider a system of equations $Ax = b$ and $A(x + \delta x) = b + \delta b$ where we may think δb as being the error in b and δx as being the resulting error in x. The definitions of M and m immediately lead to $\|b\| \le M\|x\|$ *and* $\|\delta b\| \ge m\|\delta x\|$.

Also, if $m \neq 0$, *then* $\frac{\|\delta x\|}{\|x\|} \le \psi(A)\frac{\|\delta b\|}{\|b\|}$. The quantity $\frac{\|\delta b\|}{\|b\|}$ is the relative change in the right hand side and the quantity $\frac{\|\delta x\|}{\|x\|}$ is the relative error caused by this change. This shows that the condition number is a relative error magnification factor.

Theorem 1: (Ostrowski) Let λ be an eigenvalue of a matrix A of algebraic multiplicity m. Then for any norm $\|.\|$ and all sufficiently small $\varepsilon > 0$ there is a $\delta > 0$ such that if $\|E\| < \delta$, the disk $D(\lambda, \varepsilon) = \{\zeta \in C : |\zeta - \lambda| \le \varepsilon\}$ contains exactly m eigenvalues of $\tilde{A}$.

Proof: Let ε be so small that $D(\lambda, \varepsilon)$ contains only the eigenvalue λ of A. Let $\eta(\zeta) = \phi_{\tilde{A}}(\zeta) - \phi_A(\zeta)$. By the continuity of the characteristic polynomial, as $\tilde{A} \to A$ the function $\eta(\zeta)$ converges to zero on the compact set ∂D. Since $\phi_A(\zeta)$ is non zero on ∂D, there is a $\delta > 0$ such that $|\eta(\zeta)| < |\phi_A(\zeta)|$ on ∂D

whenever $\|E\| < \delta$. By Rouche's theorem ϕ_A and $\phi_{\tilde{A}} = \phi_A + \eta$ have the same number of zeros in D.

Definition 1: Let the matrix A have eigenvalues $\lambda_1,...,\lambda_n$ and $\tilde{A}$ have eigenvalues $\tilde{\lambda}_1,...,\tilde{\lambda}_n$. Then the spectral variation of $\tilde{A}$ with respect to A is $sv_A\left(\tilde{A}\right) = \max_i \min_j \left|\tilde{\lambda}_i - \lambda_j\right|$.

Definition 2: The Hausdorff distance between the eigenvalues of A and $\tilde{A}$ is $hd_A\left(A, \tilde{A}\right) = \max\left\{ sv_A\left(\tilde{A}\right), sv_{\tilde{A}}\left(A\right)\right\}$.

Theorem 2: (Elsner) For any A and $\tilde{A}$, $hd\left(A, \tilde{A}\right) \le \left(\|A\|_2 + \|\tilde{A}\|_2\right)^{1-\frac{1}{n}} \|E\|_2^{\frac{1}{n}}$.

Proof: Since the right hand side of $hd\left(A, \tilde{A}\right) \le \left(\|A\|_2 + \|\tilde{A}\|_2\right)^{1-\frac{1}{n}} \|E\|_2^{\frac{1}{n}}$ is symmetric in A and $\tilde{A}$, it is sufficient to prove that it bounds $sv_A\left(\tilde{A}\right)$. Assume the maximum in the definition of spectral variation is attained for the eigenvalue $\tilde{\lambda}$ of $\tilde{A}$, and let $\tilde{x}_1,...,\tilde{x}_n$ be orthonormal vectors with $\tilde{A}x_1 = \tilde{\lambda}x_1$. Then

$$\begin{aligned} sv_A\left(\tilde{A}\right)^n &\le \prod_i \left|\lambda_i - \tilde{\lambda}\right| \\ &= \det(A - \tilde{\lambda}I) \\ &\le \prod_i \left\|\left(A - \tilde{\lambda}I\right)x_i\right\|_2 \quad [\textit{Hadamard inequality}] \\ &= \prod_i \left\|\left(A - \tilde{A}\right)x_1\right\|_2 \prod_{i>1} \left\|\left(A - \tilde{\lambda}I\right)x_i\right\|_2 \\ &\le \|E\|_2 \left(\|A\|_2 + \|\tilde{A}\|_2\right)^{n-1} \end{aligned}$$

The result follows on taking n-th roots in the above inequality and from the symmetry of the resulting bound.

