

BLOOD FLOW THROUGH A FLEXIBLE ARTERY IN PRESENCE OF STENOSIS - A MATHEMATICAL STUDY

By

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Abstract:

The mathematical analysis presents the study of heat transfer and magneto hydrodynamic effects on pulsatile flow of blood through geometrically irregular arterial system, and its effects on cardiovascular disorder and arterial diseases. Considering the influence of magnetic field on the stenotic artery, the effect of transverse magnetic field and multi-stenosis on the blood flow in blood vessels is studied theoretically. The blood flow is considered to be axi-symmetric with an outline of the irregular stenosis obtained from a three-dimensional casting of mild stenosed artery, so that the physical problem becomes more realistic from the physiological point of view. The MARKER AND CELL (MAC) and SUCCESSIVE –OVER-RELAXATION (SOR) methods are respectively used to solve the governing unsteady magneto-hydrodynamic equations and pressure-Poisson equation numerically. The present observations certainly have some clinical implications relating to magneto-therapy. It may help reducing the complex flow separations zones causing flow disorder and leading to the formation and propagation of the arterial diseases and cardiovascular disorders.

Keywords and phrases : stenosis, blood flow, heat transfer, magnetic field.

বিমূর্ত সার (Bengali version of the Abstract)

জ্যামিতিক বিষম ধমনী তত্ত্বের মধ্য দিয়ে রক্তের স্পন্দন প্রবাহের তাপ স্থানান্তরণ ও চৌম্বক - উদ্গতিকী এবং হৃদ-সংবহন নালিকার গোলযোগ ও ধমনীর রোগের উপর ইহার প্রভাবের অনুসন্ধানটির গাণিতিক বিশ্লেষণ উপস্থাপন করা হয়েছে। সংকুচিত ধমনীতে চৌম্বক ক্ষেত্রের প্রভাবকে বিবেচনা করে রক্ত নালীতে রক্ত প্রবাহে বহু-সংকোচনের এবং তির্যক চৌম্বক ক্ষেত্রের প্রভাবকে তত্ত্বগতভাবে অনুসন্ধান করা হয়েছে।

শরীর বৃত্তীয় দৃষ্টিকোণ থেকে বাস্তব সমস্যাটি যেন আরো বেশি বাস্তব সম্মত হয় সে জন্য রক্ত প্রবাহকে স্বল্প সংকুচিত ধমনীর ত্রি-মাত্রিক ছাঁচ নির্গিত বিষম সংকোচনের পরিলেখের সহিত অক্ষীয় প্রতিসম হিসাবে ধরা হয়েছে। নিয়ন্ত্রক পরবর্তী চৌম্বক - উদ্গতিকী সমীকরণ এবং চাপ-পোয়াঁস সমীকরণের সাংখ্যমানে সমাধান নির্ণয় করা হয়েছে যথাক্রমে মারকার এবং সেল (MARKER AND CELL) এবং সাকসেসিভ-ওভার রিলাক্সেশান (SUCCESSIVE – OVER-RELAXATION) পদ্ধতির মাধ্যমে। চৌম্বক-থেরাপী (Magnetic Therapy) সম্পর্কিত কিছু ক্লিনিকল উপযোগিতা বর্তমান পর্যবেক্ষণ থেকে অবশ্যই পাওয়া যায়।

প্রবাহ গোলযোগ এবং ধমনীর রোগ হৃদ-সংবহন নালিকার গোলযোগ গঠন ও বিস্তারনের কারণ এই পৃথক পৃথক জটিল প্রবাহকে ইহা প্রসারিত করতে পারে।

1. Introduction:

Among all the fatal diseases of the human body, circulatory disorders are a still a major cause of death. A systematic study on the rheological and hemodynamic properties of blood and blood-flow could play a significant role in the basic understanding, diagnosis and treatment of many cardiovascular, cerebro-vascular and arterial diseases. It is well known that stenosis (narrowing in the local lumen in the artery) is responsible for many cardiovascular diseases. When the degree of narrowing becomes significant enough to impede the flow of blood from the left ventricle to the arteries, heart problems develop. While the exact mechanism of the formation of stenosis in a conclusive manner remains somewhat unclear from the standpoint of physiology and pathology, the abnormal deposition of various substances like cholesterol, fat on the

endothelium of the arterial wall, and proliferation of connective tissues accelerate the growth of the disease. Plaques are thereby formed and lead to serious circulatory disorders. Plaque forms when cholesterol, fat and other substances build up in the inner lining of the artery. This process is called *Carotid circulatory disorders*. It greatly disturbs the normal blood flow leading to malfunction of the hemodynamic system (the flow of blood) and cardiovascular system. Carotid artery stenosis is a major risk factor for Ischemic stroke (most common form of stroke usually caused by blood-clot plugging an artery).

