# NEUTRAL SUBLATTICES 

## By

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#### Abstract

In this paper we introduce the notion of neutral convex sublattices of a lattice to generalize the concept of neutral ideals. Here we give several characterizations of these sublattices and include some of their properties.


Keywords and phrases: lattice, neutral ideals, neutral convex sublattices.
বিমূর্ত সার (Bengali version of the Abstract)
बই পজ্র ন্যাটিস (Lattice) - এর নিরদেক্ক উত্তল উপ-ল্যাটিস (Neutral convex sublattics) এর ধারণাকে উপস্থাপন করেছি নিরুপ্ক আইড্যেয়েলস (Neutral ideals) - এর ধারণাকে সামান্যিকরণেন জন্য। এখানে আমরা এই উপ - ন্যাট্সিজুলিকে বহবিধ বৈশিষ্ট্য প্রদান করেরি এবং ইহাদের কিছু রর্মাকে অর্ত্তযুক্ত করেছি।

## 1. Introduction:

Standard and Neutral elements and ideals have been studied by Grätzer G. [1], [2] and Grätzer G. and Schmidt E.T. [3]. By Grätzer G., and Schmidt E.T. [3], if a is an element of a lattice L, then
i) a is called standard if $x \wedge(a \vee y)=(x \wedge a) \vee(x \wedge y)$ for all $x, y \in L$, ii) a is called neutral if for all $\mathrm{x}, \mathrm{y} \in \mathrm{L}$
$(\alpha) x \wedge(a \vee y)=(x \wedge a) \vee(x \wedge y)$ that is, a is standard and
$(\beta) a \wedge(x \vee y)=(a \wedge x) \vee(a \wedge y)$.
According to G. Grätzer and E.T. Schmidt [3] an ideal $N$ of a lattice L is called neutral if N is neutral as an element of $\mathrm{I}(\mathrm{L})$, the lattice of all ideals of $L$. In response to an open problem suggested by G. Grätzer
[1], Fried E. and Schmidt E.T. [4] generalized the concept of standard ideals by introducing the notion of standard convex sublattices. On the other hand, Nieminen J. [5] introduced the notion of distributive and neutral convex sublattices. In this paper, we study the neutral convex sublattices with an approach which is different from Nieminen J. [5]. We have given several characterizations of these sublattices and included some properties of them. Throughout the paper we call these sublattices as neutral sublattices instead of neutral convex sublattices.

For a lattice $L$, the set of all convex sublattices of $L$ with the empty set $\Phi$ is a lattice, denoted by $\mathrm{CS}(\mathrm{L})$. For any A, B $\in \mathrm{CS}(\mathrm{L})$, we define $A \wedge B=\langle\{a \wedge b \mid a \in A, b \in B\}\rangle$ that is, the convex sublattice generated by the elements $\mathrm{a} \wedge \mathrm{b}$ for all $\mathrm{a} \in \mathrm{A}, \mathrm{b} \in \mathrm{B}$. We also define, $\mathrm{A} \vee \mathrm{B}=$ $\langle\{\mathrm{a} \vee \mathrm{b} \mid \mathrm{a} \in \mathrm{A}, \mathrm{b} \in \mathrm{B}\}\rangle$ and $\mathrm{A} \vee \mathrm{B}=\langle\mathrm{A}, \mathrm{B}\rangle=\langle\mathrm{A} \cup \mathrm{B}\rangle$ that is, convex sublattice generated by A and B ..

Observe that in a lattice $L$ the ideal of $L$ generated by $\{\mathrm{a} \wedge \mathrm{b} \mid \mathrm{a} \in \mathrm{A}, \mathrm{b} \in \mathrm{B}\}$, that is, $(\{\mathrm{a} \wedge \mathrm{b} \mid \mathrm{a} \in \mathrm{A}, \mathrm{b} \in \mathrm{B}\}]=(\mathrm{A}] \wedge(\mathrm{B}]$. Moreover the dual ideal of $L$ generated by $\{a \wedge b \mid a \in A, b \in B\}$, that is, $[\{a \wedge b \mid a \in A, b \in B\})=[A) \vee[B)$. Similarly $(\{a \vee b \mid a \in A, b \in B\}]=$ $(A] \vee(B]$ and $[\{a \vee b \mid a \in A, b \in B\})=[A) \wedge[B)$. Therefore when $A$ and B are ideals then $\mathrm{A} \wedge \mathrm{B}=\mathrm{A} \wedge \mathrm{B}$ and $\mathrm{A} \vee \mathrm{B}=\mathrm{A} \vee \mathrm{B}$ and when A and B are dual ideals then $\mathrm{A} \wedge \mathrm{B}=\mathrm{A} \vee \mathrm{B}$ and $\mathrm{A} \vee \mathrm{B}=\mathrm{A} \wedge \mathrm{B}$.

By above observations we find that,

$$
(\mathrm{A} \wedge \mathrm{~B}]=(\mathrm{A}] \wedge(\mathrm{B}],[\mathrm{A} \wedge \mathrm{~B})=[\mathrm{A}) \vee[\mathrm{B}),(\mathrm{A} \vee \mathrm{~B}]=(\mathrm{A}] \vee(\mathrm{B}]
$$

and $[\mathrm{A} \vee \mathrm{B})=[\mathrm{A}) \wedge[\mathrm{B})$.
If for two convex sublattices C and $\mathrm{D}, \mathrm{C} \cap \mathrm{D} \neq \phi$, then we shall write $C \cap D$ by $C \wedge D$. It is well known that for any convex sublattice $C$ of a lattice $\mathrm{L}, \mathrm{C}=(\mathrm{C}] \cap[\mathrm{C})$.

