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MODULAR AND STRONGLY DISTRIBUTIVE ELEMENTS IN A NEARLATTICE

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Abstract :

In this paper the authors have introduced the notion of modular elements in a nearlattice. We have included several characterizations of modular and strongly distributive elements with examples. We have also proved that an element in a nearlattice is standard if and only if it is both modular and strongly distributive.

Keywords and phrases : modular elements, nearlattice, strongly distributive elements

বিমূর্ত সার (*Bengali version of the Abstract*)

নিকটবর্তী ল্যাটিসের (nearlattice) মডিউলার ধারণাকে প্রবন্ধকারেরা এই পত্রে উপস্থাপন করেছেন । উদাহরণ সহ মডিউলার এবং দৃঢ় বন্টন উপাদানের বিভিন্ন চরিত্রগত বৈশিষ্ট্যকে আমরা অর্ন্তভুক্ত করেছি । আমরা এটাও প্রমাণ করেছি যে নিকটবর্তী ল্যাটিসের উপাদান প্রমাণ আকার হয় যদি এবং কেবলমাত্র যদি ইহারা উভয়ত মডিউলার এবং দৃঢ় বন্টন যুক্ত হয় ।

Introduction :

A nearlattice S is a meet semilattice together with the property that any two elements possessing a common upper bound, have a supremum. This property is known as the upper bound property. S is called a distributive nearlattice if for all $t, x, y \in S$ $t \wedge (x \vee y) = (t \wedge x) \vee (t \wedge y)$ whenever $x \vee y$ exists. By [3] this condition is equivalent to the condition that for all $t, x, y, z \in S$ $t \wedge [(x \wedge y) \vee (x \wedge z)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$.

A nearlattice S is called a modular nearlattice if for all $x, y, z \in S$ with $z \le x$ and $y \vee z$ exists implies , $x \wedge (y \vee z) = (x \wedge y) \vee z$. By [3] this condition is

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equivalent to the condition that for all $t, x, y, z \in S$ with $z \leq x$, $x \wedge [(t \wedge y) \vee (t \wedge z)] = (t \wedge x \wedge y) \vee (t \wedge z)$.

Gratzer and Schmidt [2] introduced the notion of distributive, standard and neutral elements to study a larger class of non-distributive lattices. Then Cornish and Noor in [1] extended the concepts of standard and neutral elements for nearlattices. They also studied a new type of element known as strongly distributive element.

Recently Talukder and Noor in [4] introduced the notion of modular elements in a join semi lattice directed below. This notion is also applicable for general lattices.

In this paper we introduce the concept of modular elements in a nearlattice. We have given several characterization of modular and strongly distributive elements.

Finally we prove that an element s of a nearlattice is standard if and only if it is both modular and strongly distributive.

In a lattice L an element $m \in L$ is called a *modular element* if for all $x, y \in L$ with $y \leq x$, $x \wedge (m \vee y) = (x \wedge m) \vee y$.

Of course, in a modular lattice, every element is a modular element. Moreover, if every element of a lattice is modular, then the lattice itself is a modular lattice.

In the pentagonal lattice N_5

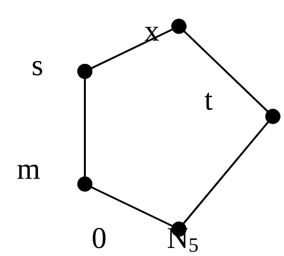

Figure-1

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Observe that here m is modular but t is not. Because, here $m < s$ and $s \wedge (t \vee m) = s$, But $(s \wedge t) \vee m = m$.

Let S be a nearlattice. An element $m \in S$ is called a *modular element* if for all $t, x, y \in S$ with $y \leq x$, $x \wedge [(t \wedge m) \vee (t \wedge y)] = (t \wedge m \wedge x) \vee (t \wedge y)$.

Of course, a nearlattice is modular if and only if its every element is modular.

Theorem 1.1. *The definition of modular element in a nearlattice* S *coincides with the definition of modular element of a lattice, when* S *is a lattice.*

Proof: Suppose m is a modular element of the lattice S. Let $t, x, y \in S$ with $y \leq x$, then $t \wedge y \leq t \wedge x$. Since m is modular,

so

$(t \wedge m \wedge x) \vee (t \wedge y) = (t \wedge x) \wedge [m \vee (t \wedge y)] = x \wedge [t \wedge (m \vee (t \wedge y))] = x \wedge [(t \wedge m) \vee (t \wedge y)]$
, which is the definition of modularity of m in a nearlattice.