5. Eigenvalue sensitivity :

To get a basic conception of sensitivity and accuracy of eigenvalues assume that A has a full set of linearly independent eigenvectors and suppose A has the eigenvalue decomposition

$$A = X \Lambda X^{-1} \qquad (1)$$

We may rewrite this as $\Lambda = X^{-1}AX$. Now suppose δA denotes a small change of
A which caused round off errors or any other kind of perturbation. Then $\Lambda + \delta A = X^{-1}(A+\delta A)X$ where $\delta A = X^{-1}\delta AX$. Taking matrix norms on both sides, we get,

$$\|\delta A\| = \|X^{-1}\delta X\|$$
$$or \ \|\delta A\| \le \|X^{-1}\|.\|X\|.\|\delta A\| = \psi(X)\|\delta A\|$$

where $\psi(X)$ is the matrix condition number. Here note that, the key factor is the condition of X , the matrix of eigenvectors, not the condition of A itself. This simple analysis tells us that, in terms of matrix norms, a perturbation $\|\delta A\|$ can be magnified by a factor as large as $\psi(X)$ in $\|\delta\Lambda\|$. However, since $\|\delta\Lambda\|$ is usually not a diagonal matrix, this analysis does not immediately say how much the eigenvalues themselves may be affected. To compute and visualize numerically, here we consider two matrices

$$\begin{bmatrix} -149 & -50 & -154 \\ 537 & 180 & 546 \\ -27 & -9 & -25 \end{bmatrix}$$

and

$$\begin{bmatrix} -9 & 11 & -21 & 63 & -252 \\ 70 & -69 & 141 & -421 & 1684 \\ -575 & 575 & -1149 & 3451 & -13801 \\ 3891 & -3891 & 7782 & -23345 & 93365 \\ 1024 & -1024 & 2048 & -6144 & 24572 \end{bmatrix}$$

which are known as `gallery(3)` and `gallery(5)` in Mat Lab language respectively. We use the function `condset` from Mat Lab to estimate the condition number of the eigenvector matrix. In this case, we use a 3-by-3 matrix `gallery (3)` to see the effect.

```
A=gallery (3)
[x , lamda]=eig(A);
condition_number=condest(x)
```

The value of condition number is 1.2002e+003. So a perturbation of the above matrix could result in perturbation in its eigenvalues that are 12×10^2 times as large which yields that the eigenvalues of the above matrix are slightly ill conditioned. A more detailed analysis can be found in the left eigenvectors which are row vectors y^H that satisfies the equation $y^H A = \lambda y^H$.
To check and verify the sensitivity of an individual eigenvalue, suppose that A varies with a perturbation parameter and let A' denote the derivative with respect to that parameter. Taking differentiation both sides of the equation $Ax = \lambda x$ we get

