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STABLE TECHNIQUE FOR OVER-DAMPEED VIBRATION IN BIOLOGICAL AND BIOCHEMICAL SYSTEMS

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Abstract

Based on the Struble technique, a simple formula is presented for obtaining approximate solutions of over-damped nonlinear differential systems when one of the roots of the unperturbed equation is much smaller than the other roots. The method is easier than the existing perturbation techniques. An example is given to biological system.

Keywords and phrases : *over-damped, perturbation techniques, biological system.*

বিমূর্ত সার (*Bengali version of the Abstract*)

স্ট্রাব্‌ল (Struble) কৃৎকৌশলের উপর ভিত্তি করে অতি-অবমন্দিত অরৈখিক অবকলন তন্ত্রের অসন্নমানে সমাধান একটি সহজ সূত্রের সাহায্যে নির্ণয় করা হয়েছে যখন ইহাদের অবিচলিত সমীকরণের একটি বীজ অন্যগুলি থেকে অনেক ক্ষুদ্র হয়। এই পদ্ধতিটি বর্তমানে বিদ্যমান বিচলিত কৃৎকৌশল পদ্ধতি থেকে অনেক সহজ। জীববিদ্যার ক্ষেত্র একটি উদাহরণ দেওয়া হয়েছে।

1. Introduction

Among the methods used to study nonlinear systems with a small non-linearity, the Krylov-Bogoliubov-Mitropolskii (KBM) [1-5] method is particularly convenient and it is widely used technique to obtain the approximate solutions. The method was originally developed by Krylov and Bogoliubov [1] for obtaining periodic solution of a second order nonlinear differential equation. The method was amplified and justified by Bogoliubov and Mitropolishkii [2-5]. Popov [6] extended the method to a damped oscillatory process in which a strong linear damping force acts. Murty, Dekshatulu and Krisna [7] extended the method to over-damped nonlinear system. Shamsul [8-10] investigated over-damped nonlinear systems and found approximate solutions of *Duffing*'s equation when one root of the unperturbed equation was respectively double or triples of the other. Recently Shamsul [18] has developed the general Struble's techniques for several damping effect. In [18], it has been shown that the general Struble's technique is identical to the unified KBM method [1-5]. In this paper, an asymptotic solution of a biological system has been found by Struble's techniques

Three such models are described below:

(i) A modified Lotka-Volterra model: Assuming in presence of predator and a logistic growth for prey one obtains the well-known equations [15,17]

$$\dot{N}_1 = N_1(k_{11} + k_{12}N_1 + k_{13}N_2), \qquad \dot{N}_2 = N_2(k_{21} + k_{22}N_1 + k_{23}N_2) \tag{1}$$

where N_1 and N_2 are two populations.

(ii) Oscillating chemical reaction: Lefever and Nicolis [14] have considered a set of chemical reactions modeled by the chemical kinetic equations

$$\dot{X} = A + X^2Y - BX - X, \qquad \dot{Y} = BX - X^2Y \tag{2}$$

where X and Y are concentrations, and A and B are initial product concentrations. Lefever and Nicolis [14] have studied the phase portrait in the phase plane (X,Y) both analytically and numerically, and shown the existence of a limit cycle.

(iii) The FitzHugh equations: To investigate the physiological state of nerve membranes, FitzHugh [11] introduce a theoretical model described by

$$\dot{x}_1 = \alpha + x_1 + x_2 - \frac{x_1}{3}, \qquad x_2 = \rho(\gamma - x_1 - \beta_2 x_2) \tag{3}$$

where it is assume that $\alpha, \gamma \in (-\infty, \infty)$ and $\beta, \rho \in (0,1)$. For $\alpha = \beta = \gamma = 0$, equation (3) reduces to a Van der Pol equation. This particular model has been studied by Troy [16], Hsu and Kazarinoff [13]. FitzHugh [11] investigated the model quantitatively in the phase plane, while Hsu and Kazarinoff [13] dealt with periodic solutions using the Poincare-Hopf bifurcation theory.

It will be shown that all the modeling equation (i)-(iii) can be presented in the neighborhood of the equilibrium position by a second order differential equation of the type [8-10].

$$\ddot{x} + 2k\,\dot{x} + c\,x = \varepsilon\, f_1(x, \dot{x}) + \varepsilon^2\, f_2(x, \dot{x}) + \ldots \tag{4}$$

where ε is a small positive parameter, and the significant damping term is expressed by the linear term $2k\,\dot{x}$. The damping coefficients k [of order O(1)], and also c, are constant. The assumption $k > \sqrt{c}$ ensures that the system is over-damped. When $\varepsilon = 0$, equation (4) has two roots, say $-\lambda, -\mu, \quad \lambda, \mu > 0$. Therefore, the unperturbed solution of the equation (4) is $x(t) = x_{1,0}\, e^{-\lambda t} + x_{-1,0}\, e^{-\mu t}$, which describes a non-oscillatory motion. Here $x_{1,0}$ and $x_{-1,0}$ are two arbitrary constants.