Therefore, $\mathrm{A} \wedge \mathrm{B}=(\mathrm{A} \wedge \mathrm{B}] \cap[\mathrm{A} \wedge \mathrm{B})=((\mathrm{A}] \wedge(\mathrm{B}]) \wedge([\mathrm{A}) \vee[\mathrm{B}))$ and
$A \vee B=(A \vee B] \cap[A \vee B)=((A] \vee(B]) \wedge([A) \wedge[B))$.
By [2] we know that $\mathrm{A} \wedge(\mathrm{B}]=(\mathrm{A}] \wedge(\mathrm{B}]$ and $\mathrm{A} \vee(\mathrm{B}]=(\mathrm{A}] \vee(\mathrm{B}]$
Following results are easily verifiable.
$A \vee(B]=((A] \vee(B]) \wedge[A), A \wedge[B)=(A] \wedge([A) \vee[B))$.
$A \vee[B)=[A) \wedge[B), A \vee[B)=[A) \vee[B)$. Also $(A \vee B]=(A] \vee(B]$
and $[A \vee B)=[A) \vee[B)$. Moreover, $(A \cap B]=(A] \wedge(B]$ and $[A \cap B)=[A) \wedge[B)$, provided $A \wedge B=A \cap B \neq \phi$

A convex sublattice S of a lattice L is called a standard convex sublattice if for all $\mathrm{I}, \mathrm{K} \in \mathrm{CS}(\mathrm{L})$,
(i) $I \wedge(S \vee K)=(I \wedge S) \vee(I \wedge K)$ and
(ii) $I \vee(S \vee K)=(I \vee S) \vee(I \vee K)$
where $\mathrm{S} \cap \mathrm{K} \neq \phi$ and $\mathrm{I} \cap(\mathrm{S} \vee \mathrm{K}) \neq \phi$.
We define a convex sublattice N of L as neutral if
(i) N is a standard sublattice and
(ii) For any $\mathrm{X}, \mathrm{Y} \in \mathrm{CS}(\mathrm{L})$ with $\mathrm{N} \cap \mathrm{X} \neq \phi$ and $\mathrm{N} \cap \mathrm{Y} \neq \phi$,
$\mathrm{N} \wedge(\mathrm{X} \vee \mathrm{Y})=(\mathrm{N} \wedge \mathrm{X}) \vee(\mathrm{N} \wedge \mathrm{Y})$ and
$N \vee(X \vee Y)=(N \vee X) \vee(N \vee Y)$.
Now we show that the concept of neutral sublattices coincides with that of neutral ideals in case of ideals.

Theorem 1. An ideal N is neutral if and only if N is a neutral sublattice.
Proof: Suppose N is a neutral sublattice. Then by definition N is a standard sublattice and hence by E. Fried, and E.T. Schmidt [4, proposition 2], it is a standard ideal.

Let $\mathrm{I}, \mathrm{J} \in \mathrm{I}(\mathrm{L})$. Then $\mathrm{N} \cap \mathrm{I} \neq \phi$ and $\mathrm{N} \cap \mathrm{J} \neq \phi$, and $N \wedge(I \vee J)=N \wedge(I \vee J)=(N \wedge I) \vee(N \wedge J)$ as $N$ is a neutral sublattice $=(\mathrm{N} \wedge \mathrm{I}) \vee(\mathrm{N} \wedge \mathrm{J})$, which implies that N is a neutral ideal.

Conversely, suppose N is a neutral ideal. Then N is a standard ideal and hence by Fried E., and Schmidt E.T. [4, Proposition 2], it is a standard sublattice.
Now, let $\mathrm{A}, \mathrm{B} \in \mathrm{CS}(\mathrm{L})$ with $\mathrm{N} \cap \mathrm{A} \neq \phi$ and $\mathrm{N} \cap \mathrm{B} \neq \phi$.
Then $\mathrm{N} \wedge(\mathrm{A} \vee \mathrm{B})=\mathrm{N} \wedge(\mathrm{A} \vee \mathrm{B}] \quad$ (as N is an ideal)
$=N \wedge((A] \vee(B])=(N \wedge(A]) \vee(N \wedge(B])$ as $N$ is neutral.

$$
=(\mathrm{N} \wedge \mathrm{~A}) \vee(\mathrm{N} \wedge \mathrm{~B}) .
$$

Moreover, since N is an ideal, then
$N \vee(A \vee B)=N \vee(A \vee B]=N \vee((A] \vee(B])$
$=(N \vee(A]) \vee(N \vee(B])=(N \vee A) \vee(N \vee B)$.
Therefore N is a neutral sublattice.
Theorem 2. A convex sublattice N is neutral if and only if N is standard and

$$
\begin{aligned}
& \mathrm{N} \wedge((\mathrm{a}] \vee(\mathrm{b}])=(\mathrm{N} \wedge(\mathrm{a}]) \vee(\mathrm{N} \wedge(\mathrm{~b}]) \\
& \text { provided } \mathrm{N} \cap(\mathrm{a}] \neq \phi, \mathrm{N} \cap(\mathrm{~b}] \neq \phi \text { and } \\
& \mathrm{N} \vee([\mathrm{c}) \vee[\mathrm{d}))=(\mathrm{N} \vee[\mathrm{c})) \vee(\mathrm{N} \vee[\mathrm{~d})) \\
& \text { provided } \mathrm{N} \cap[\mathrm{c}) \neq \phi, \mathrm{N} \cap[\mathrm{~d}) \neq \phi .
\end{aligned}
$$

Proof: If N is a neutral sublattice, then it is standard and both the given conditions hold. Conversely, let N be standard and the given conditions hold. We have to show that N is a neutral sublattice.

