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VECTOR CONE METRIC SPACES AND SOME FIXED POINT THEOREMS

By

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Abstract:

In this paper it is shown that a vector cone metric space as introduced by us bears a metric like topology. Cantor's intersection like Theorem is proved and as an application of the same a useful fixed point Theorem is obtained.

Keywords and phrases : vector cone, metric space, topology, fixed point Theorem

বিমূর্ত সার (*Bengali version of the Abstract*)

এই পত্রে আমরা দেখিয়েছি যে , যে ভেক্টর কোণ ম্যাট্রিক দেশ (vector cone metric space) উপস্থাপন করা হয়েছে তা ম্যাট্রিকের মত টপোলজিকে বহন করে । কেন্টরের (Cantor) পরস্পরচ্ছেদী সদৃশ উপপাদ্যকে প্রমাণ করেছি এবং ইহার প্রয়োগ হিসাবে একটি উপযোগী স্থির বিন্দু উপপাদ্যকে পেয়েছি ।

1. Introduction

Huang and Zhang [2] generalized the notion of metric space by replacing the set of real numbers by an ordered Banach space and they had defined a cone metric space. Huang and Zhang also established some fixed point Theorems for contractive type mappings in a normal cone metric space. Subsequently, several other authors (see [1], [3], [5]) studied problem of common fixed point of mappings satisfying a contractive type condition in a normal cone metric space. Sh. Rezapour and R. Haml barani in [5] had extended results of Huang and Zhang in [2]. Ismat Bag, Akbar Azam and Muhammad Mrshad had also extended results of Huang and Zhang for a pair of self mappings satisfying a generalized contractive type condition.

In this paper Cantor's intersection like theorem has been established in a vector cone metric space and with its aid some fixed point theorems have been established. Our findings may be treated as extension works of Huang and Zhang but in a different direction avoiding usual Picard Iterative scheme as followed by Huang and Zhang.

2.

We recall the following definitions:

2.1 Definition

A subset P of a real Banach space E is said to be a cone if and only if

$P_1 : P$ is closed, nonempty and $P \neq \{\theta\}$

$P_2 : a,b \in \mathbb{R}, a,b \geq 0, x,y \in P \Rightarrow ax+by \in P$ $\quad$ $P_3 : x \in P$ and $-x \in P \Rightarrow x = \theta$

2.2 Definition

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ on E with respect to P by the rule:

$x \leq y$ in E if and only if $y - x \in P$.

We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$; and $x << y$ will stand for $y - x \in$ Int P.

2.3(a) Definition

A cone P in E is called normal if there is a scalar $k > 0$ such that for all $x, y \in P$ with $\theta \leq x \leq y$, one has $\|x\| \leq k\|y\|$.

2.3(b) Definition

If further k_0 is the least positive number satisfying $\|x\| \leq k_0 \|y\|$ for all

$x, y \in P$ with $\theta \le x \le y$, then k_0 is called the normal constant of the normal cone *P*.

2.4 Definition

Let *M* be a nonempty set. Let $d : M \times M \to E$ satisfy,

$d_1 : \theta \le d(x; y)$ for all $x, y \in M$

and $d(x, y) = \theta$ if and only if $x = y$ in *M*.

$d_2 : d(x, y) = d(y, x)$ for all x, y in *M*

$d_3 : d(x, y) \le d(x, z) + d(y, z)$ for all x, y, z in *M*.

Then *d* is called a cone metric on *M* and (*M*, *d*) is called a cone metric space.

2.5 Definition

In a cone metric space (M, d), a member $(\alpha_1, \alpha_2 ... \alpha_n ...)$ in *E* with $\alpha_n > 0$ for all *n* is called a positive member of *E*.

Suppose *E* denotes the collection of all bounded sequence of reals. Then *E* becomes a real Banach space with norm $\|x\| = \sup_n |x_n|$; as $x \in E$ where $x = (x_1, x_2 ... x_n ...)$.

If *P* denotes the set of all bounded sequences of non-negative reals, then as a subset of *E*, we verify that *P* forms a cone in *E*.