2. The Struble's Techniques for Over-Damped Systems

Equation (4) is slightly more general than the equation initially studied by Popov [6], which does not include the term $\varepsilon^2 f_2(x, \dot{x})$. Following the Struble's techniques [18]

$$x(t, \varepsilon) = x_{1,0}e^{-\lambda t} + x_{-1,0}e^{-\mu t} + \varepsilon u_1(x_1, x_{-1}, t) + \varepsilon^2 \ldots \tag{5}$$

where $x_{1,0}$ and $x_{-1,0}$ satisfy the first order differential equations

$$\dot{x}_1 = \varepsilon A + \varepsilon^2 \ldots, \quad \dot{x}_{-1} = \varepsilon B + \varepsilon^2 \ldots, \tag{6}$$

Now differentiating (5) twice with respect to t, substituting the derivatives $\ddot{x}, \dot{x}$ and the dependent variable x in (4), utilizing (6) and comparing the coefficients of various power of ε, we obtain

$$(D+\mu)e^{-\lambda t}A+(D+\lambda)e^{-\mu t}B+(D+\lambda)(D+\mu)u_1 = f^{(0)} \qquad (7)$$

where $f^{(0)} = f^{(0)}(x_0, \dot{x}_0)$ and $x_0 = x_{1,0}e^{-\lambda t} + x_{-1,0}e^{-\mu t}$

In general, $f^{(0)}$ can be expanded in a Taylor's series as

$$f^{(0)} = \sum_{j,k=0} F_{j,k} e^{-(j\lambda+k\mu)t} \qquad (8)$$

We assume that u_1 does not contain the terms with $e^{-(j\lambda+k\mu)t}$ where $j\lambda + k\mu \le (j+k)c_1$, $c_1 = \dfrac{\lambda+\mu}{2}$, so that the coefficient in the expansion of u_1 does not become large and u_1 do not contain the secular type terms.

Substituting the values of $f^{(0)}$ from (8) into (7) using our assumption, we obtain

$$(D+\mu)e^{-\lambda t}A+(D+\lambda)e^{-\mu t}B = \sum_{j,k=0} F_{j,k} e^{-(j\lambda+k\mu)t} \qquad (9)$$

where $j\lambda + k\mu \le (j+k)c_1$, $c_1 = \dfrac{\lambda+\mu}{2}$

and $$(D+\lambda)(D+\mu)u_1 = \sum_{j,k=0} F_{j,k} e^{-(j\lambda+k\mu)t} \qquad (10)$$

Solving equation (10), we obtain

$$u_1 = \sum \frac{F_{j,k} e^{-(j\lambda+k\mu)t}}{(j\lambda+k\mu-\lambda)(j\lambda+k\mu-\mu)} \qquad (11)$$

In order to determine the unknown functions A and B, we can replace x_1 and x_{-1} in the right hand sides of (9) by their respective values obtained in the linear case and assume that the coefficients of A and B do not become large.

3. A Special Case of the Model (1)

A special case (over-damped) of the model (1) has been discussed by Goh [12].

$$\dot{N}_1 = N_1(5.6 - 0.5N_1 - 0.6N_2), \qquad \dot{N}_2 = N_2(-11 + N_1 + N_2). \tag{12}$$

where N_1 be prey density and N_2 be predator density. There exists a single steady state solution $N_1^* = 10, \quad N_2^* = 1$ of (12), obtained for $\dot{N}_1 = 0, \quad \dot{N}_2 = 0$, *i. e.*, from the equilibrium equations. Here we can show that, one of the roots is much smaller than the others. Goh [12] showed the equilibrium of the model (12) is locally stable. If the solution initially starts for $N_1^* = 11, \quad N_2^* = 3$ it rends rapidly to $N_1^* = 0$ and $N_2^* = \infty$, although the steady state solution $N_1^* = 10, \quad N_2^* = 1$ is not very far from the steady state solution. That is why we are interested to investigate quantitative solutions in the neighborhood of the steady state solution.