From biomechanics point of view, the laminar flow of blood in different arteries under certain conditions behaves like a visco-elastic fluid motion [6]. Also the blood flow effects the thermal response of living tissues which depends on the geometric structure of artery (tapered artery), and flow variation of blood due to stenosis. The main complication in describing the axisymmetric blood flow leads to develop a constitute model for unsteady non-Newtonian flow through multistenosed tapered arteries in presence of a magnetic field [1]. It has been established that once a mild stenosis is developed, the resulting flow disorder further influences the development of the disease and arterial deformity, and change the regional blood rheology [8, 9]. Steady flow through an axisymmetric stenosis has been investigated extensively by Smith using an analytical approach indicating that the flow patterns strongly depend on the geometry of the stenosis and the upstream Reynolds number (n). In recent years some studies (Katiyar and Basavarajappa, 2002; Kinouchi *et al.*, 1996; Sud and Sekhon, 1989; Tashtoush and Magableh, 2008; Tzirtzilakis, 2005) have been reported on the analysis of blood flow through single arteries in the presence of externally applied magnetic field.

It is assumed that the arterial segment is a cylindrical tube with time dependent multi-stenosis. In the proposed investigation an attempt will be made to deal with a problem, considering hemodynamic and cardiovascular disorders due to non-Newtonian flow of blood in multistenosed arteries [1, 2]. The present investigation has been devoted to the problem of blood flow through a stenosed segment of an artery where the rheology of blood is described by Herschel–Bulkley model and Bringham plastic fluid model. The dispensability of an arterial wall has been accounted for based on local fluid mechanics. Then an appropriate finite difference technique will be adopted to solve the unsteady non-Newtonian flow of blood with different boundary conditions in cylindrical co-ordinate system. A quantitative analysis will be taken based on numerical computations by taking the different values of material constants and other parameters [10, 12]. The variation of skin-friction with axial distance and impedance in the region of the stenosis are presented graphically with respect to velocity of flow of blood in arterial segment. The qualitative and quantitative changes in the skin-friction, the flow resistance and the volumetric flow rate at different stages of the growth of the stenosis have also been presented.

Nomenclature	
τ	shear stress
τ_H	yield stress
τ_R	skin-friction
$\bar{\tau}$	non-dimensional skin-friction
δ	stenosis height
λ	flow resistance
$\bar{\lambda}$	non-dimensional flow resistance
Q	volumetric flow rate
r	radial coordinate
z	axial coordinate
u	Axial average velocity of flow
R_0	radius of the artery
R(z)	radius of the artery at stenosed portion
L	half-length of the artery
L_0	half-length of the stenosis
p	pressure
k	viscosity coefficient
n	fluid index

2. The problem and its solution:

Let us consider steady laminar fully developed one-dimensional flow of blood obeying the constitutive equation given by Herschel–Bulkley model through a stenosed artery.

The equation governing the flow of blood is taken in the form

$$-\frac{dp}{dz} = \frac{1}{r} \frac{d(r\tau)}{dr} \quad (1)$$

in which τ represents the shear stress of blood considered as Herschel–Bulkley fluid and p the pressure at any point. The constitutive equation may be put as

$$\frac{du}{dr} = f(\tau) = \frac{1}{k}(\tau - \tau_H); \tau \geq \tau_H = 0; \tau \geq \tau_H \quad (2)$$

where u stands for the axial velocity of blood and τ_H is the yield shear stress and k, n are parameters which represent non-Newtonian effects.

Let us consider a bell-shaped stenosis geometry given by

$$R(z) = R_0 \left[1 - \frac{\delta}{R_0} \exp\left(-\frac{m^2 \varepsilon^2 z^2}{R_0^2}\right) \right] \quad (3)$$

Where R_0 stands for the radius of the arterial segment outside the stenosis, $R(z)$ is the radius of the stenosed portion of the arterial segment under consideration at a longitudinal distance z from the left-end of the segment; δ is the depth of the stenosis at the throat and m is a parametric constant; ε characterizes the relative length of the constriction, defined as the ratio of the radius to half length of stenosis, i.e. $\varepsilon = \frac{R}{L_0}$.