2.1 Example

Set of all bounded sequences of non-negative real numbers is a normal cone in the Banach space l_∞ (In fact $l_\infty = E$ as above).

2.6 Definition

A positive real number ε is taken to represent a positive member $(\varepsilon, \varepsilon, \varepsilon ...)$ of *E*. Let $x_0 \in E$ and *r* be a positive member of *E*. Then set denoted by $B_r(x_0) = \{x \in M : d(x, x_0) < r\}$, is called an open ball centered at x_0 with radius *r* in (*M*, *d*).

2.1 Theorem

The family $\mathcal{B}$ of all open balls in (*M*, *d*) together with empty set form a base for a topology τ_d on *X*.

Proof

Take two members $\mathcal{B}_{r_1}(x_1)$ and $\mathcal{B}_{r_2}(x_2)$ in $\mathcal{B}$ and $x_0 \in \mathcal{B}_{r_1}(x_1) \cap \mathcal{B}_{r_2}(x_2)$. It suffices to find an open ball like $\mathcal{B}_{\in}(x_0)$ to satisfy $\mathcal{B}_{\in}(x_0) \subset \mathcal{B}_{r_1}(x_1) \cap \mathcal{B}_{r_2}(x_2)$.

Suppose, $d(x_0, x_1) = (\alpha_1(x_0, x_1), \alpha_2(x_0, x_1) ... \alpha_n(x_0, x_1) ...)$ and we have $\alpha_n(x_0, x_1) < r_{1n}$ for all *n* where $r_i = (r_{i1}, r_{i2} ... r_{in}, ...)$, $i = 1, 2$ are two positive members of *E*.

If $0 < \varepsilon_n < \min\{(r_{1n} - \alpha_n(x_0, x_1)), r_{2n} - (\alpha_n(x_0, x_2))\}$ for $n = 1, 2...$ then $\varepsilon = (\varepsilon_1, \varepsilon_2 ... \varepsilon_n ...)$ is a positive member of *E* such that $\mathcal{B}_{\varepsilon}(x_0) \subset \mathcal{B}_{r_1}(x_1) \cap \mathcal{B}_{r_2}(x_2)$.

Hence the conclusion follows.

This topology τ_d is termed as a cone metric topology in (*M*, *d*).

2.2 Example

Let *M* be the collection of all real polynomials like $p(t) = \sum_{j=0}^{\infty} a_j t^j$ where a_j's are real coefficients and $a_j = 0$ eventually in *j* and let $d : M \times M \to E$ be taken as

$$d(p, q) = (|a_0 - b_0|, |a_1 - b_1|, ... |a_r - b_r|, ...)$$

where $p(t) = \sum_{j=0}^{\infty} a_j t^j$ and $q(t) = \sum_{j=0}^{\infty} b_j t^j$ as referred to above.

2.2 Theorem

The cone metric topology τ_d in (M, d) is Hausdorff.

Proof

The proof is a routine exercise and is left out.

3.

3.1 Definition

A sequence $\{x_n\}$ in (M,d) is said to be convergent in (M,d) if there is a member $x \in M$ such that if for every $c \in E$ with $\theta << c$ there is an index N such that for all $n > N$, $d(x_n, x) << c$. If $\{x_n\}$ converges to x in (M,d), we write $\lim_{n\to\infty} x_n = x$.

3.2 Definition

A sequence $\{x_n\}$ in (M,d) is said to be Cauchy in (M,d) if for any $c \in E$ with $\theta << c$, there is an index N such that we have $d(x_n, x_m) << c$ for all $m,n > N$.

3.3 Definition

(M,d) is said to be a complete cone metric space if every Cauchy sequence in (M,d) is convergent in (M,d).

3.4 (a) Definition

A subset B of M is called bounded if there is a positive member K in E such that $d(b_1,b_2) \le K$ for all $b_1,b_2 \in B$.