$$A'x + Ax' = \lambda' x + \lambda x' \tag{2}$$

Multiply both sides by the left eigenvector y^H of the equation (2) we get

$$y^H A'x + y^H Ax' = y^H \lambda' x + y^H \lambda x'$$

Since $y^H A'x = y^H \lambda' x$ it implies that

$$y^H A'x = y^H \lambda' x$$

$$\Rightarrow \lambda' = \frac{y^H A'x}{y^H x} \tag{3}$$

Taking norm on both sides of equation (3), we can get

$$\|\lambda'\| = \frac{\|y^H A'x\|}{y^H x}$$

$$|\lambda'| \le \frac{\|y\|\|x\|}{y^H x}\|A'\|$$

Define the eigenvalue condition number ψ as

$$\psi(\lambda, A) = \frac{\|y\|\|x\|}{y^H x}$$

Therefore

$$|\lambda'| \le \psi(\lambda, A)\|A'\| \tag{4}$$

So, we can conclude that $\psi(\lambda, A)$ is the magnification factor relating to a perturbation in the matrix A to the resulting perturbation in an eigenvalue λ. If we wish to compute the left eigenvectors then consider X is a matrix whose columns are the eigenvectors. To compute the left eigenvectors set $y^H = X^{-1}$. Therefore, $Y^H A = \Lambda Y^H$, the rows of Y^H are the left eigenvectors. In this case, the left eigenvectors are normalized so that $Y^H X = I$, consequently the denominator in $\psi(\lambda, A)$ is $y^H x = 1$. Therefore, $\psi(\lambda, A) = \|y\|\|x\|$. Since $\|x\| \le \|X\|$ and $\|y\| \le \|Y\|$, we have $\psi(\lambda, A) = \psi(X)$. This shows that the condition of the eigenvector matrix is an upper bound for the individual eigenvalue condition numbers. We compute the eigenvalue condition numbers with the MatLab function `condeig` of the matrix `gallery (3)` which are shown in Table I. This indicates that $\lambda_1 = 1$ is effectively

TABLE I.

VALUES OF THE CONDITION NUMBERS

λ	ψ
1.0000	603.6390
2.0000	395.2366
3.0000	219.2920

more sensitive than $\lambda_2 = 2$ or $\lambda_3 = 2$. A perturbation in the matrix `gallery (3)` may result in perturbations in its eigenvalues that are 200 to 600 times as large. This is consistent with the harder estimates 12×10^2 obtained from `condest(x)`.We make a small random perturbation in A = `gallery (3)` to see and check what happens to its eigenvalues.

```
format long
delta=1.e-6
A=gallery(3); lamda=eig(A)
lamda_bar=eig(A+delta*rand(3))
perturb_eigenvalue=lamda_bar-lamda
Perturb_condition=delta*condeig(A)
```

The last column of the Table-II is the perturbation of the eigenvalue which is smaller than we estimated by `condeg` and the perturbation analysis, but roughly the same size. If A is real and symmetric, or complex and Hermitian, then it's left and right eigenvectors are the same. In this case, $y^H x = \|y\|\|x\|$, therefore for symmetric and Hermitian matrices, $\psi(\lambda, A) = 1$. Thus the eigenvalues of symmetric and Hermitian matrices are perfectly well conditioned. For multiple eigenvalues, roughly it is also true that the perturbations in the matrix lead to perturbations in the eigenvalues with the same size. In our discussion we considered, *A* has a full set of linearly independent eigenvectors. If λ_k is a multiple eigenvalue that does not have a corresponding full set of linearly independent eigenvectors, then the previous analysis does not apply. In this case, the characteristic polynomial for an *n*-by-*n* matrix can be written $p(\lambda) = \det(A - \lambda I) = (\lambda - \lambda_k)^m q(\lambda)$ where *m* is the multiplicity of λ_k and $q(\lambda)$ is a polynomial of degree $(n-m)$ that does not vanish at λ_k. Now make a perturbation of size δ in the matrix A. The result will be change and the characteristic polynomial $p(\lambda) = 0$ would be something like $p(\lambda) = O(\delta)$. We can write it in this form

$$(\lambda - \lambda_k)^m = \frac{O(\delta)}{q(\lambda)} \qquad (5) \; .$$

Thus the roots of this equation are

TABLE II.