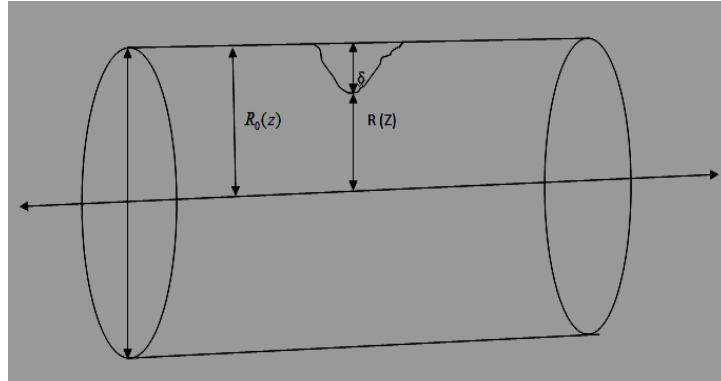


Figure: 1

(Picture of arterial stenosed segments & stenosis throat)

Considering the stenosis geometry to be of the form (cf. Fig. 1)

$$\frac{R(z)}{R_0} = 1 - a e^{-b z^2} \quad (4)$$

With $a = \frac{\delta}{R_0}$ and $b = \frac{m^2 \varepsilon^2}{R_0^2}$

Equations (1) and (2) are to be solved subject to the boundary conditions

$$u=0 \text{ at } r=R(z) \quad (\text{no slip condition}) \quad (5)$$

$$\tau \text{ is finite at } r=0 \quad (\text{regularity condition}) \quad (6)$$

Integrating Equation (1) and using (6) we get

$$\tau = -\frac{r}{2} \frac{dp}{dz} \quad (7)$$

The skin-friction τ_R is given by

$$\tau_R = -\frac{R}{2} \frac{dp}{dz} \quad (8)$$

Where $R=R(z)$

The volumetric flow rate Q is given by the Rabinowitsch equation

$$Q = \frac{\pi R^3}{\tau_R^3} \int_0^{\tau_R} \tau^2 f(\tau) d\tau \quad (9)$$

Where τ and τ_R given by the equations (7) and (8) respectively.

Therefore substituting the value of $f(\tau)$ from equation (2) we get.

$$Q = \frac{\pi R^3}{\tau_R^3} \int_0^{\tau_R} \tau^2 \frac{1}{k} (\tau - \tau_H)^n d\tau \quad (10)$$

(Where n =fluid index parameter)

$$= \frac{\pi R^3 \tau_R^n}{k(n+3)} \left(1 - \frac{\tau_H}{\tau_R}\right)^{n+1} \left[1 + \left(\frac{2}{n+2}\right) \frac{\tau_H}{\tau_R} + \frac{2}{(n+1)(n+2)} \left(\frac{\tau_H}{\tau_R}\right)^2\right] \quad (11)$$

When $\left(\frac{\tau_H}{\tau_R}\right) \leq 1$ the above equation reduces to

$$Q = \frac{\pi R^3}{k(n+3)} \left\{ \tau_R - \left(\frac{n+3}{n+2}\right) \tau_H \right\}^n \quad (12)$$

Again assuming that the flowing blood is representing a Newtonian fluid

$$\text{Then } \frac{dp}{dz} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \mu \frac{\partial u}{\partial r} \right) \quad (13)$$

Where u is the velocity of flow.

Now the volumetric flow flux Q is thus calculated as

$$Q = 2\pi \int_0^{R(z)} r u dr$$

$$Q = \pi \{R(z)\}^2 u$$
(14)

Now from (12) and (14) we have

$$Q = \frac{\pi R^3}{k(n+3)} \left[\tau_R - \left(\frac{n+3}{n+2} \right) \tau_H \right]^n = \pi \{R(z)\}^2 u$$
(15)

$$u = \frac{R^3 \left\{ \tau_R - \left(\frac{n+3}{n+2} \right) \tau_H \right\}^n}{k(n+3) R^2(z)}$$
(16)