3.4 (b) Definition

Diameter of a bounded set B in M denoted by Diam B is defined as

Diam $B = \left(\sup_{b_1,b_2 \in B} \alpha_1(b_1,b_2), \sup_{b_1,b_2 \in B} \alpha_2(b_1,b_2), \ldots, \sup_{b_1,b_2 \in B} \alpha_n(b_1,b_2), \ldots \right)$ where $d(b_1,b_2) = (\alpha_1, \alpha_2 \ldots) \in E$; where for each i, $\sup_{b_1,b_2 \in B} \alpha_i(b_1,b_2) < +\infty$, by virtue of B being bounded. Therefore Diam (B) of a non-empty bounded set containing more than one member is always a positive member of E.

3.1 Theorem

A necessary and sufficient condition for a vector cone metric space (M,d) to be complete is that every nested sequence of nonempty closed subsets $\{G_n\}$ with Diameter $(G_n) \to \theta \in E$ as $n \to \infty$ has $\bigcap_{n=1}^{\infty} G_n$ as a singleton.

We use the following lemma. Proof of which is easy and left out.

3.1 Lemma

If G is a nonempty subset of (M,d) then Diam $G =$ Diam $\overline{G}$, $\overline{G}$ denoting τ_d-closure of G in (M,d).

Proof of Theorem

For necessary part, suppose (M,d) is complete and take $a_n \in G_n$, then for $p \geq 1, a_{n+p} \in G_{n+p} \subset G_n$. $d(a_n, a_{n+p}) \leq \text{Diam } G_n$.

Then $\|d(a_n, a_{n+p})\| \leq k \|\text{Diam } G_n\|$, where k is a normal constant. This implies $d(a_n, a_{n+p}) \to \theta$ as $n \to \infty$. Then $\{a_n\}$ becomes Cauchy in (M, d) and by completeness of (M, d) $\lim_{n\to\infty} a_n = u \in M$. Now $a_{n+p} \in G_n$ and by closure property of G_n, we have $\lim_{n\to\infty} a_{n+p} = u \in G_n$. Therefore $u \in \bigcap_{n=1}^{\infty} G_n$. If v is another member of $\bigcap_{n=1}^{\infty} G_n$, for all n, we have $u, v \in G_n$ and $d(u,v) \leq \text{Diam } G_n$. Hence $\|d(u,v)\| \leq k\|\text{Diam } G_n\|$, where k is a normal constant. This implies $d(u,v) \to \theta$ as $n \to \infty$. Therefore $u = v$. Hence $\bigcap_{n=1}^{\infty} G_n$ is a singleton.

For sufficiency, let $\{x_n\}$ be a Cauchy sequence in (M,d). Put $H_n = (x_n, x_{n+1}, x_{n+2}, ...)$. Then $\overline{H_n}$ is a decreasing sequence of non empty closed sets in (M,d) such that $\text{Diam}(\overline{H_n}) = \text{Diam}(H_n) \to \theta$ as $n \to \infty$. Then, $\bigcap_{n=1}^{\infty} \overline{H_n}$ is a singleton (say) $\{u\}$; Now $d(x_n, u) \leq \text{Diam}(H_n)$. Therefore $\|d(x_n, u)\| \leq k\|\text{Diam}(H_n)\|$, where k is a normal constant. This implies $d(x_n, u) \to \theta$ as $n \to \infty$.

Therefore $\lim_{n\to\infty} x_n = u \in M$. So ($M$, d) is complete.

3.2 Theorem

Let (*M*, *d*) be a complete vector cone metric space and $T : M \to M$ be an operator to satisfy, $d\left(T(x),T(y)\right) \le \alpha d\left(x,T(x)\right)+\beta d\left(y,T(y)\right)+\upsilon d(x,y)$ with $0 \le \alpha,\beta,\upsilon$ and $\alpha+\beta+\upsilon<1$ for all $x,y \in X$. Then T has a unique fixed point in M.