Suppose A, B $\in \operatorname{CS}(\mathrm{L})$ with $\mathrm{N} \cap \mathrm{A} \neq \phi$ and $\mathrm{N} \cap \mathrm{B} \neq \phi$.
Obviously $(\mathrm{N} \wedge \mathrm{A}) \vee(\mathrm{N} \wedge \mathrm{B}) \subseteq \mathrm{N} \wedge(\mathrm{A} \vee \mathrm{B})$.
Let $x \wedge t$ be a generator of $N \wedge(A \vee B)$ for some $x \in N$ and $t \in A \vee$ B. Now $t \in A \vee B$ implies that $a_{1} \wedge b_{1} \leq t \leq a_{2} \vee b_{2}$ for some $a_{1}, a_{2} \in A$ and $b_{1}, b_{2} \in B$. Let $a \in N \cap A$ and $b \in N \cap B$, then $t \leq\left(a \vee a_{2}\right) \vee\left(b \vee b_{2}\right)$ $=\mathrm{a}^{\prime} \vee \mathrm{b}^{\prime}$ where $\mathrm{a}^{\prime}=\mathrm{a} \vee \mathrm{a}_{2}$ and $\mathrm{b}^{\prime}=\mathrm{b} \vee \mathrm{b}_{2}$. Now, $\mathrm{t} \leq \mathrm{a}^{\prime} \vee \mathrm{b}^{\prime}$ implies $t \in\left(a^{\prime}\right] \vee\left(b^{\prime}\right]$. Then $x \wedge t \in N \wedge\left(\left(a^{\prime}\right] \vee\left(b^{\prime}\right]\right)$.

But $\mathrm{N} \wedge\left(\left(\mathrm{a}^{\prime}\right] \vee\left(\mathrm{b}^{\prime}\right]\right)=\mathrm{N} \wedge\left(\left(\mathrm{a}^{\prime}\right] \vee\left(\mathrm{b}^{\prime}\right]\right)$
$=\left(\mathrm{N} \wedge\left(\mathrm{a}^{\prime}\right]\right) \vee\left(\mathrm{N} \wedge\left(\mathrm{b}^{\prime}\right]\right)$ by the given condition.

$$
\left.=(\mathrm{N}] \wedge\left(\mathrm{a}^{\prime}\right]\right) \vee\left((\mathrm{N}] \wedge\left(\mathrm{b}^{\prime}\right]\right) .
$$

This implies that $\left.x \wedge t \in\left((N] \wedge\left(a^{\prime}\right]\right) \vee(N] \wedge\left(b^{\prime}\right]\right)$, hence, $x \wedge t \leq$ $\left(\mathrm{n}_{1} \wedge \mathrm{a}^{\prime}\right) \vee\left(\mathrm{n}_{2} \wedge \mathrm{~b}^{\prime}\right)$ where $\mathrm{n}_{1}, \mathrm{n}_{2} \in(\mathrm{~N}]$. Then $\mathrm{n}_{1} \leq \mathrm{n}^{\prime}$ and $\mathrm{n}_{2} \leq \mathrm{n}^{\prime \prime}$ for some $n^{\prime}, n^{\prime \prime} \in N$, thus $x \wedge t \leq\left(n^{\prime} \wedge a^{\prime}\right) \vee\left(n^{\prime \prime} \wedge b^{\prime}\right)$. But $\left(n^{\prime} \wedge a^{\prime}\right) \vee\left(n^{\prime \prime} \wedge\right.$ $\left.b^{\prime}\right) \in(N \wedge A) \vee(N \wedge B)$. Again $a_{1} \wedge b_{1} \leq x \wedge t$ and so $\left(n \wedge a_{1}\right) \wedge\left(n \wedge b_{1}\right)$ $\leq \quad x \wedge t$ for all $n \in N$. But $\left(n \wedge a_{1}\right) \wedge\left(n \wedge b_{1}\right) \in(N \wedge A) \vee(N \wedge B)$. Hence by convexity $x \wedge t \in(N \wedge A) \vee(N A B)$.

