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SOME PROPERTIES OF STANDARD SUBLATTICES OF A LATTICE

By

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Abstract :

In this paper we study some properties and give some characterizations of these sublattices. Also we prove that for a central element n of a lattice, the standard n-congruences are permutable.

Keywords and phrases : *lattice, sublattices, central element, congruences.*

বিমূর্ত সার *(Bengali version of the Abstract)*

এই পত্রে উপ-ল্যাটিসের (Sub Lattice)- এর কিছু ধর্মকে অনুসন্ধান করা হয়েছে এবং ইহাদের কিছু চরিত্রগত বৈশিষ্ট্য প্রদান করা হয়েছে। ইহাও প্রমাণ করা হয়েছে যে ল্যাটিসের n- কেন্দ্রীয় উপাদানের জন্য প্রমিত (Standard) n- সর্বসমতাগুলি বিন্যাস যোগ্য।

1. Introduction:

Distributive lattices have a lot of important properties that lattice in general do not have. This fact gives the reason why researchers have tried to define different types of elements and ideals of lattices which preserve some properties of distributive lattices. Several authors including G. Grätzer and E.T. Schmidt [1], G. Grätzer [2], W.H. Cornish and A.S.A Noor [3] have studied these elements and ideals in different contexts, which are known as Distributive, Standard and Neutral elements (ideals).

By G. Grätzer and E.T. Schmidt [1], if a is an element of a lattice L, then

i) a is called *standard* if $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$ for all $x, y \in L$,

ii) a is called *neutral* if for all $x, y \in L$

(α) $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$ that is, a is standard and

(β) $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$.

An ideal S of a lattice L is called a *standard ideal* if

$I \wedge (S \vee J) = (I \wedge S) \vee (I \wedge J)$ for all $I, J \in I(L)$.

That is, S is a standard ideal if it is a standard element of I(L). Standard ideals play important roles in establishing some beautiful results in non-distributive lattices. For example, they play the same role for lattices as invariant subgroups for groups.

G. Grätzer [2] in his book presents the problem "Generalize the concepts of distributive, standard and neutral ideals to convex sublattices". E. Fried and E.T. Schmidt [4] have done the generalization work for standard ideals. Also J. Nieminen [5] have made an attempt to study distributive, standard and neutral convex sublattices, although there are several certain errors in his work.

For a lattice L, the set of all convex sublattices of L with the empty set Φ is a lattice, denoted by CS(L). For any $A, B \in CS(L)$, we define

$A \dot{\wedge} B = \langle \{a \wedge b \mid a \in A, b \in B\} \rangle$ that is, the convex sublattice generated by the elements $a \wedge b$ for all $a \in A$, $b \in B$.We also define $A \dot{\vee} B = \langle \{a \vee b \mid a \in A, b \in B\} \rangle$ and $A \vee B = \langle A, B \rangle = \langle A \cup B \rangle$ that is, convex sublattice generated by A and B.

By E. Fried and E.T. Schmidt [4], A convex sublattice S of a lattice L is called a *standard convex sublattice* if for all $I, K \in CS(L)$,

(i) $I \dot{\wedge} (S \vee K) = (I \dot{\wedge} S) \vee (I \dot{\wedge} K)$ and

(ii) $I \dot{\vee} (S \vee K) = (I \dot{\vee} S) \vee (I \dot{\vee} K)$

where $S \cap K \neq \phi$ and $I \cap (S \vee K) \neq \phi$.

It should be mentioned that all through this paper, we write standard sublattice for standard convex sublattice. In this paper we study some properties of these sublattices with several characterizations.At the end we include a result on the permutability of standard n-congruences.

Observe that in a lattice L the ideal generated by $\{a \wedge b \mid a \in A, b \in B\}$, that is, $(\{a \wedge b \mid a \in A, b \in B\}] = (A] \wedge (B]$. Moreover the dual ideal of L generated by $\{a \wedge b \mid a \in A, b \in B\}$, that is, $[\{a \wedge b \mid a \in A, b \in B\}) = [A) \vee [B)$. Similarly $(\{a \vee b \mid a \in A, b \in B\}] = (A] \vee (B]$ and $[\{a \vee b \mid a \in A, b \in B\}) = [A) \wedge [B)$. Therefore when A and B are ideals then $A \dot{\wedge} B = A \wedge B$ and $A \dot{\vee} B = A \vee B$ and when A and B are dual ideals then

$A \dot{\wedge} B = A \vee B$ and $A \dot{\vee} B = A \wedge B$.