The solution in the neighborhood of the steady state are presented by x and y

where $$N_1(t) = N_1^* + \varepsilon\, x(t), \qquad N_2(t) = N_2^* + \varepsilon\, y(t) \tag{13}$$

Using (12) and (13), we obtain

$$\begin{aligned} \dot{x} &= -(10 + \varepsilon\, x)(0.5\, x + 0.6\, y), \\ \dot{y} &= x + y + \varepsilon\ (xy + y^2). \end{aligned} \tag{14}$$

Eliminating y from equation (13) and (14) leads to a second order differential equation for x

$$\ddot{x} + 4\,\dot{x} + x = \varepsilon\,[\frac{11}{15}x^2 - \frac{7}{6}x\,\dot{x} - \frac{1}{15}\dot{x}^2] + O(\varepsilon^2) \tag{15}$$

Here the unperturbed equation, *i. e*. $\ddot{x} + 4\,\dot{x} + x = 0$ has the roots -2+√3 and -2-√3. It is clear that the ratio of the roots is 12.34. *i. e*. one of the roots is much smaller than the others.

Therefore, for modeling equation (12), equation (7) becomes

$$\begin{aligned}&(D+\mu)e^{-\lambda t}A+(D+\lambda)e^{-\mu t}B+(D+\lambda)(D+\mu)u_1\\&=\frac{{x_1}^2}{30}e^{-2\lambda t}(22+35\lambda-2\lambda^2)+\frac{{x_{-1}}^2}{30}e^{-2\mu t}(22+35\mu-2\mu^2)\\&+\frac{x_1x_{-1}}{30}e^{-(\lambda+\mu)t}(44+35\lambda+35\mu-4\lambda\mu)\end{aligned} \tag{16}$$

Here

$$(D+\mu)e^{-\lambda t}A=\frac{x_1x_{-1}}{30}e^{-(\lambda+\mu)t}(44+35\lambda+35\mu-4\lambda\mu) \tag{17}$$

$$(D+\lambda)e^{-\mu t}B=\frac{{x_{-1}}^2}{30}e^{-2\mu t}(22+35\mu-2\mu^2) \tag{18}$$

$$(D+\lambda)(D+\mu)u_1=\frac{{x_1}^2}{30}e^{-2\lambda t}(22+35\lambda-2\lambda^2) \tag{19}$$

and

Solving equations (17)-(19), we obtain

$$A=\frac{x_1x_{-1}}{30\lambda}e^{-\mu t}(44+35\lambda+35\mu-4\lambda\mu) \tag{20}$$

$$B=\frac{{x_{-1}}^2}{30(2\mu-\lambda)}e^{\mu t}(22+35\mu-2\mu^2) \tag{21}$$

$$u_1=n_1{x_1}^2e^{-2\lambda t} \quad \text{where } n_1=\frac{1}{30\lambda(2\lambda-\mu)}(22+35\lambda-2\lambda^2) \tag{22}$$

Substituting the values of A and B into equation (6), we obtain

$$\dot{x}_1=\varepsilon l_1x_1x_{-1}e^{-\mu t} \qquad \dot{x}_{-1}=\varepsilon m_1{x_{-1}}^2e^{-\mu t} \tag{23}$$

where $l_1=\frac{1}{30\lambda}(44+35\lambda+35\mu-4\lambda\mu)$, $m_1=\frac{1}{30(2\mu-\lambda)}(22+35\mu-2\mu^2)$

First equation of (23) has an exact equation but the second equation of (23) has no exact solution. So, we solved the first equation analytically and assume that x_1 is constant in the right hand side of the second equation of (23).

Therefore,

$$x_1 = x_{1,0} e^{\varepsilon l_1 x_{-1,0}(1-e^{-\mu t})/\mu}, \qquad x_{-1} = \frac{x_{-1,0}}{1-\varepsilon m_1 x_{-1,0}(1-e^{-\mu t})/\mu} \tag{24}$$

Thus the first order approximate solution of (15) is

$$x = x_1 e^{-\lambda t} + x_{-1} e^{-\mu t} + \varepsilon u_1 \tag{25}$$

Where x_1 and x_{-1} are given by (24) and u_1 is given by (22).

4. Results and Discussion

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we compare the approximate solution to the numerical solution. With regard to such a comparison concerning the presented Struble's techniques of this article, we refer to the work of Murty and Deekshatulu [7], Shamsul [18]. In this article, we have compare the approximate solution (25) (when $k = 2, c = 1$ and $\varepsilon = 0.5$) to those obtained by a fourth order Runge-Kutta method.

Here we have considered the equation (12) [special case of modeling equation (1)] in which $N_1^* = 10,\ \ N_2^* = 1$ (in lakh, 1 lakh=1,00,000). Let us assume that 20 thousands prey have been added to this population. For that we have chosen $y(0) = 0.0$ and $\varepsilon = 0.5$. First of all, $x(t)$ has been computed by our asymptotic solution (25) with initial condition $x(0) = 0.4$ and $\dot{x}(0) = -2.04$. Then $N_1(t) = N_1^* + \varepsilon\, x(t)$ has been computed. To verify the results, corresponding numerical solutions of $N_1(t)$ has been computed by fourth order Runge-Kutta method. All the results are shown in Fig. 1(a). From Fig. 1(a) it is clear that the asymptotic solution (25) shows good agreement with the numerical solution.