Thus, $N \wedge(A \vee B) \subseteq(N \wedge A) \vee(N \wedge B)$.
Therefore, $\mathrm{N} \wedge(\mathrm{A} \vee \mathrm{B})=(\mathrm{N} \wedge \mathrm{A}) \vee(\mathrm{N} \wedge \mathrm{B})$.
Dually we can show that $N \vee(A \vee B)=(N \vee A) \vee(N \vee B)$.
Hence N is a neutral sublattice.
Theorem 3. Every singleton sublattice is neutral.
Proof: Let $\mathrm{a} \in \mathrm{L}$ then by E. Fried, and E.T. Schmidt [4, Proposition 1], $\{\mathrm{a}\}$ is standard.Choose $\mathrm{b}, \mathrm{c} \in \mathrm{L}$ with $\{\mathrm{a}\} \cap(\mathrm{b}] \neq \phi$ and $\{\mathrm{a}\} \cap(\mathrm{c}] \neq \phi$, then $\mathrm{a} \in(\mathrm{b}]$ and $\mathrm{a} \in(\mathrm{c}]$.

Now $\{\mathrm{a}\} \wedge((\mathrm{b}] \vee(\mathrm{c}])=(\mathrm{a}] \wedge(\mathrm{b}] \vee(\mathrm{c}])=(\mathrm{a}]$ and
$(\{a\} \wedge(b]) \vee(\{a\} \wedge(c])=((a] \wedge(b]) \vee((a] \wedge(c])=(a] \vee(a]=(a]$.
Therefore, $\{\mathrm{a}\} \wedge((\mathrm{b}] \vee(\mathrm{c}])=(\{a\} \wedge(b]) \vee(\{a\} \wedge(c])$.
Again if for $\mathrm{x}, \mathrm{y} \in \mathrm{L}$ with $\{\mathrm{a}\} \cap[\mathrm{x}) \neq \phi$ and $\{\mathrm{a}\} \cap[\mathrm{y}) \neq \phi$.
Then $\mathrm{a} \in[\mathrm{x})$ and $\mathrm{a} \in[\mathrm{y})$.
So $\{\mathrm{a}\} \vee([\mathrm{x}) \vee[\mathrm{y}))=[\mathrm{a}) \wedge([\mathrm{x}) \vee[\mathrm{y}))=[\mathrm{a})$ and
$(\{a\} \vee[x)) \vee(\{a\} \vee[y))=([a) \wedge[x)) \vee([a) \wedge[y))=[a) \vee[a)=[a)$.
Therefore, $\{a\} \vee([x) \vee[y))=(\{a\} \vee[x)) \vee(\{a\} \vee[y))$.
Hence by Theorem 2, $\{a\}$ is neutral.

In Grätzer G. [2], has characterized a neutral element by a single equation. Following result gives the similar type of characterization for neutral sublattices.

Theorem 4. A convex sublattice N is neutral if and only if

$$
\begin{aligned}
& (\mathrm{N} \wedge(\mathrm{a}]) \vee(\mathrm{N} \wedge(\mathrm{~b}]) \vee((\mathrm{a}] \wedge(\mathrm{b}]) \\
& =(\mathrm{N} \vee(\mathrm{a}]) \wedge(\mathrm{N} \vee(\mathrm{~b}]) \wedge(\mathrm{a}] \vee(\mathrm{b}]) \text { and } \\
& (\mathrm{N} \vee[\mathrm{c})) \vee(\mathrm{N} \vee[\mathrm{~d})) \vee([\mathrm{c}) \vee[\mathrm{d})) \\
& =(\mathrm{N} \vee[\mathrm{c})) \vee(\mathrm{N} \vee[\mathrm{~d})) \vee([\mathrm{c}) \vee[\mathrm{d})) \\
& \quad \text { provided } \mathrm{N} \cap(\mathrm{a}] \neq \phi, \mathrm{N} \cap(\mathrm{~b}] \neq \phi \text { and } \mathrm{N} \cap[\mathrm{c}) \neq \phi, \mathrm{N} \cap(\mathrm{~d}] \neq \phi .
\end{aligned}
$$

Proof: Suppose $N$ is neutral. Then, $(\mathrm{N} \vee(\mathrm{a}]) \wedge(\mathrm{N} \vee(\mathrm{b}]) \wedge((\mathrm{a}] \vee(\mathrm{b}])$

$$
\begin{aligned}
& =(\mathrm{N} \vee(\mathrm{a}]) \wedge[(((\mathrm{a}] \vee(\mathrm{b}]) \wedge \mathrm{N}) \vee((\mathrm{a}] \vee(\mathrm{b}]) \wedge(\mathrm{b}])] \text { as } \mathrm{N} \text { is standard. } \\
& =(\mathrm{N} \vee(\mathrm{a}]) \wedge[(\mathrm{N} \wedge(\mathrm{a}]) \vee(\mathrm{N} \wedge(\mathrm{~b}]) \vee(\mathrm{b}]] \\
& =(\mathrm{N} \vee(\mathrm{a})] \wedge[(\mathrm{N} \wedge(\mathrm{a}]) \vee(\mathrm{b}]] \\
& =[((\mathrm{N} \wedge(\mathrm{a}]) \vee(\mathrm{b}]) \wedge \mathrm{N}] \vee[((\mathrm{N} \wedge(\mathrm{a}]) \vee(\mathrm{b}]) \wedge(\mathrm{a}]] \\
& =[(\mathrm{N} \wedge(\mathrm{a}]) \vee(\mathrm{N} \wedge(\mathrm{~b}])] \vee[((\mathrm{N} \wedge(\mathrm{a}]) \vee(\mathrm{b}]) \wedge(\mathrm{a}]] .
\end{aligned}
$$