Conversely, Let m be modular according to the definition given for a nearlattice. Let $x, y \in S$ with $y \leq x$.

Choose $t = m \vee y$. Then $x \wedge (m \vee y) = x \wedge ((t \wedge m) \vee (t \wedge y))$

$$= (t \wedge m \wedge x) \vee (t \wedge y)$$
$$= (m \wedge x) \vee y$$

Hence m is modular according to the definition of modular element in a nearlattice.

Here is a characterization of modular elements in a lattice.

Theorem 1.2. *Let* L *be a lattice and* $m \in L$. *Then the following conditions are equivalent.*

(i) m *is modular*.

(ii) *For* $y \leq x$ *with* $m \vee x = m \vee y$ *and* $m \wedge x = m \wedge y$ *implies* $x = y$.

Proof: $(i) \Rightarrow (ii)$ Suppose m is modular $y \leq x$ and $m \vee x = m \vee y$, $m \wedge x = m \wedge y$.

Then $x = x \wedge (m \vee x) = x \wedge (m \vee y) = (x \wedge m) \vee y$ (by modularity of m)

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$$=(y \wedge m) \vee y = y .$$

$(ii) \Rightarrow (i)$ suppose (ii) holds.

Let $y \leq x$, then $(x \wedge m) \vee y \leq x \wedge (m \vee y)$ always holds.

Let $x \wedge (m \vee y) = p$ and $(x \wedge m) \vee y = q$. Then $q \leq p$

Now $p \wedge m = x \wedge m$

Also,

$$q \wedge m = m \wedge [(x \wedge m) \vee y] = m \wedge [(x \wedge m) \vee (x \wedge y)] = (m \wedge x) \wedge [(x \wedge m) \vee (x \wedge y)] = x \wedge m .$$

Thus $p \wedge m = q \wedge m$

Again, $q \vee m = y \vee m$

$$p \vee m = [x \wedge (m \vee y)] \vee m \leq (m \vee y) \vee m$$
$$= y \vee m = q \vee m \leq p \vee m$$

as $q \leq p$. Thus $p \vee m = q \vee m = y \vee m$

Hence by (ii) $p = q$, that is $x \wedge (m \vee y) = (x \wedge m) \vee y$ and so m is modular.

Now we extend the above result and give a characterization of a modular element m in a nearlattice.

Theorem 1.3. *Let* S *be a nearlattice and* $m \in S$. *Then the following conditions are equivalent.*

(i) m *is modular*.

(ii) *For* $t, x, y \in S$ *with* $y \leq x$, $(t \wedge m) \vee (t \wedge x) = (t \wedge m) \vee (t \wedge y)$ *and* $t \wedge m \wedge x = t \wedge m \wedge y$ *implies* $t \wedge x = t \wedge y$.

Proof: $(i) \Rightarrow (ii)$ suppose m is modular, let $t, x, y \in S$ with $y \leq x$, $(t \wedge m) \vee (t \wedge x) = (t \wedge m) \vee (t \wedge y)$ and $t \wedge m \wedge x = t \wedge m \wedge y$.

Then $t \wedge x = (t \wedge x) \wedge [(t \wedge m) \vee (t \wedge x)] = (t \wedge x) \wedge [(t \wedge m) \vee (t \wedge y)]$

$= (t \wedge m \wedge x) \vee (t \wedge y)$ (by modularity of m)

$= (t \wedge m \wedge y) \vee (t \wedge y) = t \wedge y$.

$(ii) \Rightarrow (i)$ suppose (ii) holds. Let $t, x, y \in S$ with $y \leq x$

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Now $x \wedge [(t \wedge m) \vee (t \wedge y)] \geq (t \wedge m \wedge x) \vee (t \wedge y)$ always holds.

Let $x \wedge [(t \wedge m) \vee (t \wedge y)] = p$ and $(t \wedge m \wedge x) \vee (t \wedge y) = q$. Then $p \geq q$.