Proof

If x_0 be an arbitrary point in M and $x_n = T^n(x_0), n = 1,2,...$ where $T^0(x_0)=x_0$, we have

$$\begin{aligned} d(x_2,x_1) &= d\left(T(x_1),T(x_0)\right) \\ &\le \alpha d(x_1,x_2)+\beta d(x_0,x_1)+\upsilon d(x_0,x_1) \end{aligned}$$

or, $d(x_2,x_1) \le \dfrac{\beta+\upsilon}{1-\alpha} d(x_0,x_1)$

Similarly, $d(x_3,x_2) \le \left(\dfrac{\beta+\upsilon}{1-\alpha}\right)^2 d(x_0,x_1)$ and by induction we have

$$\begin{aligned} d(x_n,x_{n+1}) &\le \left(\frac{\beta+\upsilon}{1-\alpha}\right)^n d(x_0,x_1) \\ &= \delta^n d\left(x_0,T(x_0)\right) \qquad (1) \end{aligned}$$

where $\delta = \dfrac{\beta+\upsilon}{1-\alpha} < 1$.

If $\{h_k\}$ is a decreasing sequence of positive members of E such that $\lim_{k\to\infty} h_k = \theta$.

Put $G_k = \left\{x \in M : d\left(x,T(x)\right) \le h_k\right\}$, where $h_k = (h_k,h_k,...) \in E$. From (1) it follows that for large $k, G_k \ne \phi$. Suppose $G_k \ne \phi$ for all k. Clearly $\{G_k\}$ forms a decreasing chain of non-empty set in M. We show that T maps G_k into itself. If $x \in G_k$, then

$$\begin{aligned} d\left(T(x),T\left(T(x)\right)\right) \le \alpha d\left(x,T(x)\right)+\beta d\left(T(x),T\left(T(x)\right)\right) \\ +\upsilon d\left(x,T(x)\right) \end{aligned}$$

$$(1-\beta) d\left(T(x),T\left(T(x)\right)\right) \le (\alpha+\upsilon) d\left(x,T(x)\right)$$

$$d\left(T(x),T\left(T(x)\right)\right)\leq\frac{\alpha+\upsilon}{1-\beta}d\left(x,T(x)\right)$$

$$<d\left(x,T(x)\right)\quad\text{where}\quad\frac{\alpha+\upsilon}{1-\beta}<1,\quad\text{since}\quad\alpha+\beta+\upsilon<1$$

$$<h_k.$$

Hence $T:G_k\rightarrow G_k$.

Next we show that each G_k is closed.

Let $\lim_j x_j=u\in M$ where $x_j\in G_k$, then

$$d\left(u,T(u)\right)\leq d\left(u,x_j\right)+d\left(x_j,T\left(x_j\right)\right)+d\left(T\left(x_j\right),T(u)\right)$$

$$\leq d\left(u,x_j\right)+d\left(x_j,T\left(x_j\right)\right)+\alpha d\left(x_j,T\left(x_j\right)\right)$$
$$+\beta d\left(u,T(u)\right)+\upsilon d\left(x_j,u\right)$$

$$(1-\beta)d\left(u,T(u)\right)\leq d\left(u,x_j\right)+h_k+\alpha h_k+\upsilon d\left(x_j,u\right)$$

$$d\left(u,T(u)\right)\leq\frac{1+\alpha}{1-\beta}h_k+(1+\upsilon)d\left(x_j,u\right)$$

Passing on limit $j\rightarrow\infty$, noting that $d\left(x_j,T\left(x_j\right)\right)\leq h_k$, we find

$$d\left(u,T(u)\right)\leq\left(\frac{1+\alpha}{1-\beta}\right)h_k.$$

Now $0\leq\alpha,\beta,\upsilon<1$ and $\alpha+\beta+\upsilon<1$ gives $\frac{\alpha+\upsilon}{1-\beta}<1$ and so

$\sup_{\upsilon}\left\{\frac{\alpha+\upsilon}{1-\beta}\right\}\leq 1$ or $\frac{\alpha+1}{1-\beta}\leq 1$ and hence

$$d\left(u,T(u)\right)\leq h_k.$$

Thus $u\in G_k$ and G_k is shown to be closed.

Finally, we show that each G_k is bounded.