Now $(((\mathrm{N}] \wedge(\mathrm{a}]) \vee(\mathrm{b}]) \wedge(\mathrm{a}] \subseteq((\mathrm{N}] \vee(\mathrm{b}]) \wedge(\mathrm{a}]$
But $((\mathrm{N}] \vee(\mathrm{b}]) \wedge(\mathrm{a}]=(\mathrm{N} \vee(\mathrm{b}]) \wedge(\mathrm{a}]=((\mathrm{a}] \wedge \mathrm{N}) \vee((\mathrm{a}] \wedge(\mathrm{b}])$
Thus, $((N \wedge(a]) \vee(b]) \wedge(a] \subseteq((a] \wedge N) \vee((a] \wedge(b])$.
Hence, $((N \vee(a]) \wedge(N \vee(b]) \wedge((a] \vee(b])$

$$
\begin{aligned}
& \subseteq(\mathrm{N} \wedge(\mathrm{a}]) \vee(\mathrm{N} \wedge(\mathrm{~b}]) \vee(\mathrm{N} \wedge(\mathrm{a}]) \vee((\mathrm{a}] \wedge(\mathrm{b}]) \\
& =(\mathrm{N} \wedge(\mathrm{a}]) \vee(\mathrm{N} \wedge(\mathrm{~b}]) \vee(\mathrm{a}] \wedge(\mathrm{b}]) .
\end{aligned}
$$

The reverse inclusion is trivial.
Therefore,
$(\mathrm{N} \wedge(\mathrm{a}]) \vee(\mathrm{N} \wedge(\mathrm{b}]) \vee((\mathrm{a}] \wedge(\mathrm{b}])=(\mathrm{N} \vee(\mathrm{a}]) \wedge(\mathrm{N} \vee(\mathrm{b}]) \wedge((\mathrm{a}] \vee(\mathrm{b}])$.
A dual proof of above shows that,
$(N \vee[c)) \vee(N \vee[d)) \vee([c) \vee[d))=(N \vee[c)) \vee(N \vee[d)) \vee([c) \vee[d))$

Conversely, suppose the given conditions hold. We have to show that N is neutral. Suppose $(\mathrm{x}] \subseteq(\mathrm{N}]$ and $\mathrm{N} \cap(\mathrm{y}] \neq \phi$ Put $(\mathrm{x}]=(\mathrm{a}]$ and $(\mathrm{y}]=$ (b] in the first condition. Then we get,

$$
\begin{aligned}
& N \wedge((x] \vee(y])=(N] \wedge((x] \vee(y])=((N] \vee(x]) \wedge((N] \vee(y]) \wedge((x] \vee(y]) \\
& =(N \vee(x]) \wedge(N \vee(y]) \wedge((x] \vee(y])=(N \wedge(x]) \vee(N \wedge(y]) \vee((x] \wedge(y]) \\
& =((N] \wedge(x]) \vee((N] \wedge(y]) \vee((x] \wedge(y])=(x] \vee((N] \wedge(y]) \vee((x] \wedge(y]) \\
& =(x] \vee((N] \wedge(y])=(x] \vee(N \wedge(y]) .
\end{aligned}
$$

Then it is not hard to prove that for any ideals, I and J with I $\subseteq$ ( N ] and $\mathrm{N} \cap \mathrm{J} \neq \phi, \mathrm{N} \wedge(\mathrm{I} \vee \mathrm{J})=(\mathrm{N} \wedge \mathrm{I}) \vee(\mathrm{N} \wedge \mathrm{J})$.

Hence for any ( x$],(\mathrm{y}]$ with $\mathrm{N} \cap(\mathrm{x}] \neq \phi, \mathrm{N} \cap(\mathrm{y}] \neq \phi$,
$\mathrm{N} \wedge((\mathrm{x}] \vee(\mathrm{y}])=(\mathrm{N}] \wedge((\mathrm{x}] \vee(\mathrm{y}])=$
$(\mathrm{N}] \wedge((\mathrm{N}] \vee(\mathrm{x}]) \wedge((\mathrm{N}] \vee(\mathrm{y}]) \wedge((\mathrm{x}] \vee(\mathrm{y}])$
$=(\mathrm{N}] \wedge[((\mathrm{N}] \vee(\mathrm{x}]) \wedge((\mathrm{N}] \vee(\mathrm{y}]) \wedge((\mathrm{x}] \vee(\mathrm{y}])]$
$=(N] \wedge[(N \vee(x]) \wedge(N \vee(y]) \wedge((x] \vee(y])]$
$=(\mathrm{N}] \wedge[(\mathrm{N} \wedge(\mathrm{x}]) \vee(\mathrm{N} \wedge(\mathrm{y}]) \wedge((\mathrm{x}] \wedge(\mathrm{y}])]$ (by the first condition)
$=\mathrm{N} \wedge[((\mathrm{N} \wedge(\mathrm{x}]) \vee(\mathrm{N} \wedge(\mathrm{y}]) \vee((\mathrm{x}] \wedge(\mathrm{y}])]$
Since, $(\mathrm{N} \wedge(\mathrm{x}]) \vee(\mathrm{N} \wedge(\mathrm{y}])$ is an ideal contained in ( N$]$, so by above part of the proof $\mathrm{N} \wedge((\mathrm{x}] \vee(\mathrm{y}])=((\mathrm{N} \wedge \mathrm{x}]) \vee(\mathrm{N} \wedge(\mathrm{y}])) \vee(\mathrm{N} \wedge((\mathrm{x}] \wedge(\mathrm{y}]))$