Choose $r = (t \wedge m) \vee (t \wedge y)$. Then $r \wedge p = p$ and $r \wedge q = q$.

$$r \wedge m = m \wedge [(t \wedge m) \vee (t \wedge y)] = (t \wedge m) \wedge [(t \wedge m) \vee (t \wedge y)] = t \wedge m.$$

Thus, $(r \wedge m) \vee (r \wedge q) = (t \wedge m) \vee q = (t \wedge m) \vee (t \wedge m \wedge x) \vee (t \wedge y) = (t \wedge m) \vee (t \wedge y) = r$

Then $(r \wedge m) \vee (r \wedge p) \leq r = (r \wedge m) \vee (r \wedge q) \leq (r \wedge m) \vee (r \wedge p)$ as $q \leq p$

Hence $(r \wedge m) \vee (r \wedge p) = (r \wedge m) \vee (r \wedge q) = r$

Also,

$$r \wedge m \wedge p = m \wedge p = m \wedge x \wedge [(t \wedge m) \vee (t \wedge y)] = x \wedge (t \wedge m) \wedge [(t \wedge m) \vee (t \wedge y)] = x \wedge t \wedge m$$

and

$$r \wedge m \wedge q = m \wedge q = m \wedge [(t \wedge m \wedge x) \vee (t \wedge y)] = m \wedge t \wedge x \wedge [(t \wedge m \wedge x) \vee (t \wedge y)] = x \wedge t \wedge m$$

Thus $r \wedge m \wedge p = r \wedge m \wedge p$ and so by (ii) $r \wedge p = r \wedge q$ Hence $p = q$ and so m is modular.

Now we include the following result in a nearlattice which is parallel to the characterization theorem for modular elements in a lattice given in theorem 1.2. But this cannot be considered as a definition of a modular element in a nearlattice.

Theorem 1.4. *Let* S *be a nearlattice and* $m \in S$. The *following conditions are equivalent.*

(i) *For all* $x, y \in S$ *with* $y \leq x$

$x \wedge (m \vee y) = (x \wedge m) \vee y$ *provided* $m \vee y$ *exists*.

(ii) *For all* $x, y \in S$ *with* $y \leq x$ *if* $m \vee x$, $m \vee y$ *exist and* $m \vee x = m \vee y$, $m \wedge x = m \wedge y$, *then* $x = y$.

Proof: $(i) \Leftrightarrow (ii)$ holds by the proof similar to the proof of Th.1.2 For the last part,

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Consider the following nearlattice.

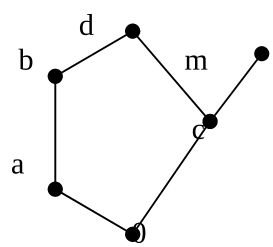

Figure-2

Observe that m satisfies the condition of theorem 1.4

Here $a < b$ and $b \wedge [(d \wedge m) \vee (d \wedge a)] = b \wedge (c \vee a) = b \wedge d = b$.

But $(b \wedge d \wedge m) \vee (d \wedge a) = 0 \vee a = a$, so m is not modular.

In a lattice L, an element d is called a *distributive element* if for all $x, y \in L$, $d \vee (x \wedge y) = (d \vee x) \wedge (d \vee y)$.

In order to introduce this notion for nearlattices, Cornish and Noor [1] could not give a suitable definition for distributive elements. But they discovered an element $d \in S$, such that $t \wedge d$ is a distributive element in the lattice $(t]$ for every $t \in S$. They found that these elements are also new even in casa of lattices, and in fact, they are much stronger than the distributive elements. So they referred them as "*strongly distributive*" elements.

An element d of a nearlattice S is called a *strongly distributive* element if for all $t, x, y \in S$

$$(t \wedge d) \vee (t \wedge x \wedge y) = [(t \wedge d) \vee (t \wedge x)] \wedge [(t \wedge d) \vee (t \wedge y)]$$

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In other words $t \wedge d$ is distributive in $(t]$ for each $t \in S$.

Theorem 1.5. *In a Lattice, every strongly distributive element is distributive but the converse is not necessarily true.*

Proof. Let d be a strongly distributive element of a lattice L. Suppose $x, y \in L$ and $t = x \vee y \vee d$.

Then $d \vee (x \wedge y) = (t \wedge d) \vee (t \wedge x \wedge y) = [(t \wedge d) \vee (t \wedge x)] \wedge [(t \wedge d) \vee (t \wedge y)]$

$= (d \vee x) \wedge (d \vee y)$, and so d is distributive.

Now consider the lattice in figure 3.

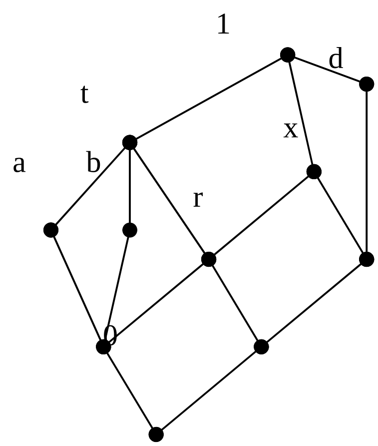

Figure 3

Here d is distributive but

$(t \wedge d) \vee (t \wedge a \wedge b) = r < t = [(t \wedge d) \vee (t \wedge a)] \wedge [(t \wedge d) \vee (t \wedge b)]$ and so it is not strongly distributive.