Again resistance to flow λ defined by

$$\lambda = \frac{P_1 - P_2}{Q}$$

$$\text{Using } \tau_R = -\frac{R}{2} \frac{dp}{dz}$$

Where $R=R(z)$ in equation

$$Q = \frac{\pi R^3}{k(n+3)} \left[-\frac{R}{2} \frac{dp}{dz} - \left(\frac{n+3}{n+2} \right) \tau_H \right]^n$$
(17)

$$-\frac{dp}{dz} = \left\{ \frac{2^n k Q (n+3)}{\pi R^{n+3}} \right\}^{\frac{1}{n}} + \frac{2(n+3)}{(n+2)} \frac{\tau_H}{R}$$
(18)

Now integrating equation (18) along the length of the artery and using the

Conditions that $P = P_1$ at $z = -L$ and $P = P_2$ at $z=L$ we obtain,

$$P_1 - P_2 = \left\{ \frac{2^n kQ(n+3)}{\pi R^{n+3}} \right\}^{\frac{1}{n}} \int_{z=-L}^{z=L} \frac{dz}{\left(\frac{R}{R_0}\right)^{\frac{3}{n}+1}} + \frac{2(n+3)}{(n+2)} \frac{\tau_H}{R_0} \int_{z=-L}^{z=L} \frac{dz}{\left(\frac{R}{R_0}\right)} \quad (19)$$

Thus the resistance of flow is defined by

$$\lambda = \frac{P_1 - P_2}{Q}$$

$$\lambda = \frac{\left\{ \frac{2^n kQ(n+3)}{\pi R_0^{n+3}} \right\}^{\frac{1}{n}} \int_{z=-L}^{z=L} \frac{dz}{\left(\frac{R}{R_0}\right)^{\frac{3}{n}+1}} + \frac{2(n+3)}{(n+2)} \frac{\tau_H}{R_0} \int_{z=-L}^{z=L} \frac{dz}{\left(\frac{R}{R_0}\right)}}{\left\{ \frac{2^n kQ(n+3)}{\pi R^{n+3}} \right\}^{\frac{1}{n}} + \frac{2(n+3)}{(n+2)} \frac{\tau_H}{R}} \quad (20)$$

In absence of any constriction the resistance to flow λ_N is defined by

$$\lambda_N = \frac{4(n+3)L}{QR_0} \left[\left\{ \frac{(n+3)^{1-n} kQ}{\pi R_0^3} \right\}^{1/n} + \frac{\tau_H}{(n+2)} \right] \quad (21)$$

Now in dimensionless form the flow resistance may be expressed as

$$\bar{\lambda} = \frac{\lambda}{\lambda_N} = \frac{\left\{ \frac{2^n kQ(n+3)}{\pi R_0^{n+3}} \right\}^{\frac{1}{n}} \int_{z=-L}^{z=L} \frac{dz}{\left(\frac{R}{R_0}\right)^{\frac{3}{n}+1}} + \frac{2(n+3)}{(n+2)} \frac{\tau_H}{R_0} \int_{z=-L}^{z=L} \frac{dz}{\left(\frac{R}{R_0}\right)}}{\frac{4(n+3)L}{R_0} \left[\left\{ \frac{(n+3)^{1-n} kQ}{\pi R_0^3} \right\}^{1/n} + \frac{\tau_H}{(n+2)} \right]} \quad (22)$$

Therefore substituting the value of $Q = \pi \{R(z)\}^2 u$ we have from (22),

$$\bar{\lambda} = \frac{\left\{ \frac{k(n+3)R^2u}{R_0^3} \right\}^{\frac{1}{n}} \int_{z=-L}^{z=L} \frac{dz}{(R/R_0)^{\frac{3}{n}+1}} + \frac{(n+3)}{(n+2)} \tau_H \int_{z=-L}^{z=L} \frac{dz}{(R/R_0)}}{\left[\left\{ \frac{(n+3)kL^n R^2u}{\pi R_0^3} \right\}^{1/n} + \frac{(n+3)\tau_H}{(n+2)} \right]}$$

$$\bar{\lambda} = \frac{\left\{ \frac{k(n+3)R^2u}{R_0^3} \right\}^{\frac{1}{n}} \left[(L-L_0) + \int_0^{L_0} \frac{dz}{(R/R_0)^{\frac{3}{n}+1}} \right] + \frac{(n+3)}{(n+2)} \tau_H \left[(L-L_0) + \int_0^{L_0} \frac{dz}{(R/R_0)} \right]}{\left[\left\{ \frac{(n+3)kL^n R^2u}{\pi R_0^3} \right\}^{1/n} + \frac{(n+3)\tau_H}{(n+2)} \right]} \quad (23)$$

