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SOME CHARACTERIZATIONS OF THE RADICAL OF GAMMA RINGS

By

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Abstract.

In this paper we have developed some properties of nilpotent ideals and radical of Γ-rings. At last we have prove that an external direct sum of finitely many matrix gamma rings over division gamma rings is a semi-simple Γ_n-ring.

Keywords and phrases : *gamma rings, nilpotent ideals, radical of gamma rings*

বিমূর্ত সার (*Bengali version of the Abstract*)

এই পত্রে শূন্য ফলদ আইডিয়েল (nilpotent ideals) এবং গামা রিং (Γ-rings) - এর মূলকের (radical) কিছু র্ধমের উন্নয়ন করেছি । পরিশেষে আমরা প্রমাণ করেছি যে গামা রিং - এর বিভিন্ন অংশের উপর সসীম বহু ম্যাট্রিক্স গামা রিং - এর বহিঃস্থ প্রত্যক্ষ যোগফল হচ্ছে অর্ধ - সরল Γ_n - রিং ।

1. Introduction

The concepts of a Γ-ring was first introduced by Nobusawa [7] in 1964. His concept is more general than a ring. Now a day, his Γ-ring is called a Γ-ring in the sense of Nobusawa.

W. E. Barnes [2] gave a definition of a Γ-ring which is more general. He introduced the notation of Γ- homomorphisms, Prime and Primary ideals, m-systems and the radical of an ideal for Γ-rings.

The notion of Jacobson radical, nil radical and strongly nilpotent radical for Γ-rings were introduced by Coppage and Luh [4] and they developed some radical properties. Also inclusion relation for these radicals were obtained and it was shown that the radicals all coincide in the case of a Γ-ring which satisfies the descending chain condition (DCC) on one-sided ideals. They studied Barnes prime radicals for a Γ-ring.

The general radical theory for rings had been introduced by A. Kurosh [6] and S.A. Amitsur [1]. They studied the generalizations of a general radical. Divinsky [5] studied the general radical theory, the upper radical and the lower radical. Various kinds of radicals were studied here and he had also shown that these radicals are equal by minimum condition. He also characterized special class of rings and special radicals.

G. L. Booth [3] studied radicals of matrix gamma rings. He developed various properties of radicals of matrix gamma rings. He also studied the properties of some radical classes of matrix gamma rings which were not N-radicals.

Hiram Paley and Paul M. Weichsel [8] studied the theory of radical of rings in a classical notion. They were characterizing semi-simple rings in terms of matrices. They developed some important characterizations in ring theories. Some characterizing of the radical of rings are studied by them. In this paper, we generalized some important result of the radical of Γ-rings of Hiram Paley and Paul M. Weichsel [8].

2. Preliminaries.

2.1 Definitions.

Gamma Ring. Let M and Γ be two additive abelian groups. Suppose that there is a mapping from $M \times \Gamma \times M \rightarrow M$ (sending (x, α, y) into $x\alpha y$) such that

(i) $(x + y)\alpha z = x\alpha z + y\alpha z$

$x(\alpha + \beta)z = x\alpha z + x\beta z$

$x\alpha(y + z) = x\alpha y + x\alpha z$

(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

where $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then M is called a Γ-ring in the sense of Barnes [1].

Ideal of Γ-rings. A subset A of the Γ-ring M is a left (right) ideal of M if A is an additive subgroup of M and $M\Gamma A = \{c\alpha a \mid c\in M, \alpha\in\Gamma, a\in A\}(A\Gamma M)$ is contained in A. If A is both a left and a right ideal of M, then we say that A is an ideal or two-sided ideal of M.

If A and B are both left (respectively right or two-sided) ideals of M, then A + B = $\{a + b|a\in A, b\in B\}$ is clearly a left (respectively right or two-sided) ideal, called the sum of A and B.We can say every finite sum of left (respectively right or two-sided) ideal of a Γ-ring is also a left (respectively right or two-sided) ideal.

It is clear that the intersection of any number of left (respectively right or two-sided) ideal of M is also a left (respectively right or two-sided) ideal of M.