By E. Fried and E.T. Schmidt [4] we know that $A \dot{\wedge} (B] = (A] \wedge (B]$ and $A \vee (B] = (A] \vee (B]$

Also by above observations we find that,

$(A \dot{\wedge} B] = (A] \wedge (B]$, $[A \dot{\wedge} B) = [A) \vee [B)$, $(A \dot{\vee} B] = (A] \vee (B]$

and $[A \dot{\vee} B) = [A) \wedge [B)$.

If for two convex sublattices C and D, $C \cap D \neq \phi$, then we shall write $C \cap D$ by $C \wedge D$. It is well known that for any convex sublattice C of a lattice L, $C = (C] \cap [C)$.

Therefore, $A \dot{\wedge} B = (A \dot{\wedge} B] \cap [A \dot{\wedge} B) = ((A] \wedge (B]) \wedge ([A) \vee [B))$ and

$A \dot{\vee} B = (A \dot{\vee} B] \cap [A \dot{\vee} B) = ((A] \vee (B]) \wedge ([A) \wedge [B))$.

Following results are easily verifiable.

$A \dot{\vee} (B] = ((A] \vee (B]) \wedge [A), \ A \dot{\wedge} [B) = (A] \wedge ([A) \vee [B)).$

$A \dot{\vee} [B) = [A) \wedge [B), \ A \vee [B) = [A) \vee [B)$. Also $(A \vee B] = (A] \vee (B]$ and $[A \vee B) = [A) \vee [B)$. Moreover, $(A \cap B] = (A] \wedge (B]$ and $[A \cap B) = [A) \wedge [B)$, provided $A \wedge B = A \cap B \neq \phi$.

Following result is due to E. Fried and E.T. Schmidt [4], that is, the concept of standard sublattices coincides with standard ideals in case of ideals.

Proposition 1. *An ideal* S *of a lattice* L *is standard if and only if it is a standard sublattice*.

Next theorem gives nice characterizations of standard sublattices which is due to E. Fried and E.T. Schmidt [4].

Theorem 2. *The following four conditions are equivalent for each convex sublattice* S *of a lattice* L.

(α) S *is a standard sublattice*

(β) *Let* K *be any convex sublattice of* L *such that* $S \cap K \neq \phi$. *Then, to each* $x \in \langle S, K\rangle$ *there exist* $s_1, s_2 \in S$ $a_1, a_2 \in K$ *such that;*

$$x = (x \wedge s_1) \vee (x \wedge a_1) = (x \vee s_2) \wedge (x \vee a_2).$$

(β') *Let* K *be as before. Then, for each* S *and to each elements* $x \in \langle S, K\rangle$ *and to each* $s_2, s_1' \in S$ *there are elements* $s_1, s_2' \in S$, $a_1, a_2 \in K$ *such that*

$$x = (x \wedge s_1) \vee (x \wedge (a_1 \vee s_2)) = (x \vee s'_2) \wedge (x \vee (a_2 \wedge s_1'))$$

(γ) *The relation* $\theta[S]$ *on* L *defined by* " $x \equiv y\ (\theta[S])$ *if and only if* $x \wedge y = ((x \wedge y) \vee t) \wedge (x \vee y)$ *and* $x \vee y = ((x \vee y) \wedge s) \vee (x \wedge y)$ *with suitable* t, s *in* S " *is a congruence relation.*

Corollary 3. *If* S *is standard sublattice then* S *is a congruence class by the congruence relations* $\theta[S]$.

In E. Fried and E.T. Schmidt [4], Fried and Schmidt proved that non empty intersection of two standard sublattices is a standard sublattice. Moreover, the meet of a standard ideal and a standard dual ideal is a standard sublattice.