$$
=(\mathrm{N} \wedge(\mathrm{x}]) \vee(\mathrm{N} \wedge((\mathrm{y}]) .
$$

A dual proof shows that, $N \vee([c) \vee[d))=(N \vee[c)) \vee(N \vee[d))$
with $\mathrm{N} \cap[\mathrm{c}) \neq \phi$ and $\mathrm{N} \cap[\mathrm{d}) \neq \phi$.
Now we show the standardness of N .
Observe that for all $\mathrm{x}, \mathrm{y} \in \mathrm{L}$ with $\mathrm{N} \cap(\mathrm{x}] \neq \phi, \mathrm{N} \cap(\mathrm{y}] \neq \phi$ (Then of course $\mathrm{N} \cap((\mathrm{x}] \wedge(\mathrm{y}]) \neq \phi)$. Then, $(\mathrm{x}] \wedge[((\mathrm{x}] \wedge(\mathrm{y}]) \vee \mathrm{N}]$

$$
\begin{aligned}
& =((x] \vee((x] \wedge(y])) \wedge((x] \vee N) \wedge(((x] \wedge(y]) \vee N) \\
& =((x] \wedge((x] \wedge(y])) \vee((x] \wedge N) \vee((x] \wedge(y]) \wedge N) \text { (by first condition) }
\end{aligned}
$$

$$
=((x] \wedge(y]) \vee((x] \wedge N) \vee(((x] \wedge(y]) \wedge N)=((x] \wedge(y]) \vee((x] \wedge N)
$$

Thus, for $\mathrm{a}, \mathrm{b} \in \mathrm{L}$ with $\mathrm{N} \cap(\mathrm{a}] \neq \phi$, and $\mathrm{N} \cap(\mathrm{b}] \neq \phi$, we get
$(\mathrm{a}] \wedge((\mathrm{b}] \vee \mathrm{N})=[(\mathrm{a}] \wedge((\mathrm{a}] \vee \mathrm{N}) \wedge((\mathrm{a}]) \vee(\mathrm{b}])] \wedge((\mathrm{b}] \vee \mathrm{N})$
$=(a] \wedge[((a] \vee N) \wedge((a] \vee(b]) \wedge((b] \vee N)]$
$=(a] \wedge[((a] \wedge N) \vee((a] \wedge(b]) \vee((b] \wedge N)]$
$\subseteq(\mathrm{a}] \wedge(((\mathrm{a}] \wedge(\mathrm{b}]) \vee \mathrm{N})=((\mathrm{a}] \wedge(\mathrm{b}]) \vee((\mathrm{a}] \wedge \mathrm{N})($ by above $)$
$\subseteq(\mathrm{a}] \wedge((\mathrm{b}] \vee \mathrm{N})$.
Therefore, $(\mathrm{a}] \wedge(\mathrm{b}] \vee \mathrm{N})=(\mathrm{a} \wedge \mathrm{b}] \vee((\mathrm{a}] \wedge(\mathrm{N}])$.
A dual proof shows that $[\mathrm{c}) \vee(\mathrm{N} \wedge[\mathrm{d}))=([\mathrm{c}) \wedge[\mathrm{N})) \vee[\mathrm{c} \vee \mathrm{d})$
for $\mathrm{c}, \mathrm{d} \in \mathrm{L}$ with $\mathrm{N} \cap[\mathrm{d}) \neq \phi$.Therefore by Hafizur Rahman R.M. [6, Theorem 8], N is standard and hence by Theorem $2, \mathrm{~N}$ is neutral.
Lemma 5. Let S and T be two neutral sublattices. Then $\mathrm{S} \cap \mathrm{T}$ is either a neutral sublattice or it is empty.

Proof: We assume that $\mathrm{S} \cap \mathrm{T} \neq \phi$. By the definition of neutral sublattice both S and T are standard sublattices and hence by Hafizur Rahman R.M. [6] , $\mathrm{S} \cap \mathrm{T}$ is standard.

Let $\mathrm{a}, \mathrm{b} \in \mathrm{L}$ with $(\mathrm{S} \cap \mathrm{T}) \cap(\mathrm{a}] \neq \phi$ and $(\mathrm{S} \cap \mathrm{T}) \cap(\mathrm{b}] \neq \phi$. Then obviously $\mathrm{S} \cap(\mathrm{a}] \neq \phi, \mathrm{S} \cap(\mathrm{b}] \neq \phi, \mathrm{T} \cap(\mathrm{a}] \neq \phi$ and $\mathrm{T} \cap(\mathrm{b}] \neq \phi$.