Therefore from (3) we have, substituting the value of $\varepsilon = \frac{R_0}{L_0}$

$$\left(\frac{R(z)}{R_0} \right) = \left[1 - \left(\frac{\delta}{R_0} \right) e^{\left(-\frac{m^2 z^2}{L_0^2} \right)} \right] \quad (24)$$

In order to have an estimate of the quantitative effects of the various parameters involved in the analysis, it is necessary to evaluate the analytical results obtained for dimensionless resistance to flow, λ . It is based on area-axial average velocity of flow on constant tube diameter, where the constitutive coefficient $m=0.1260$ g/cm .s in Power law fluid model and $n=0.8$ (Power law fluid model), $u=(0.5,1,1.5,2,2.5,3)$, $\frac{\delta}{R_0}=(0.1,0.2,0.3,0.4,0.5)$, $L_0=1$ cm, $L=(1,2,5)$ cm $\tau_H=0.05$ then $k=4$ and when $\tau_H=0.10$ then $k=7$.

$\frac{\delta}{R_0}=0.1$		$\frac{\delta}{R_0}=0.2$		$\frac{\delta}{R_0}=0.3$		$\frac{\delta}{R_0}=0.4$		$\frac{\delta}{R_0}=0.5$	
Z	$\frac{R(z)}{R_0}$	z	$\frac{R(z)}{R_0}$	z	$\frac{R(z)}{R_0}$	z	$\frac{R(z)}{R_0}$	z	$\frac{R(z)}{R_0}$
-2.5	0.9093	-2.5	0.8188	-2.5	0.7283	-2.5	0.6377	-2.5	0.5472
-2.0	0.9061	-2.0	0.8123	-2.0	0.7184	-2.0	0.6246	-2.0	0.5307
-1.5	0.9034	-1.5	0.8070	-1.5	0.7105	-1.5	0.6140	-1.5	0.5175
-1.0	0.9015	-1.0	0.8031	-1.0	0.7047	-1.0	0.6063	-1.0	0.5078
-0.5	0.9003	-0.5	0.8007	-0.5	0.7011	-0.5	0.6015	-0.5	0.5019
0.0	0.9	0.0	0.8	0.0	0.7	0.0	0.6	0.0	0.5
0.5	0.9003	0.5	0.8007	0.5	0.7011	0.5	0.6015	0.5	0.5019
1.0	0.9015	1.0	0.8031	1.0	0.7047	1.0	0.6063	1.0	0.5078
1.5	0.9034	1.5	0.8070	1.5	0.7105	1.5	0.6140	1.5	0.5175
2.0	0.9061	2.0	0.8123	2.0	0.7184	2.0	0.6246	2.0	0.5307
2.5	0.9093	2.5	0.8188	2.5	0.7283	2.5	0.6377	2.5	0.5472
average	0.9037	average	0.8076	average	0.7115	average	0.6152	average	0.5191

With the data the graphical representation of different values of $\frac{R(z)}{R_0}$ for

different values of $(\frac{\delta}{R_0})$ is shown below:

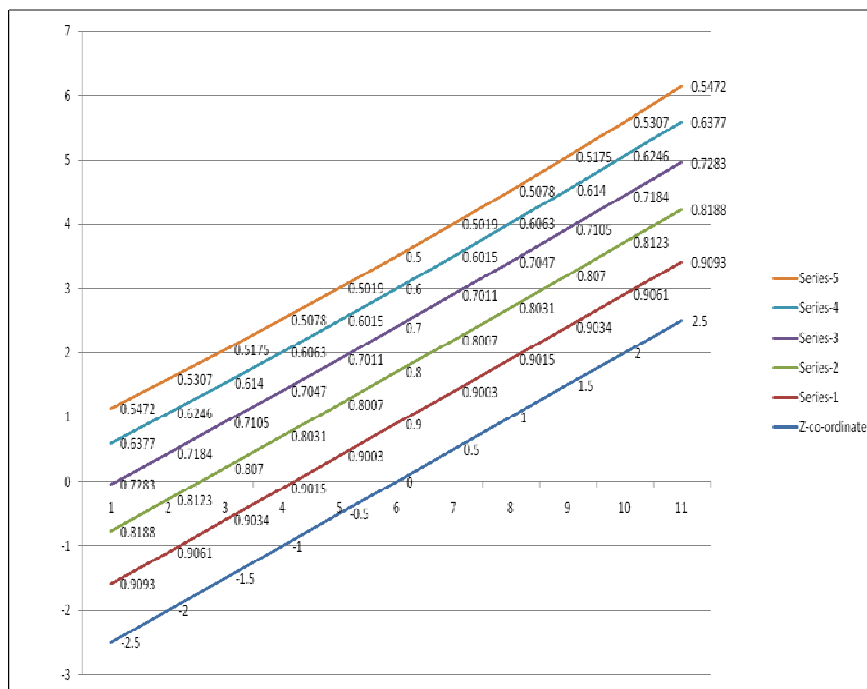


Figure: 2

(Series-1)-values of $\frac{R(z)}{R_0}$ when $\frac{\delta}{R_0}=0.1$, (series-2)-values of $\frac{R(z)}{R_0}$ when $\frac{\delta}{R_0}=0.2$, (series-3)-values of $\frac{R(z)}{R_0}$ when $\frac{\delta}{R_0}=0.3$, (Series-4)-values of $\frac{R(z)}{R_0}$ when $\frac{\delta}{R_0}=0.4$, (series-5)-values of $\frac{R(z)}{R_0}$ when $\frac{\delta}{R_0}=0.5$.

$\frac{R(z)}{R_0} = .9037$ when $\frac{\delta}{R_0} = 0.1$		$\frac{R(z)}{R_0} = .8076$ when $\frac{\delta}{R_0} = 0.2$		$\frac{R(z)}{R_0} = .5191$ when $\frac{\delta}{R_0} = 0.5$	
U	$\bar{\lambda}$	u	$\bar{\lambda}$	u	$\bar{\lambda}$
0.5	1.6141	0.5	2.7455	0.5	21.9644
1.0	1.6161	1.0	2.7535	1.0	22.2823
1.5	1.6167	1.5	2.7558	1.5	22.3759
2.0	1.6170	2.0	2.7569	2.0	22.4191
2.5	1.6171	2.5	2.7575	2.5	22.4434

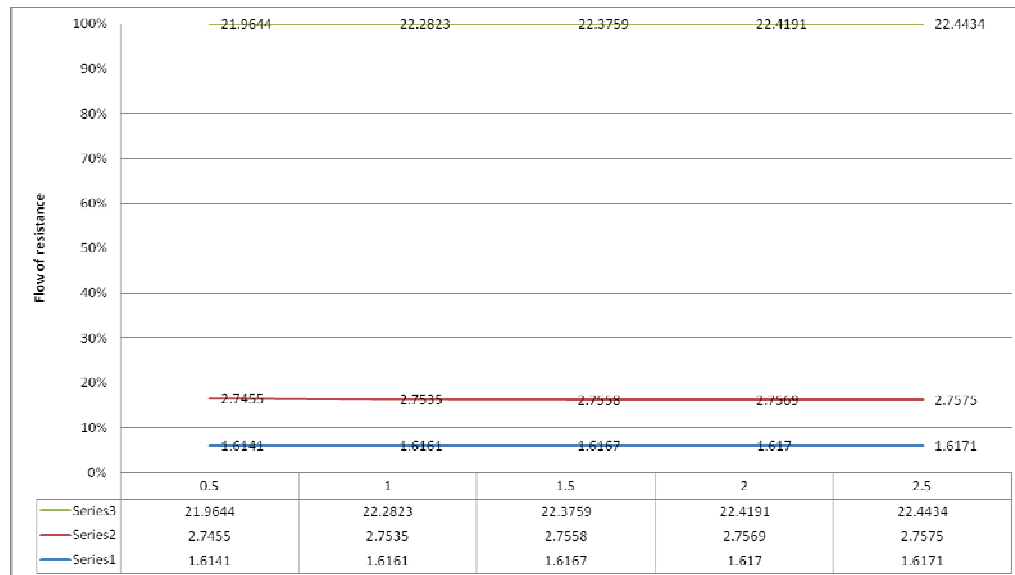


Figure: 3

3. Conclusion:

The variation of flow resistance in the stenosed portion of the artery with average flow velocity is exhibited in figures 1 and 2. For a given set of the values of different parameters and material constants, it is observed from the theoretical investigation that the flow resistance at the stenosed portion of the artery increases marginally with the increase of the average flow velocity. The corresponding results (flow resistance) are seen to increase with the decrease in the radius of the artery in the stenosed portion and the stenosis depth.

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