Remark. J. Nieminen [5] proved that if a sublattice S is standard in CS(L) then (S] is standard in I(L) and [S) is standard in D(L). But this is completely wrong, as there are certain errors in his proof. In his proof he showed that if I, K $\in$ I (L),

$$I \wedge ((S] \dot{\vee} K) = I \mathbin{\dot{\wedge}} ((S] \vee K) = I \mathbin{\dot{\wedge}} (S \vee K) = (I \mathbin{\dot{\wedge}} S) \vee (I \mathbin{\dot{\wedge}} K)$$

$$= (I \wedge (S]) \vee (I \wedge K).$$

But notice that in I $\dot{\wedge}$ (S $\vee$ K) of above proof, there is no guarantee that S $\cap$ K $\neq \phi$. Hence (S] is not necessarily standard in I(L), similarly [S) is not necessarily standard in D(L), the lattice of dual ideals of L.

In this connection it is notable that by E. Fried and E.T. Schmidt [4] every singleton set {s}, where s $\in$ L of a lattice L is a standard sublattice. So according to J. Nieminen's result [5, Lemma 1], (s] for all s $\in$ L must be standard which is absurd for a non-distributive lattice. But as S = (S] $\cap$ [S) for all S $\in$ CS(L), so the converse of Nieminen's result obviously holds; that is,

Lemma 4. *If for any* S $\in$ CS(L), (S] *is standard in* I(L) *and* [S) *is standard in* D(L), *then* S *is a standard sublattice.*

It is well known that for a standard element s in a lattice, $s \wedge a = s \wedge b$ and $s \vee a = s \vee b$ imply $a = b$. We now prove a similar result for standard sublattices.

Lemma 5. *Suppose* S *is a standard sublattice. If for any* $A, B \in CS(L)$ *with* $S \cap A \neq \phi$, $S \cap B \neq \phi$ *and* $S \dot{\wedge} A = S \dot{\wedge} B$, $S \dot{\vee} A = S \dot{\vee} B$ *and* $S \vee A = S \vee B$ *hold, then* $A = B$.

Proof: We have, $A = (A] \cap [A) = (A] \wedge [A) = (A] \wedge [A \vee S) \wedge [A)$

$= (A \dot{\wedge} (A \vee S)) \wedge [A) = (A \dot{\wedge} (B \vee S)) \wedge [A)$

$= ((A \dot{\wedge} B) \vee (A \dot{\wedge} S)) \wedge [A)$ as S is standard.

$= ((A \dot{\wedge} B) \vee (B \dot{\wedge} S)) \wedge [A) = (B \dot{\wedge} (A \vee S)) \wedge [A)$ as S is standard

$= (B \dot{\wedge} (B \vee S)) \wedge [A) = (B] \wedge [B \vee S) \wedge [A) = (B] \wedge [A \vee S) \wedge [A)$

Thus, $A = (B] \wedge [A)$ . . . . (1)

Again, $A = (A] \cap [A) = (A] \wedge [A) = (A] \wedge (A \vee S] \wedge [A)$

$= (A] \wedge (A \dot{\vee} (A \vee S)) = (A] \wedge (A \dot{\vee} (B \vee S)) = (A] \wedge ((A \dot{\vee} B) \vee (A \dot{\vee} S))$ as S is standard, $= (A] \wedge ((A \dot{\vee} B) \vee (B \dot{\vee} S)) = (A] \wedge (B \dot{\vee} (A \vee S))$ as S is standard, $= (A] \wedge (B \dot{\vee} (B \vee S)) = (A] \wedge (B \vee S] \wedge [B) =$

$(A] \wedge (A \vee S] \wedge [B)$. Thus, $A = (A] \wedge [B)$ . . . . (2)

From (1) and (2) and by unique representation of convex sublattices

$(A] = (B]$ and $[A) = [B)$. Therefore, $A = B$.

By G. Grätzer and E.T. Schmidt [1, Lemma 2.4.8.] we know that if for any ideal I and a standard ideal S of a lattice, both $I \wedge S$ and $I \vee S$ are principal, then I itself is principal. Now we give a generalization of this result.

Proposition 6. Let S be a standard sublattice. Suppose I is an ideal of a lattice L such that $I \dot{\wedge} S = (a]$ and $I \vee S = (b]$ then I is a principal ideal provided $I \cap S \neq \phi$.