Now $(\mathrm{S} \cap \mathrm{T}) \wedge((\mathrm{a}] \vee(\mathrm{b}])=(\mathrm{S} \wedge \mathrm{T}) \wedge((\mathrm{a}] \vee(\mathrm{b}])$ as $\mathrm{S} \cap \mathrm{T} \neq \varphi$ $=(\mathrm{S} \wedge \mathrm{T}] \wedge((\mathrm{a}] \vee(\mathrm{b}])=((\mathrm{S}] \wedge(\mathrm{T}]) \wedge((\mathrm{a}] \vee(\mathrm{b}])=(\mathrm{S}] \wedge((\mathrm{T}] \wedge((\mathrm{a}] \vee(\mathrm{b}]))$ $=(\mathrm{S}] \wedge(\mathrm{T} \wedge((\mathrm{a}] \vee(\mathrm{b}]))=(\mathrm{S}] \wedge((\mathrm{T} \wedge(\mathrm{a}]) \vee(\mathrm{T} \wedge(\mathrm{b}]))$ as T is neutral.
$=(\mathrm{S}] \wedge(((\mathrm{T}] \wedge(\mathrm{a}]) \vee((\mathrm{T}] \wedge(\mathrm{b}]))=\mathrm{S} \wedge((\mathrm{T}] \wedge(\mathrm{a}]) \vee((\mathrm{T}] \wedge(\mathrm{b}]))$
$=(\mathrm{S} \wedge((\mathrm{T}] \wedge(\mathrm{a}])) \vee(\mathrm{S} \wedge((\mathrm{T}] \wedge(\mathrm{b}]))$ as S is neutral.
$=((\mathrm{S}] \wedge((\mathrm{T}] \wedge(\mathrm{a}]) \vee((\mathrm{S}] \wedge((\mathrm{T}] \wedge(\mathrm{b}]))$
$=((S \wedge T] \wedge(a]) \vee((S \wedge T] \wedge(b])=((S \wedge T) \wedge(a]) \vee((S \wedge T) \wedge(b])$
$($ as $\mathrm{S} \cap \mathrm{T} \neq \phi)=((\mathrm{S} \cap \mathrm{T}) \wedge(\mathrm{a}]) \vee((\mathrm{S} \cap \mathrm{T}) \wedge(\mathrm{b}])$

A dual proof shows that,
$(S \cap T) \vee([c) \vee[d))=((S \cap T) \vee[c)) \vee((S \cap T) \vee[d))$
provided $(\mathrm{S} \cap \mathrm{T}) \cap[\mathrm{c}) \neq \phi,(\mathrm{S} \cap \mathrm{T}) \cap[\mathrm{d}) \neq \phi$.
Hence $\mathrm{S} \cap \mathrm{T}$ is a neutral sublattice.
Lemma 6. The meet of a neutral ideal and a neutral dual ideal is a neutral sublattice provided their intersection is non-empty.

Proof: By Theorem1 and by the duality, all neutral ideals and neutral dual ideals are neutral sublattices. Then Lemma 5 completes the proof.

To prove the next theorem we need the following lemma, which is due to Cornish W.H. J. [7].

Lemma 7. Let $\mathrm{L}_{1}$ be a sublattice of a lattice L . Suppose $\mathrm{I}_{1}$ is an ideal in $\mathrm{L}_{1}$. Then there exists an ideal I of L such that $\mathrm{I}_{1}=\mathrm{I} \cap \mathrm{L}_{1}$.

Theorem 8. Let N be a neutral sublattice of a lattice L and I be any convex sublattice, then $\mathrm{N} \cap \mathrm{I}$ is neutral in I Provided $\mathrm{N} \cap \mathrm{I} \neq \phi$.

Proof: If N is a neutral sublattice then by definition N is standard and hence by Fried E., and Schmidt E.T. [4, Lemma 2] $\mathrm{N} \cap \mathrm{I}$ is a standard sublattice of the lattice I.

Now let (a $]_{\mathrm{I}}$ and (b] $]_{\mathrm{I}}$ be two ideals of the sublattice I with (a) $]_{I} \cap \mathrm{~N} \neq \phi$, (b] $]_{\mathrm{I}} \cap \mathrm{N} \neq \phi$. Then by Lemma 7, there exist ideals P and Q of L such that $(\mathrm{a}]_{\mathrm{I}}=\mathrm{P} \cap \mathrm{I}$ and $(\mathrm{b}]_{\mathrm{I}}=\mathrm{Q} \cap \mathrm{I}$.

Now, $(\mathrm{N} \cap \mathrm{I}) \wedge\left((\mathrm{a}]_{\mathrm{I}} \vee(\mathrm{b}]_{\mathrm{I}}\right) .=(\mathrm{N} \cap \mathrm{I}) \wedge((\mathrm{P} \cap \mathrm{I}) \vee(\mathrm{Q} \cap \mathrm{I}))$.
Since P, Q are ideals, so by routine calculation,

$$
\begin{aligned}
& (\mathrm{N} \wedge \mathrm{I}) \wedge\left((\mathrm{a}]_{\mathrm{I}} \vee(\mathrm{~b}]_{\mathrm{I}}\right)=(\mathrm{N}] \wedge(\mathrm{I}] \wedge[(\mathrm{P} \wedge(\mathrm{I}]) \vee(\mathrm{Q} \wedge(\mathrm{I}])] \\
& =(\mathrm{I}] \wedge[(\mathrm{N}] \wedge((\mathrm{P} \wedge(\mathrm{I}]) \vee(\mathrm{Q} \wedge(\mathrm{I}]))]=(\mathrm{I}] \wedge[\mathrm{N} \wedge((\mathrm{P} \wedge \mathrm{I}) \vee(\mathrm{Q} \wedge \mathrm{I}))] \\
& =(\mathrm{I}] \wedge[(\mathrm{N} \wedge(\mathrm{P} \wedge \mathrm{I})) \vee(\mathrm{N} \wedge(\mathrm{Q} \wedge \mathrm{I}))](\text { as } \mathrm{N} \text { is neutral in } \mathrm{L}) \\
& =(\mathrm{I}] \wedge[((\mathrm{N}] \wedge(\mathrm{P} \wedge(\mathrm{I}]) \vee((\mathrm{N}] \wedge \mathrm{Q} \wedge(\mathrm{I}])] .
\end{aligned}
$$