If A is a left ideal of M, B is a right ideal of M and S is any non-empty subset of M, then the set, $A\Gamma S = \{\sum_{i=1}^{n} a_i\gamma s_i \mid a_i\in A, \gamma\in\Gamma, s_i\in S$, n is a positive integer$\}$ is a left ideal of M and SΓB is a right ideal of M. AΓB is a two-sided ideal of M.

Nilpotent element. Let M be a Γ-ring. An element x of M is called nilpotent if for some $\gamma\in\Gamma$, there exists a positive integer $n = n(\gamma)$ such that $(x\gamma)^n x = (x\gamma x\gamma \ldots \gamma x\gamma)x = 0$.

The descending chain condition (DCC). A Γ-ring M is said to have the descending chain condition on left ideals or DCC on left ideals if every descending sequence of left ideals $M \supseteq A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$ terminates after a finite number steps, that is, there exists an integer n such that $A_n = A_{n+1} = A_{n+2} = \ldots\ldots$.

The ascending chain condition (ACC). A Γ-ring M is said to have the ascending chain condition on left ideals or ACC on left ideals if every ascending sequence of left ideals $A_1 \subseteq A_2 \subseteq \ldots\ldots \subseteq A_n \subseteq \ldots\ldots$ terminates after a finite number of steps, that is, there exists an integer n such that $A_n = A_{n+1} = A_{n+2} = \ldots\ldots$.

Matrix Gamma Ring. Let M be a Γ-ring and let $M_{m,n}$ and $\Gamma_{n,m}$ denote, respectively, the set of all m × n matrices with entries from M and the set of all n × m matrices with entries from Γ, then M_{mn} is a Γ_{nm} -ring and multiplication defined by

$(a_{ij})(\gamma_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = \sum_p \sum_q a_{ip}\gamma_{pq}b_{qj}$. If m = n, then M_n is a Γ_n-ring.

Division gamma ring. Let M be a Γ-ring. Then M is called a division Γ-ring if it has an identity element and its only non-zero ideal is itself.

Internal direct sum. Let M be a **Γ-ring** and let N_1 and N_2 be two left ideals of M such that

(i) $M = N_1 + N_2 = \{n_1+n_2 \mid n_1 \in N_1, n_2 \in N_2\}$

(ii) $N_1 \cap N_2 = \{0\}$.

Then we say M is the internal direct sum or simply direct sum of N_1 and N_2 and we write $M = N_1 \oplus N_2$.

External direct sum. Let M and N be Γ-rings. Then the external direct sum of M and N denoted by $M \dot{+} N$ is $\{(m, n) \mid m \in M, n \in N\}$, where for $(m_1, n_1), (m_2, n_2) \in M \dot{+} N$, $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$ and $(m_1, n_1)\gamma(m_2, n_2) = (m_1\gamma m_2, n_1\gamma n_2)$ all $\gamma \in \Gamma$.

Quotient Γ-ring. Let M be a Γ-ring. Let A be an ideal of M. Then the set $\{m + A \mid m \in M\}$ is called the quotient Γ-ring of M by A. It is denoted by M/A, where $(m_1 + A)\gamma(m_2 + A) = m_1\gamma m_2 + A$ and $(m_1 + A) + (m_2 + A) = (m_1 + m_2) + A$ for all $m_1, m_2 \in M$ and $\gamma \in \Gamma$.

2.2 Theorem. Let M_1 and M_2 Γ-rings with DCC (ACC) on left ideals. Then $M_1 \dot{+} M_2$ has also DCC (ACC) on left ideals.

2.3 Theorem. Let Δ be a division Γ-ring. Then Δ_n, the Γ_n-ring of all n × n matrices over Δ, satisfies the ACC and the DCC on left ideals.

3. Nilpotent ideal of Γ-rings.

3.1 Definition. An ideal A of a Γ-ring M is called **nilpotent** if $(A\Gamma)^n A = (A\Gamma A\Gamma \ldots \Gamma A\Gamma)\quad A = 0$, where n is the least positive integer. In addition, if A is nilpotent, then every element in A is nilpotent.

3.2 Theorem. Let M be a Γ-ring and let N_1 and N_2 be two nilpotent left (right) ideals. Then $N_1 + N_2$ is a nilpotent left (right) ideal.