Proof: Since S is standard, so by Theorem 2, $b = (b \wedge s) \vee (b \wedge i)$ for some $s \in S$ and $i \in I$. Since $I, S \subseteq (b]$, so $s, i \leq b$ and $b = s \vee i$. Also $a \in I$ as I is an ideal. We claim that $I = (i \vee a]$.

Here $(b] = I \vee S \supseteq (i \vee a] \vee S \supseteq (i] \vee S = (i] \vee (S] \supseteq (i \vee s] = (b]$.

Therefore $I \vee S = S \vee (i \vee a]$. Again $(a] = I \dot{\wedge} S = I \wedge (S] \supseteq (i \vee a] \wedge (S] \supseteq (a] \wedge (S] \supseteq (a]$. Hence $S \dot{\wedge} I = (S] \wedge (i \vee a] = S \dot{\wedge} (i \vee a]$

Finally, $S \dot{\vee} I = ((S] \vee (I]) \wedge [S) = (b] \wedge [S)$ and

$S \dot{\vee} (I \vee a] = ((S] \vee (i \vee a]) \wedge [S) = ((S] \vee (s] \vee (i \vee a]) \wedge [S)$

$= ((S] \vee (i \vee s \vee a]) \wedge [S) = ((S] \vee (b]) \wedge [S) = (b] \wedge [S)$.

Thus $S \dot{\vee} I = S \dot{\vee} (i \vee a]$.

Moreover, $S \cap I \neq \phi$, let $u \in S \cap I$, then $u \in I \dot{\wedge} S = (a]$. Hence $u \in S \cap (i \vee a]$, that is, $S \cap (i \vee a] \neq \phi$. Hence by Lemma 5, $I = (i \vee a]$.

Similarly we can prove the following result.

Proposition 7. Let S be a standard sublattice. If for a dual ideal D, both $D \dot{\vee} S$ and $D \vee S$ are principal dual ideals, then D itself is principal, if $D \cap S \neq \phi$.

Here is another characterization of standard sublattices.

Theorem 8. A convex sublattice S is standard if and only if

$(x] \dot{\wedge} [S \vee (y]) = ((x] \wedge (S]) \vee (x \wedge y]$ and

$[p) \dot{\vee} (S \vee [q)) = ([p) \wedge [S)) \vee ([p) \wedge [q))$

$= ([p) \wedge [S)) \vee [p \wedge q)$,

for all $x, y, p, q \in L$ with $S \cap (y] \neq \phi$ and $S \cap (q] \neq \phi$.

Proof : If S is standard, then clearly above relations hold.To prove the converse, let K be any convex sublattice with $S \cap K \neq \phi$. Suppose $b \in S \cap K$. Now choose $a \in S \vee K$, then $s_1 \wedge k_1 \leq a \leq s_2 \vee k_2$ for some $s_1, s_2 \in S$ and $k_1, k_2 \in K$. Then $a \in (a]$ and $a \leq s_2 \vee k_2 \vee b$ implies that $a \in (S] \vee (k_2 \vee b] = S \vee (k_2 \vee b]$. Hence, $a \in (a] \curlywedge (S \vee (k_2 \vee b])$ and $b \in S \cap (k_2 \vee b]$. Then by first relation $a \in (a] \curlywedge (S \vee (k_2 \vee b]) = ((a] \wedge (S]) \vee (a \wedge k]$, where $k = k_2 \vee b \in K$. Thus $a \leq (a \wedge s) \vee (a \wedge k) \leq a$ for some $s \in S$ implies that $a = (a \wedge s) \vee (a \wedge k)$. Similarly $a \geq s_1 \wedge (k_1 \wedge b)$ implies $a \in (a] \dot{\vee} (S \vee [k_1 \wedge b))$ and $b \in S \cap [k_1 \wedge b)$. Then using the second relation, we can similarly show that $a = (a \vee s_1) \wedge (a \wedge k')$ for some $s_1 \in S$, $k' \in K$. Therefore by Theorem 2, S is a standard sublattice.

Theorem 9. A lattice L is distributive if and only if its every convex sublattice is a standard sublattice.

Proof: Let L be a distributive lattice and S be any convex sublattice of L. We have to show that S is a standard sublattice. Let $I, K \in CS(L)$ with $S \cap I \neq \phi$ and $I \cap (S \vee K) \neq \phi$.