Take $u,v\in G_k$. So

$$d(u,v) \le d(u,T(u)) + d(T(u),T(v)) + d(T(v),v)$$

$$\le h_k + h_k + \alpha d(u,T(u)) + \beta d(v,T(v)) + \upsilon d(u,v) \text{ So, } (1-\upsilon) d(u,v) \le (\alpha+\beta+2) h_k$$

implying that

$$d(u,v) \le \frac{\alpha+\beta+2}{1-\upsilon} h_k .$$

That means G_k is bounded and if $x, y \in G_k$, we have

$$d(x,y) \le d(x,T(x)) + d(T(x),T(y)) + d(T(y),y)$$

$$\le 2h_k + \alpha d(x,T(x)) + \beta d(y,T(y)) + \upsilon d(x,y) \qquad \le \frac{\alpha+\beta+2}{1-\upsilon} h_k .$$

So, $\|d(x,y)\| \le k_0 . \dfrac{\alpha+\beta+\upsilon}{1-\upsilon} \|h_k\|$, where k_0 is a normal constant.

Right hand side $\to \theta$ as $k \to \infty$. So $d(x,y) \to \theta$ as $k \to \infty$ so $\text{Diam}\, G_k \to \theta$ as $k \to \infty$.

Thus $\{G_k\}$ is a decreasing chain of nonempty closed set in (M,d) with diam

$(G_k) \to \theta$ as $k \to \infty$.

By the theorem above, $\bigcap_{k=1}^{\infty} G_k$ is a singleton, say $\{u\}$ for some $u \in M$. Hence $T(u) = u$. Uniqueness of u is also clear and proof is complete.

3.1 Corollary

If T is a contraction mapping from a complete vector cone metric space into itself, then T has a unique fixed point in M.

Taking $\alpha = \beta = 0$ and $0 < \upsilon < 1$ in theorem above, corollary follows.

3.2 Corollary

If T is a Kannan like mapping from a complete vector cone metric space into itself, then T has a unique fixed point in M.

Taking $\alpha = \beta$ and $\upsilon = 0$ in 3.2 theorem, corollary follows

4.

4.1 Definition

Let (M, d) be a vector cone metric space and $P \subseteq M$ denote the set of all bounded sequences of non-negative reals. A mapping $\phi : P \to P$ is said to be upper semi continuous from right if for each j, $\overline{\lim_n} \phi\left(\alpha_j^{(n)}\right) \le \phi\left(\lim_n \alpha_j^{(n)}\right)$ where

$\left\{\alpha_j^{(n)}\right\}_{j=1,2,...} = \left(\alpha_1^{(n)}, \alpha_2^{(n)}, ..., \alpha_j^{(n)}, ...\right) \in P$, n =1,2…

4.1 Theorem

If (M, d) is a complete vector cone metric space and $T : M \to M$ satisfies the condition that for all $x, y \in X$,

$d\left(T(x), T(y)\right) \le \phi\left[\max\left\{d(x, y), d\left(x, T(x)\right), d\left(y, T(y)\right)\right\}\right]$ where ϕ is an upper semicontinuous function from right from P to P $\left(P = (x) = (x_1, x_2 ... x_n ...) : x_i > 0 \forall i\right)$ such that $\phi(t) \ne t$ and $\sup\limits_{t>0} \dfrac{t}{t - \phi(t)} < \infty$. Then T has a fixed point in M.

The proof rests upon applying the following lemma.

4.1 Lemma

If $d(x_n, x_{n+1}) = \left(\alpha_1^{(n)}, \alpha_2^{(n)}, ..., \alpha_n^{(n)}, ...\right) = \left(\alpha_j^{(n)}\right), j = 1, 2, ...$

where $x_n = T^n(x)$ and $x \in X$, then $\lim\limits_{n\to\infty} \alpha_j^{(n)} = \theta, j = 1, 2, ...$

Proof

Suppose $\alpha^{(n)} > \theta$ for all n, where $\alpha^{(n)} = \left(\alpha_1^{(n)}, \alpha_2^{(n)}, ..., \alpha_n^{(n)}, ...\right)$