Following characterization of strongly distributive elements in a nearlattice is due to [1].

Theorem 1.6. *Let* S *be a nearlattice and* $d \in S$. *Then the following conditions are equivalent.*

(i) d *is strongly distributive.*

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(ii) *For all* $x, y, t \in S$ $(x \wedge [(t \wedge y) \vee (t \wedge d)]) \vee (t \wedge d) = (t \wedge x \wedge y) \vee (t \wedge d)$.

An element $s \in S$ is called a *standard element* if for all $t, x, y \in S$ $t \wedge [(x \wedge y) \vee (x \wedge s)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$.

In a distributive nearlattice every element is standard. If every element of S is standard then S is itself a distributive nearlattice.

Theorem 1.7. *Every standard element in a nearlattice* S *is modular but a modular element may not be standard.*

Proof: Let $s \in S$ be standard, let $t, x, y \in S$ with $y \leq x$

$$x \wedge [(t \wedge s) \vee (t \wedge y)] = x \wedge [(t \wedge y) \vee (t \wedge s)]$$

$$= (t \wedge x \wedge y) \vee (t \wedge s \wedge x)$$

$$= (t \wedge s \wedge x) \vee (t \wedge y)$$

So s is modular.

Conversely, consider the lattice of Figure 1.

Here m is modular

But $s \wedge (m \vee t) = s \wedge x = s$

$$(s \wedge m) \vee (s \wedge t) = m \vee 0 = m$$

So m is not standard

Theorem 1.8. *Every standard element is strongly distributive but the converse may not be true.*

Proof. Suppose s is standard in S. Let $t, a, b \in S$

Then, $[(t \wedge s) \vee (t \wedge a)] \vee [(t \wedge s) \vee (t \wedge b)]$

$$= ([(t \wedge s) \vee (t \wedge a)] \wedge (t \wedge s)) \vee ([(t \wedge s) \vee (t \wedge a)] \wedge (t \wedge b))$$ (as s is standard.)

$$= (s \wedge [(t \wedge a) \vee (t \wedge s)]) \vee (b \wedge [(t \wedge a) \vee (t \wedge s)])$$

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$$=(t \wedge a \wedge s) \vee (t \wedge s) \vee (t \wedge a \wedge b) \vee (t \wedge a \wedge s)$$

$$=(t \wedge s) \vee (t \wedge a \wedge b)$$

so s is strongly distributive.

In Figure 3. observes that t is strongly distributive, but it is not standard, because

$d \wedge (x \vee t) > (d \wedge x) \vee (d \wedge t)$.

Remark:

In the pentagonal lattice of figure 1, m is modular and t is strongly distributive .

Observe that $m \leq s$ and $s \wedge (t \vee m) = s \wedge x = s$ but $(s \wedge t) \vee m = 0 \vee m = m$. Thus t is not modular. On the other hand, $(x \wedge m) \vee (x \wedge s \wedge t) = m \vee 0 = m$, but $[(x \wedge m) \vee (x \wedge s)] \wedge [(x \wedge m) \vee (x \wedge t)] = (m \vee s) \wedge (m \vee t) = s \wedge x = s$ implies m is not strongly distributive.

We conclude the paper with the following characterization of standard elements in a nearlattice.

Theorem 1.9. *Let* S *be a nearlattice. An element* $s \in S$ *is standard if and only if it is both modular and strongly distributive.*

Proof: If s is standard then by theorem 1.7 and theorem 1.8, s is both modular and strongly distributive. Conversely, suppose s is both modular and strongly distributive. Let $t, x, y \in S$.

Then, $(t \wedge x \wedge y) \vee (t \wedge x \wedge s) = (t \wedge x) \wedge [(x \wedge s) \vee (t \wedge x \wedge y)]$ (as s is modular)

$= (t \wedge x) \wedge [(x \wedge s) \vee (t \wedge x)] \wedge [(x \wedge s) \vee (x \wedge y)]$ (as s is strongly distributive)

$= t \wedge x \wedge [(x \wedge s) \vee (x \wedge y)] = t \wedge [(x \wedge s) \vee (x \wedge y)]$

so s is standard.

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