To compute $N_2(t)$ or $y(t)$, we have to compute $\dot{x}(t)$. Differentiating $x(t)$ from (25) and then substituting the values of $\dot{x}(t)$ and $x(t)$ into the first equation of (14) and simplifying, we have computed $y(t)$ and then $N_2(t) = N_2^* + \varepsilon\, y(t)$. Corresponding numerical results of $N_2(t)$ have been computed and both the results are shown in Fig. 1(b). From Fig. 1(b), we see that the perturbation results of $N_2(t)$ also agree with the numerical results.

Fig. 1(a): Perturbation solutions (solid line) and numerical solutions (dotted line) of N_1 are computed when $N_1(0) = 10.2$ and $N_2(0) = 1.0$ [or $N_1 = 10.2$ and $N_2 = 1.0$]. In this case, $x(0) = 0.4$, $\dot{x}(0) = -2.04$ and $\varepsilon = 0.5$ or $a_{1,0} = 0.529798$ and $a_{1,2} = -.153345$.

Fig. 1(b): Perturbation solutions (solid line) and numerical solutions (dotted line) of N_2 are computed with the same initial conditions as in Fig. 1(b).

4. Conclusion

An asymptotic solution is found based on the Struble technique for non-oscillatory nonlinear biological systems, when one of the roots of the unperturbed equation is much smaller than the other. The results agree with the numerical solutions nicely.

References

1) Krylov N.N and Bogoliubov N.N, *Introduction to Nonlinear Mechanics*, Princeton University press, New Jersey 1947.

2) Bogoliubov N.N, Mitropolskii Yu. A., *Asymptotic Methods in the Theory of Nonlinear Oscillation*, Gordan and Breach, New York, 1961.

3) Cunningham W.J., *Introduction to Nonlinear Analysis*, McGraw-Hill Book Company, 1958.

4) MinorskN. y, *Introduction to Nonlinear Mechanics*, J .E. Edwards, Ann Arbor, Michigan, 1947.

5) Nayfeh A. H., *Perturbation Method*s, J. Wiley, New York, 1973.

6) Popov I. P., “A generalization of the Bogoliubov asymptotic method in the theory of nonlinear oscillations”, Dokl.Akad. Nauk SSSR 111 (1956), 308-310 (in Russian).

7) Murty I. S. N., Deekshatulu B. L. and Krisna G., “General asymptotic method of Krylov-Bogoliubov for over-damped nonlinear system”, J. Frank Inst. 288 (1969), 49-46.

8) Alam Shamsul M., “Asymptotic methods for second-order over-damped and critically damped nonlinear system”, Soochow J. Math, 27 (2001), 187-200.

9) Alam Shamsul M., “Method of solution to the *n*-th order over-damped nonlinear systems under some special conditions", Bull. Call. Math. Soc., 94(6) (2002), 437-440.

10) Alam Shamsul M.,, “Method of solution to the order over-damped nonlinear systems with varying coefficients under some special conditions", Bull. Call. Math. Soc., 96(5) (2004), 419-426.

11) FitzHugh, R., Impulse and physiological states in theoretical models of nerve membrane”, J. Biophys., Vol. 1, pp. 445-466, 1961.

12) Goh, B. S., Global stability in many species systems, The American Naturalist, Vol. 111, pp. 135-143, 1977.

13) Hsu, I. D. and Kazarinoff, “An applicable Hopf bifurcation formula and instabity of small periodic solution of the field-noise model”, J. math. Anal. Applic., Vol. 55, pp. 61-89, 1976.

14) Lefever, R. and Nicolis G., “Chemical instabilities and sustained oscillations”, J. Theor. Biol., Vol. 30, pp. 267-248, 1971.

15) Lotka, A. J., “The growth of mixed population”, J. Wsah. Acad. Sci. Vol. 22, pp. 461-469, 1932.

16) Troy, W. C., “Oscillating phenomena in nerve condition equation”, Ph.D. Dissertation, SUNY at Buffelo, 1974.

17) Volterra, V., “Variazioni e fluttuazioni del numero d’individue in species animali conviventi”, Memorie del R. Comitato Talassografico Italiano, Vol. 131, pp. 1-142, 1927.

18) Alam Shamsul M., Azad Abul Kalam M. and Hoque M. A., “A general Struble’s technique for solving an *n*-th order weakly non-linear differential system with damping”, Journal of Non-Linear Mechanics, Vol. 41, pp.905-918, 2006