VALUES OF THE CONDITION NUMBERS

λ	$\tilde{\lambda}$	Perturbed eigenvalue	Perturbed condition number
1.0000	0.999977883966814	1.0e-003×0.493725272157297	1.0e-03× 0.6036389649562
2.0000	1.999994195745145	1.0e-003×0.288054643827751	1.0e-03×0.3952366379896
3.0000	3.000029282827517	1.0e-003×0.208036628212671	1.0e-03×0.2192920427184

$$\lambda = \lambda_k + O\left(\delta^{\frac{1}{m}}\right) \quad (6).$$

From this equation the behavior of the m-th root, concludes that the multiple eigenvalues without a full set of eigenvectors are extremely sensitive to perturbation. Here we provide an illustrative example. We consider a 16-by-16 matrix with 2's on the main diagonal, 1's on the superdiagonal, δ in the lower left-hand corner, and 0's elsewhere.

$$A = \begin{bmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & \ddots & \ddots & \\ & & & 2 & 1 \\ \delta & & & & 2 \end{bmatrix}$$

The characteristic equation of A is $(\lambda-2)^{16} = \delta$. If $\delta = 0$, this matrix has an eigenvalue of multiplicity 16 at $\lambda = 2$, but there is only one eigenvector to go along with this multiple eigenvalue. If $\delta \approx 10^{-16}$, i.e. the floating point of round off error, then the eigenvalues are on a circle in the complex plane with center at 2 and radius $\left(10^{-16}\right)^{\frac{1}{16}} = 0.1$. Therefore, a perturbation on the size of round off error changes the eigenvalue from 2.0000 to16 different values. A great small change in the matrix elements causes a much larger change in the eigenvalues. Here, we discuss the behavior of another important matrix A = `gallery (5),` which corroborates the same phenomena. The matrix A provides an interesting eigenvalue structure which is related with exact eigenvalues and eigenvectors problem. By the naive approach, the computed eigenvalues of A yields

-0.040520367667793

-0.011779333466727 + 0.038286113829096i

-0.011779333466727 - 0.038286113829096i

0.032039517299891 + 0.022811592233697i

0.032039517299891 - 0.022811592233697i

Can we guess, how much accurate are these computed eigenvalues? Somewhat, more reliable information can be obtained from polynomial matrix of A, by the function `poly (A).` This may lead to the conjecture that $p(\lambda) = \lambda^5$ is the characteristic polynomial of A, which would imply that $\lambda = 0$ is a 5-fold eigenvalue, i.e. $\lambda = 0$ has the algebraic multiplicity 5. According to the Cayley-Hamilton theorem every matrix satisfies its characteristic equation, which would mean $A^5 = 0$ if the above conjecture is true. We can solve the characteristic equation manually by hand and this can be easily verified by noting that $A^5 = 0$, which is computed without any round off error, is the zero matrix. We clearly find that five eigenvalues are actually equal to zero. The computed eigenvalues exhibit a little warning to indicate that the "correct" eigenvalues are all zero. We must have to confess