Proof. Let M be a Γ-ring. Let N_1 and N_2 be nilpotent left ideals of M. Then there exist two least positive integers q and n such that

$$(N_1\Gamma)^q N_1 = (N_1\Gamma N_1\Gamma \ldots\ldots \Gamma N_1\Gamma)N_1 = 0 \text{ and}$$

$$(N_2\Gamma)^n N_2 = (N_2\Gamma N_2\Gamma \ldots\ldots \Gamma N_2\Gamma)N_2 = 0.$$

Then N_1+N_2 is also a left ideal of M. Every element of $\{(N_1+N_2)\Gamma\}^{q+n+1}(N_1+N_2)$ is a sum of products $x_1\gamma x_2\gamma \ldots\ldots\gamma x_{q+n+2}$ in which either at least (s+1) factors belong to N_1 and (r+1) factors belong to N_2. In the former case, the above product may be written as

$$(x_1\gamma x_2\gamma\ldots\gamma x_{i_1})\gamma(x_{i_1+1}\gamma x_{i_1+2}\gamma\ldots\gamma x_{i_2})\gamma(x_{i_2+1}\gamma x_{i_2+2}\gamma\ldots\gamma x_{i_3})\gamma\ldots$$
$$\ldots\gamma(x_{i_s+1}\gamma x_{i_s+2}\gamma\ldots\ldots\gamma x_{i_{s+1}})\gamma\ \ldots\ldots,$$

where $x_{i_1}, x_{i_2}, \ldots x_{i_{s+1}} \in N_1$ and $s+1 \geq n+1$. Each group in parenthesis belongs to N_1, since N_1 is a left ideal of M. However, the product of any s+1 elements of N_1 is 0 and so the above product is 0. A similar argument holds when at least (r +1) factors belong to N_2.

Thus $\{(N_1+N_2)\Gamma\}^{s+n+1}(N_1+N_2) = \{(N_1+N_2)\Gamma(N_1+N_2)\Gamma\ldots\Gamma(N_1+N_2)\Gamma\}(N_1+N_2) = 0$.

Hence (N_1+N_2) is nilpotent. Thus the theorem is proved.

3.3 Corollary. Let M be a Γ-ring and let $A_1, A_2, \ldots, A_n$ be nilpotent left (right) ideals in M. Then ΣA_λ is a nilpotent left (right) ideal in M.

3.4 Theorem. Let A be a nilpotent left (right) ideals in a Γ-ring M. Then AΓM (MΓA) is a nilpotent ideal in M.

Proof. Since A is a left ideal, so is AΓM, and since M is a right ideal so is AΓM. Thus AΓM is and ideal in M. If $(A\Gamma)^n A = 0$, then

$$\begin{aligned}\{(A\Gamma M)\,\Gamma\}^n (A\Gamma M) &= (A\Gamma M)\,\Gamma\,(A\Gamma M)\,\Gamma \ldots\ldots\ \Gamma(A\Gamma M)\,\Gamma(A\Gamma M)\\ &= A\Gamma[(M\Gamma A)\Gamma\,(M\Gamma A)\Gamma \ldots\ldots\ \Gamma(M\Gamma A)\Gamma]\,M\\ &= A\Gamma\{(M\Gamma A)\Gamma\}^{n-1}(M\Gamma A)\Gamma M\\ &\subseteq A\Gamma\{(A\Gamma)^{n-1} A\Gamma M\\ &= (A\Gamma)^n\, A\Gamma M\\ &= 0\Gamma M\\ &= 0.\end{aligned}$$

Hence AΓM is nilpotent.

4. Radical of a Γ-ring.

4.1 Definition. Let M be a Γ-ring with DCC on left ideals. Let $\{A_\lambda\}$ be the collection of all nilpotent left ideals of M. Then $N = \Sigma A_\lambda$ is called the **radical** of M.

We shall show that N possesses the following properties :

(1) N is a nilpotent ideal

(2) N contains all nilpotent right ideals, as well as all nilpotent left ideals; thus N is

the unique ideal of M maximal with respect to being nilpotent.

4.2 Theorem. Let M be a Γ-ring with DCC on left ideals and let N be the radical of M. Then N is a nilpotent left ideal of M.