Then $\alpha^{(n)} = d(x_n, x_{n+1})$

$$= d\left(T^n(x), T^{n+1}(x)\right)$$

$$= d\left(T\left(T^{n-1}(x)\right), T\left(T^n(x)\right)\right)$$

$$= d\left(T(x_{n-1}), T(x_n)\right)$$

$$\le \phi\left[\max\left\{d(x_{n-1}, x_n), d\left(x_{n-1}, T(x_{n-1})\right), d\left(x_n, T(x_n)\right)\right\}\right]$$

$$=\phi\left[\max\left\{d\left(x_{n-1},x_n\right),d\left(x_{n-1},x_n\right),d\left(x_n,x_{n+1}\right)\right\}\right]$$

$$=\phi\left[\max\left\{d\left(x_{n-1},x_n\right),d\left(x_n,x_{n+1}\right)\right\}\right]$$

Now if max value $=d\left(x_n,x_{n+1}\right)$, then one has

$\alpha^{(n)}\le\phi\left(\alpha^{(n)}\right)$, which is untenable.

Hence max value $=d\left(x_{n-1},x_n\right)=\alpha^{(n-1)}$

So we have $\alpha^{(n)}=d\left(x_{n-1},x_n\right)\le\phi\left(\alpha^{(n-1)}\right)$ (1)

$$<\alpha^{(n-1)}$$

That means $\left\{\alpha^{(n)}\right\}$ is a decreasing sequence and let $\lim_{n\to\infty}\alpha^{(n)}=\alpha$ where $\alpha=(\alpha,\alpha,\alpha,...)$. If $\alpha>0$, we have $\phi(\alpha)<\alpha$.

Since ϕ is an upper semicontinuous function from right we get $\overline{\lim_{n\to\infty}}\phi\left(\alpha^{(n)}\right)\le\phi\left(\lim_{n\to\infty}\alpha^{(n)}\right)=\phi(\alpha)<\alpha$ contradicting (1) namely, $\alpha^{(n)}\le\phi\left(\alpha^{(n-1)}\right)$.

Hence one concludes $\lim_{n}\alpha^{(n)}=\theta$.

Proof of Theorem

Put $G_n=\left\{x\in X,d\left(x,T(x)\right)\le\frac{1}{n}\right\}$

First we show that for large values of *n*, $G_n\ne\phi$. Take $x\in M$ and consider $x_n=T^n(x)$.

Lemma above shows that $d\left(x_n,T\left(x_n\right)\right)\to\theta$ as $n\to\infty$. Clearly $G_n\ne\phi$ for large values of *n*.

Let us suppose that $G_n\ne\phi$ for all *n*. We verify that *T* maps G_n into G_n. Take $x\in G$,

$$d\left(T(x),T\left(T(x)\right)\right)\le\phi\left[\max\left\{d\left(x,T(x)\right),d\left(x,T(x)\right),\right.\right.$$
$$\left.\left.d\left(T(x),T\left(T(x)\right)\right)\right\}\right]$$

$$= \phi\left[\max\left\{d\left(x,T(x)\right), d\left(T(x),T\left(T(x)\right)\right)\right\}\right]$$

Now if max value is $d\left(T(x),T^2(x)\right)$; then

$$d\left(T(x),T\left(T(x)\right)\right) \le \phi\left\{d\left(T(x),T\left(T(x)\right)\right)\right\}$$

$$< d\left(T(x),T^2(x)\right) \text{ which is a contradiction.}$$

Hence max value $= d\left(x,T(x)\right)$. Therefore

$$d\left(T(x),T\left(T(x)\right)\right) \le \phi\left(d\left(x,T(x)\right)\right) < d\left(x,T(x)\right) \le \frac{1}{n}.$$

That means $T(x) \in G_n$. Hence T maps G_n into itself.

Now we check that G_n is closed in vector cone metric space.

Let $\left\{x_{n_k}\right\} \in G_n$ satisfy $\lim\limits_{k\to\infty} x_{n_k} = x_0 \in M$.