First we show that $I \curlywedge (S \vee K) = (I \curlywedge S) \vee (I \curlywedge K)$.

Clearly, $(I \curlywedge S) \vee (I \curlywedge K) \subseteq I \curlywedge (S \vee K)$.

To show the reverse inclusion, consider $i \wedge r$, where $i \in I$ and $r \in S \vee K$. Now $r \in S \vee K$, then $s_1 \wedge k_1 \leq r \leq s_2 \vee k_2$ for some $s_1, s_2 \in S$ and $k_1, k_2 \in K$ implies that $(i \wedge s_1) \wedge (i \wedge k_1) \leq i \wedge r \leq i \wedge (s_2 \vee k_2)$ and so $(i \wedge s_1) \wedge (i \wedge k_1) \leq i \wedge r \leq (i \wedge s_2) \vee (i \wedge k_2)$ as L is distributive. Hence $i \wedge r \in (I \curlywedge S) \vee (I \curlywedge K)$. Since $I \curlywedge (S \vee K) = \langle \{i \wedge r \mid i \in I, r \in S \vee K\} \rangle$. Therefore, $I \curlywedge (S \vee K) \subseteq (I \curlywedge S) \vee (I \curlywedge K)$, and this implies that,

$I \dot{\wedge} (S \vee K) = (I \dot{\wedge} S) \vee (I \dot{\wedge} K)$. By a similar proof we show that $I \dot{\vee} (S \vee K) = (I \dot{\vee} S) \vee (I \dot{\vee} K)$. Hence S is a standard sublattice.

Conversely, suppose every convex sublattice of L is a standard sublattice. Since every ideal I of L is a convex sublattice of L so I is a standard sublattice and hence by Proposition1, I is standard in I(L). That is, every element of I(L) is a standard element, which implies that I(L) is a distributive lattice and so L is a distributive lattice .

Following result is also due to E. Fried and E.T. Schmidt [4] .

Theorem 10. Let n be an element of a lattice L. If every convex sublattice containing n is standard then L is a distributive lattice.

We call the convex sublattices containing the element n as n-sublattices and the standard sublattices containing n as standard n-sublattices. It should be mentioned that by A.S.A. Noor and M.A. Latif [6], standard n-sublattices are known as standard n-ideals and theorem 10 above, is a very trivial result of A.S.A. Noor and M.A. Latif [6].

The congruence relations Θ and Φ of a lattice L are said to be permutable if for a, b, c $\in$ L with $a \equiv b\ (\Theta)$, $b \equiv c\ (\Phi)$ imply that there exists d $\in$ L such that $a \equiv d\ (\Phi)$ and $d \equiv c\ (\Theta)$.

An element n of a lattice L is called central if it is neutral and is complemented in each interval containing it. For a standard n-sublattice S, we call the congruence $\Theta(S)$ as standard n-congruence. Now we give a theorem on the permutability of standard n-congruence. To prove this we need the following lemmas.

Lemma 11. Let S and T be two standard n-sublattices and n be a neutral element of a lattice L. If $x \le y \le z$ and $x \equiv y\ \Theta(S)$, $y \equiv z\ \Theta(T)$, then there exists r with $x \vee n \le r \le z \vee n$ such that $x \vee n \equiv r\ \Theta(T)$ and $r \equiv z \vee n\ \Theta(S)$.

Proof: Since x $\vee$ n $\equiv$ y $\vee$ n Θ(S) so by Theorem 2, we have $y \vee n = ((y \vee n) \wedge s) \vee (x \vee n) = (y \wedge s) \vee (n \wedge s) \vee (x \vee n) = ((y \wedge s) \vee n) \vee (x \vee n) =$ $x \vee n \vee a$ for some s $\in$ S where $a = (y \wedge s) \vee n$. Now $n \leq (y \wedge s) \vee n \leq s \vee n$ impies that $a \in S$. Also, since y $\vee$ n $\equiv$ z $\vee$ n Θ(T), so proceeding as above we get b $\in$ T such that z $\vee$ n = y $\vee$ n $\vee$ b. Set r = x $\vee$ n $\vee$ b. Then x $\vee$ n $\equiv$ x $\vee$ n $\vee$ b = r Θ(T) as n, n $\vee$ b $\in$ T and r = x $\vee$ n $\vee$ b $\equiv$ x $\vee$ n $\vee$ a $\vee$ b Θ(S) (as n, n $\vee$ a $\in$ S) = y $\vee$ n $\vee$ b = z $\vee$ n. Moreover, x $\vee$ n $\leq$ r $\leq$ z $\vee$ n and this completes the proof.