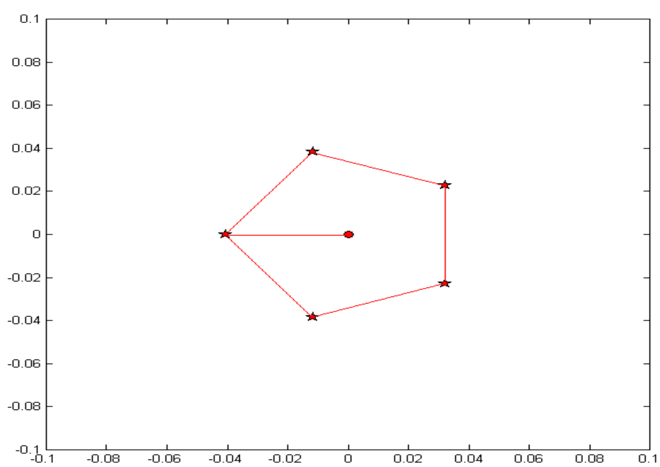

Figure 1. Plot of eigenvalues

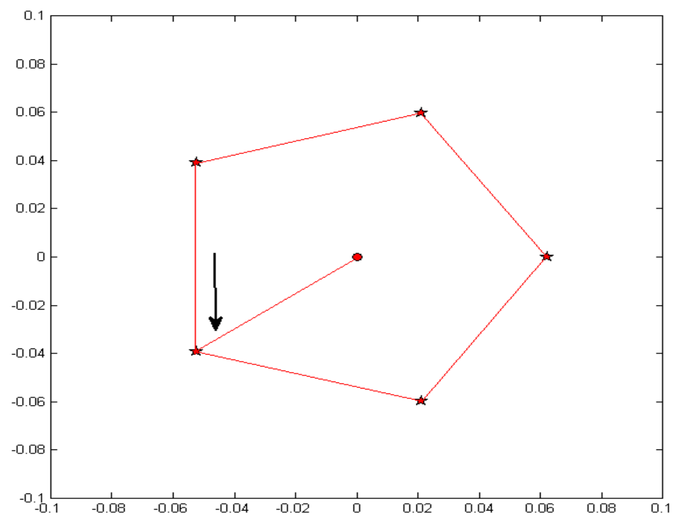

Figure 2. The orientation of eigenvalues

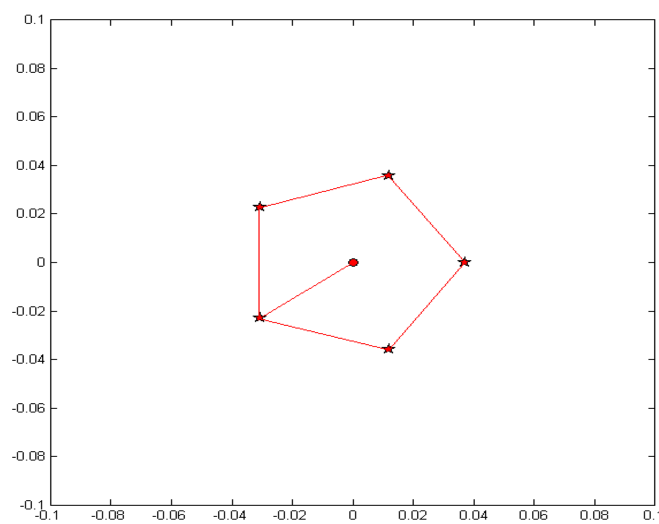

Figure 3. Variation of the radius

that the computed eigenvalues are not very accurate. This problem can be addressed with the Mat Lab function `eig`. The inaccuracy of the computed eigenvalues is caused by their sensitivity, not by anything wrong with `eig`. We can demonstrate this in a graphical representation. We plot the eigenvalues in Figure-1 which shows that the computed eigenvalues are the vertices of a regular pentagon in the complex plane, centered at the origin.

The radius is about 0.04. Now, we repeat this experiment with a matrix where each element is perturbed by a single round off error. We will clearly see that the pentagon flips orientation (Figure 2) and that its radius varies (Figure 3) between 0.035 and 0.070, but that the computed eigenvalues of the perturbed problems behave pretty much like the computed eigenvalues of the original matrix. So the matrix A with which we are presented is an approximation to the matrix which corresponds to exact measurements. It can be asserted that the error in every element of A is bounded by a positive number δ (say), then we can say that the true matrix is (A+E), where E is some matrix for which $|e_{ij}| \leq \delta$.

6. CONCLUSIONS :

In this paper, we have addressed some aspects of sensitivity and accuracy of eigenvalues with their perturbations. The structure on the errors for eigenvalues, the effect on their perturbation and what kinds of paths do they follow in the complex plane are also discussed. It is seen that the sensitivity of eigenvalues is estimated by the condition number of the matrix of eigenvectors. Our demonstration shows that the computed eigenvalues of a particular five-by-five matrix named `gallery (5)` are the vertices of a regular pentagon in the complex plane, centered at the origin. The pentagon flips in orientation and its radius varies in some range. Our experiment provides evidence for the fact that the computed eigenvalues are the exact eigenvalues of a matrix $A + E$, where the elements of E are on the order of round off error compared to the elements of A. This is the best we can expect to achieve a solid intuition behind the sensitivity and accuracy of eigenvalues with floating point computation.

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