Now $d\left(x_0,T(x_0)\right) \le d\left(x_0,x_{n_k}\right) + d\left(x_{n_k},T(x_0)\right)$

$$\le d\left(x_0,x_{n_k}\right) + d\left(T\left(x_{n_{k-1}}\right),T(x_0)\right)$$

$$\le d\left(x_0,x_{n_k}\right) + \phi\Big[\max\Big\{d\left(x_{n_{k-1}},x_0\right), d\left(x_{n_{k-1}},x_{n_k}\right), d\left(x_0,T(x_0)\right)\Big\}\Big]$$

Since $d\left(x_0,x_{n_k}\right), d\left(x_{n_{k-1}},x_0\right) \to \theta$ as $k\to\infty$ and $d\left(x_{n_{k-1}},x_{n_k}\right) \le \frac{1}{n}$

$$d\left(x_0,T(x_0)\right) \le \phi\left[\max\left\{\frac{1}{n}, d\left(x_0,T(x_0)\right)\right\}\right]$$

If $\max\left\{\frac{1}{n}, d\left(x_0,T(x_0)\right)\right\} = d\left(x_0,T(x_0)\right)$

then $d\left(x_0,T(x_0)\right) \le \phi\left\{d\left(x_0,T(x_0)\right)\right\}$

$$< d\left(x_0,T(x_0)\right) \text{ which is untenable.}$$

So, we conclude that $d\left(x_0,T(x_0)\right) \le \phi\left(\frac{1}{n}\right)$

$$< \frac{1}{n}.$$

Thus $x_0 \in G_n$ and G_n is shown to be closed.

Finally take $x, y \in G_n$, then $d\left(x, T(x)\right) \le \frac{1}{n}$ and $d\left(y, T(y)\right) \le \frac{1}{n}$.

Now $\quad d(x,y) \le d\left(x,T(x)\right) + d\left(y,T(y)\right) + d\left(T(x),T(y)\right)$

$$\le \frac{1}{n} + \frac{1}{n} + \phi\left[\max\left\{d(x,y), d\left(x,T(x)\right), d\left(y,T(y)\right)\right\}\right]$$

$$\le \frac{2}{n} + \phi\left[\max\left\{d(x,y), \frac{1}{n}\right\}\right]$$

Now there are two cases to examine.

Case I

Suppose $d(x,y) \le \frac{1}{n}$ then $\max\left\{d(x,y), \frac{1}{n}\right\} \le \frac{1}{n}$ and in this case,

$$d(x,y) \le \frac{2}{n} + \phi\left(\frac{1}{n}\right)$$

$$< \frac{3}{n} < \infty \text{ and hence } \operatorname{Diam}(G_n) \le \frac{3}{n}.$$

Case II

Suppose $d(x,y) > \frac{1}{n}$,

then $\max\left\{d(x,y), \frac{1}{n}\right\} = d(x,y)$ and in this case we have

$$d(x,y) \le \frac{2}{n} + \phi\left(d(x,y)\right)$$

$$d(x,y)\left(1 - \frac{\phi\left(d(x,y)\right)}{d(x,y)}\right) \le \frac{2}{n}$$

or, $d(x,y) \le \frac{2}{n} \cdot \frac{d(x,y)}{d(x,y) - \phi\left(d(x,y)\right)}$

$$\le \frac{2}{n} \sup_{t>0} \frac{t}{t - \phi(t)}$$

$$= \frac{2R}{n} \text{ where } \sup_{t>0} \frac{t}{t - \phi(t)} = R < \infty.$$

This gives $\left\| Diam(G_n) \right\| \le k \left\| \frac{2R}{n} \right\|$, where k is a normal constant. This gives $\lim_{n\to\infty} Diam(G_n) = \theta$.

Hence $\lim_{n\to\infty} G_n = \theta$.

Now Cantor's intersection Theorem as proved before applies to give $\bigcap_n G_n = \{w\}$ say, for some $w \in X$ and $T(w) = w$ is the fixed point of *T*.

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