$=((\mathrm{N}] \wedge \mathrm{P} \wedge(\mathrm{I}]) \vee((\mathrm{N}] \wedge \mathrm{Q} \wedge(\mathrm{I}])=((\mathrm{N} \wedge \mathrm{I}] \wedge \mathrm{P} \wedge(\mathrm{I}]) \vee((\mathrm{N} \wedge \mathrm{I}] \wedge \mathrm{Q}$
$\wedge(\mathrm{I}])$
$=((\mathrm{N} \wedge \mathrm{I}) \wedge(\mathrm{P} \cap \mathrm{I})) \vee((\mathrm{N} \wedge \mathrm{I}) \wedge(\mathrm{Q} \cap \mathrm{I}))$
$=\left((\mathrm{N} \cap \mathrm{I}) \wedge(\mathrm{a}]_{\mathrm{I}}\right) \vee\left((\mathrm{N} \cap \mathrm{I}) \wedge(\mathrm{b}]_{\mathrm{I}}\right)$.
Similarly for dual ideals $[a)_{\mathrm{I}}$ and $[\mathrm{b})_{\mathrm{I}}$ of sublattice I with $[\mathrm{a})_{\mathrm{I}} \cap \mathrm{N} \neq \phi$, $[\mathrm{b})_{\mathrm{I}} \cap \mathrm{N} \neq \phi$, by a dual proof of above we can show that,
$(N \cap I) \vee\left([a)_{I} \vee[b)_{I}\right)=\left((N \cap I) \vee[a)_{I}\right) \vee\left((N \cap I) \vee[b)_{I}\right)$.
Therefore by Theorem 2, $\mathrm{N} \cap \mathrm{I}$ is neutral in I .
We conclude this paper with the following result.
Theorem 9. In a modular lattice L, every standard sublattice is neutral.
Proof: Let N be a standard sublattice of a modular lattice L. We need to show that N is neutral.As the lattice L is modular then both $\mathrm{I}(\mathrm{L})$ and $\mathrm{D}(\mathrm{L})$ are modular which implies that every elements in $I(L)$ and $D(L)$ are modular. Let $\mathrm{a}, \mathrm{b} \in \mathrm{L}$ with $\mathrm{N} \cap(\mathrm{a}] \neq \phi$ and $\mathrm{N} \cap(\mathrm{b}] \neq \phi$.

Then $(\mathrm{N} \wedge(\mathrm{a}]) \vee(\mathrm{N} \wedge(\mathrm{b}])=((\mathrm{N}] \wedge(\mathrm{a}]) \vee((\mathrm{N}] \wedge(\mathrm{b}])$

$$
\begin{aligned}
& =(\mathrm{N}] \wedge[(\mathrm{a}] \vee((\mathrm{N}] \wedge(\mathrm{b}])](\text { as } \mathrm{I}(\mathrm{~L}) \text { is modular }) . \\
& =(\mathrm{N}] \wedge[(\mathrm{a}] \vee((\mathrm{a}] \wedge(\mathrm{b}]) \vee((\mathrm{N}] \wedge(\mathrm{b}])] \\
& =(\mathrm{N}] \wedge[(\mathrm{a}] \vee((\mathrm{a}] \wedge(\mathrm{b}]) \vee(\mathrm{N} \wedge(\mathrm{~b}]))]=(\mathrm{N}] \wedge[(\mathrm{a}] \vee((\mathrm{b}] \wedge(\mathrm{N} \vee
\end{aligned}
$$

(a])], as N is a standard sublattice.
$=(\mathrm{N}] \wedge[(\mathrm{a}] \vee((\mathrm{b}] \wedge((\mathrm{N}] \vee(\mathrm{a}]))]=(\mathrm{N}] \wedge[((\mathrm{a}] \vee(\mathrm{N}]) \wedge((\mathrm{a}] \vee(\mathrm{b}])]$
( by the modularity of $\mathrm{I}(\mathrm{L}))=(\mathrm{N}] \wedge((\mathrm{a}] \vee(\mathrm{N}]) \wedge((\mathrm{a}] \vee(\mathrm{b}])$

$$
=(\mathrm{N}] \wedge((\mathrm{a}] \vee(\mathrm{b}])=\mathrm{N} \wedge((\mathrm{a}] \vee(\mathrm{b}])
$$

By using the modularity of $\mathrm{D}(\mathrm{L})$ a dual proof of above gives that $\mathrm{N} \vee([\mathrm{c}) \vee[\mathrm{d}))=(\mathrm{N} \vee[\mathrm{c})) \vee(\mathrm{N} \vee[\mathrm{d}))$ provided $\mathrm{N} \cap(\mathrm{c}) \neq \phi$ and $\mathrm{N} \cap(\mathrm{d}) \neq \phi$.

Therefore by Theorem 2, N is a neutral sublattice.

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