A dual proof of above lemma gives the following result.

Lemma 12. Let S and T be two standard n-sublattices and n be a neutral element of a lattice L. If x $\leq$ y $\leq$ z and x $\equiv$ y Θ(S), y $\equiv$ z Θ(T), then there exists s with x $\wedge$ n $\leq$ s $\leq$ z $\wedge$ n such that x $\wedge$ n $\equiv$ s Θ(T), s $\equiv$ z $\wedge$ n Θ(S).

Theorem 13. If n is a central element of a lattice L, then any two standard n-congruences are permutable.

Proof: Suppose S and T are standard n-sublattices of L. Let x, y, z $\in$ L with x $\equiv$ y Θ(S), y $\equiv$ z Θ(T). First consider x $\leq$ y $\leq$ z then by above lemmas there exist r, s $\in$ L with x $\vee$ n $\leq$ r $\leq$ z $\vee$ n and x $\wedge$ n $\leq$ s $\leq$ z $\wedge$ n such that x $\vee$ n $\equiv$ r Θ(T), r $\equiv$ z $\vee$ n Θ(S) and x $\wedge$ n $\equiv$ s Θ(T), s $\equiv$ z $\wedge$ n Θ(S). Now s $\leq$ n $\leq$ r. Since n is central, there exists p $\in$ L such that p $\wedge$ n = s and p $\vee$ n = r. Set u = z $\wedge$ (p $\vee$ x), then x = x $\vee$ (x $\wedge$ n) $\equiv$ x $\vee$ s Θ(T) = x $\vee$ (p $\wedge$ n) = (x $\vee$ p) $\wedge$ (x $\vee$ n) = (x $\vee$ p) $\wedge$ r Θ(T) = x $\vee$ p (as r $\geq$ p, x $\vee$ n) Thus x = z $\wedge$ x $\equiv$ z $\wedge$ (x $\vee$ p) = u Θ(T). Also z = z $\wedge$ (z $\vee$ n) $\equiv$ z $\wedge$ r Θ(S) = z $\wedge$ (r $\vee$ x $\vee$ n) = z $\wedge$ (p $\vee$ x $\vee$ n) = (z $\wedge$ (p $\vee$ x)) $\vee$ (z $\wedge$ n) $\equiv$ (z $\wedge$ (p $\vee$ x)) $\vee$ s Θ(S) = z $\wedge$ (p $\vee$ x) (as s $\leq$ p, z $\wedge$ n) $\equiv$ u Θ(S).

In the general case consider x, $x \vee y$, $x \vee y \vee z$. We have $x \equiv x \vee y$ $\Theta(S)$ and $x \vee y \equiv x \vee y \vee z$ $\Theta(T)$. Then by first part, there exists $v \in L$ with $x \leq v \leq x \vee y \vee z$, such that $x \equiv v$ $\Theta(T)$ and $v \equiv x \vee y \vee z$ $\Theta(S)$. Similarly from $z \equiv z \vee y$ $\Theta(T)$ and $z \vee y \equiv z \vee y \vee x$ $\Theta(S)$, we find $w \in L$ with $z \leq w \leq x \vee y \vee z$ such that $z \equiv w$ $\Theta(S)$ and $w \equiv x \vee y \vee z$ $\Theta(T)$. Set $q = v \wedge w$ then $q = v \wedge w \equiv v \wedge (x \vee y \vee z)$ $\Theta(T) = v$. Since $v \equiv x$ $\Theta(T)$, so We have $q \equiv x$ $\Phi(T)$. Similarly $q = v \wedge w \equiv w \wedge (x \vee y \vee z)$ $\Theta(S) = w$. Since $w \equiv z$ $\Theta(S)$, so we have $q \equiv z$ $\Theta(S)$. In other words Θ (S) and Θ (T) are permutable.

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