Proof. Let M be a Γ-ring with DCC. Then clearly N is a left ideal of M. So $N \supseteq N\Gamma N \supseteq (N\Gamma)^2 N \supseteq$ .... is a descending sequence of left ideals. By the DCC, there exists an integer n such that $(N\Gamma)^n N = (N\Gamma)^{n+1} N =$ .... $= (N\Gamma)^{2n+1} N$.

Thus $(N\Gamma)^n N = (N\Gamma)^{2n+1} N = (N\Gamma)^n N\Gamma(N\Gamma)^n N$. If $(N\Gamma)^n N = 0$,then N is nilpotent. So the theorem is proved. If $(N\Gamma)^n N \neq 0$, then there exists a left ideal A of M such that $(N\Gamma)^n N\Gamma A \neq 0$. By the DCC, there exists a left ideal A_0 of M such that $(N\Gamma)^n N\Gamma A_0 \neq 0$. Now since $(N\Gamma)^n N\Gamma A_0 \neq 0$, there exists an element x $(\neq 0) \in A_0$ such that$(N\Gamma)^n N\Gamma x \neq 0$.Then $(N\Gamma)^n N\Gamma(N\Gamma)^n N\Gamma x = (N\Gamma)^{2n+1} N\Gamma x = (N\Gamma)^n N\Gamma x \neq 0$. Since $x \in A_0$, $(N\Gamma)^n N\Gamma x \subset A_0$. So by the minimality of $A_{0,}$ $(N\Gamma)^n N\Gamma x = A_0$. Thus there exists an element $y \in (N\Gamma)^n N$ such that $y\gamma x = x$ for some $\gamma \in \Gamma$. Therefore $y \in N$. So y is also contained in the sum of finitely many nilpotent left ideals of M. By Corollary **3.3,** the sum of finitely many nilpotent left ideals of M is also a nilpotent ideal of M. Hence y is nilpotent. So $(y\gamma)^m y = 0$ for some positive integer m. We have $y\gamma x = x$, then $y\gamma y\gamma x = y\gamma(y\gamma x) = y\gamma x = x$. Continue this process, we get $x = y\gamma x = (y\gamma y)\gamma x = (y\gamma)^2 y\gamma x = \ldots = (y\gamma)^m y\gamma x = \ldots$. Since $(y\gamma)^m y = 0$, $(y\gamma)^m y\gamma x = 0\gamma x = 0$. Thus x = 0, which contradicts the fact that $x \neq 0$. Hence $(N\Gamma)^n N = 0$. Therefore N is nilpotent. Hence the theorem is proved.

4.3 Theorem. Let M be a Γ-ring. Let N be the sum of all nilpotent left ideals of M. Then N contains all nilpotent right ideal of M also.

Proof. Let A be a nilpotent right ideal. By Theorem **3.4,** $M\Gamma A$ is nilpotent, whence $A + M\Gamma A$ is also nilpotent by Theorem **3.2.** But $A + M\Gamma A$ is clearly a left ideal, so $A + M\Gamma A \subseteq N$, whence $A \subseteq N$.

We summarize Theorems **4.2** and **4.3** in the next theorem, which implies that properties **(1)** and **(2)** given prior to Theorem **4.2** hold.

4.4 Theorem. Let M be a Γ-ring with DCC on left ideals. Let N be the radical of M. Then

(i) N is a nilpotent ideal

(ii) N is the sum of all nilpotent right ideals

(iii) N is the unique ideal of M maximal with respect to being nilpotent.

Proof. (i) By Theorem **4.2,** N is a nilpotent left ideal. Since $N\Gamma M \oplus N$ is a nilpotent left ideals, $N\Gamma M \oplus N \subseteq N$, that is, N is also a right ideal, whence N is an ideal.

(ii) By Theorem **4.2,** N contains all nilpotent right ideals of M. Since N is also a nilpotent right ideal, N is clearly the sum of all nilpotent right ideals.

(iii) This follows from the definition of N and (ii).

We note that if M has DCC on right ideals, then we could define the radical of M as the sum of the nilpotent right ideals, and prove that it is equal to the sum of the nilpotent left ideals. That is, if M has either DCC on left ideals or DCC on right ideals, then we get the same radical whether we define it as the sum of the nilpotent left ideals or the nilpotent right ideals.

Now let M be a Γ-ring with DCC on left ideals and let N be the radical of M. Since N is an ideal in M, we may form the quotient Γ-ring M/N and it is easy to see that M/N also has DCC on left ideals. We shall prove that the radical of M/N is the zero element of M/N.

4.5 Theorem. Let M be a Γ-ring with DCC on left ideals. Then the radical of M/N is zero.

Proof. Let A' be a nilpotent left ideal of M/N and let

$A = \{m \in M \mid m + N \in A'\}$. Then A is a left ideal of M/N.

Since A' is nilpotent and since N is nilpotent, there exist integers s, t such that

(i) $(A'\Gamma)^s A' = 0$ in M/N, that is, $(m_1 + N)\gamma(m_2 + N)\gamma \ldots \gamma(m_s + N)\gamma(a + N) = N$

where $m_1, m_2, \ldots, m_s, a \in A$ and $\gamma \in \Gamma$

(ii) $(N\Gamma)^t N = 0$.

Now let $m_1, m_2, \ldots, m_{st} \in A$. Then

$a_1 = m_1\gamma m_2\gamma \ldots \gamma m_s \in N$, $a_2 = m_{s+1} \in N \gamma m_{s+2}\gamma \ldots \gamma m_{2s} \in N, \ldots,$

$a_t = m_{(t-1)s+1}\gamma \ldots \gamma m_{st} \in N$, whence the product $a_1\gamma a_2\gamma \ldots \gamma a_t \gamma a$ of these (t + 1) elements of N is zero, that is, $m_1\gamma m_2\gamma \ldots \gamma m_{st}\gamma a = 0$ and so $(A\Gamma)^{st} A = 0$. Since A is nilpotent, $A \subseteq N$ and so A' equals zero in M/N. Thus M/N has radical zero.

4.6 Definition. Let M be a Γ-ring with DCC on left ideals. We say M is **semi-simple** if the radical of M is 0.

We see immediately that if M has DCC on left ideals, then M/N is semi-simple by Theorem **4.5**. Moreover, it is easy to prove that a direct sum of finitely many matrix gamma rings over division Γ-rings, say $\Delta^{(1)}_{n_1} \dot{+} \Delta^{(2)}_{n_2} \dot{+} \ldots \dot{+} \Delta^{(k)}_{n_k}$, where $\Delta^{(i)}$ is a division Γ-ring, is a semi-simple Γ_n-ring.

4.7 Theorem. Let $\Delta^{(i)}$ be a division Γ-ring, i, = 1, 2, ..., k, Let $n_1, n_2, \ldots, n_k$ be integers that are greater than 0. Then

$$S = \Delta_{n_1} \dot{+} \Delta_{n_2} \dot{+} \ldots \dot{+} \Delta^{(k)}_{n_k} \text{ is semi-simple.}$$

Proof. By Theorem **2.2** and Theorem **2.3,** the Γ-ring S satisfies the DCC on left ideals. Thus we need only show that the radical N of S is zero. If $N \neq 0$, then there exists an element

$(a_1, a_2, \ldots, a_i, \ldots, a_k)$ in N, $a_i \neq 0$. Since N is a two-sided ideal in S,

$(0, 0, \ldots, 0, b_i, 0, \ldots, 0)\gamma(a_1, a_2, \ldots, a_i, \ldots, a_k)\gamma(0, 0, \ldots, 0, c_i, 0, \ldots, 0)$ is in N for all

b_i, c_i, $\in \Delta_{n_i}$ and $\gamma \in \Gamma$. Thus N contains all the elements of the form

(0, 0,. . ., 0, $b_i\gamma\ a_i\gamma c_i$,..., 0,..., 0). Since $\Delta_{n_i}^{(i)}$ has no proper two-sided ideals, S contains the set T = {(0, 0,..., 0, x_i, 0,..., 0) | $x_i \in \Delta_{n_i}^{(i)}$}. But this ideal T of S is contained in the radical of S and T is clearly not nilpotent, contradicting that N is nilpotent. Thus N = 0 and the proof is completed.

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