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DEDUCTION OF SOME RESULTS ON THE MAXIMUM TERMS OF COMPOSITE ENTIRE FUNCTIONS

By

[1]Sanjib Kumar Datta, [2]Tanmay Biswas and [3]Soumen Kanti Deb

[1]Department of Mathematics, University of Kalyani, Nadia, West Bengal, India.
[2]Krishnagar, Nadia, West Bengal, India.
[3]Bahin High School, Uttar Dinajpur, West Bengal, India.

Abstract.

In this paper we compare the maximum term of composition of two entire functions with their corresponding left and right factors.

Keywords and phrases : entire function, complex plane, composite entire functions.

বিমূর্ত সার (*Bengali version of the Abstract*)

এই পত্রে ডান ও বাম অনুসঙ্গী উৎপাদক সহ দু'টি এন্টায়ার অপেক্ষকের (entire functions) সংযোজনের বৃহত্তম পদ সংখ্যার তুলনা করা হয়েছে ।

1. Introduction, Definitions and Notations.

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum term $\mu(r,f)$ of $f=\sum_{n=0}^{\infty} a_n z^n$ on $|z|=r$ is defined by $\mu(r,f)=\max_{n\geq 0}(|a_n|r^n)$. To start our paper we just recall the following definitions.

Definition 1 The order ρ_f and lower order λ_f of an entire function f are defined as

$$\rho_f=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r} \qquad and \qquad \lambda_f=\liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r}$$

where

$$\log^{[k]}x=\log(\log^{[k-1]}x) \text{ for } k=1,2,3,\dots \text{ and } \log^{[0]}x=x.$$

If $\rho_f < \infty$ then f is of finite order. Also $\rho_f = 0$ means that f is of order zero. In this connection Liao and Yang [6] gave the following definition.

Definition 2 [6] Let f be an entire function of order zero. Then the quantities ρ_f^* and λ_f^* of an entire function f are defined as:

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r} \quad and \quad \lambda_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}.$$

Datta and Biswas [2] gave an alternative definition of zero order and zero lower order of an entire function in the following way:

Definition 3 [2] Let f be an entire function of order zero. Then the quantities ρ_f^{**} and λ_f^{**} of f are defined by:

$$\rho_f^{**} = \limsup_{r\to\infty} \frac{\log M(r,f)}{\log r} \quad and \quad \lambda_f^{**} = \liminf_{r\to\infty} \frac{\log M(r,f)}{\log r}.$$

Since for $0 \leq r < R$,

$$\mu(r,f) \leq M(r,f) \leq \frac{R}{R-r}\mu(R,f) \quad \{cf. [9]\} \tag{1}$$

it is easy to see that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} \mu(r,f)}{\log r}, \qquad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} \mu(r,f)}{\log r};$$

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} \mu(r,f)}{\log^{[2]} r}, \qquad \lambda_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} \mu(r,f)}{\log^{[2]} r};$$

and

$$\rho_f^{**} = \limsup_{r\to\infty} \frac{\log \mu(r,f)}{\log r}, \qquad \lambda_f^{**} = \liminf_{r\to\infty} \frac{\log \mu(r,f)}{\log r}.$$

Definition 4 The type σ_f and lower type $\bar{\sigma}_f$ of an entire function f are defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}} \quad and \quad \bar{\sigma}_f = \liminf_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

Definition 5 [7] A function $\rho_f(r)$ is called a proximate order of f relative to $T(r,f)$ if

(i) $\rho_f(r)$ is non-negative and continuous for $r > r_0$, say,

(ii) $\rho_f(r)$ is differentiable for $r > r_0$ except possibly at isolated points at which $\rho'_f(r-0)$ and $\rho'_f(r+0)$ exist,

(iii) $\lim_{r\to\infty} \rho_f(r) = \rho_f < \infty$,

(iv) $\lim_{r\to\infty} r\rho'_f(r)\log r = 0$ *and*

(v) $\limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f(r)}} = 1$.

In the line of Definition 5 the following definition can be given.

Definition 6 A function $\lambda_f(r)$ is called a proximate order of f relative to $T(r,f)$ if

(i) $\lambda_f(r)$ is non-negative and continuous for $r > r_0$, say,

(ii) $\lambda_f(r)$ is differentiable for $r > r_0$ except possibly at isolated points at which $\lambda'_f(r-0)$ and $\lambda'_f(r+0)$ exist,

(iii) $\lim_{r\to\infty} \lambda_f(r) = \lambda_f < \infty$,

(iv) $\lim_{r\to\infty} r\lambda'_f(r)\log r = 0$ *and*

(v) $\limsup_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f(r)}} = 1$.

In this paper we investigate some aspects of the comparative growths of maximum terms of two entire functions with their corresponding left and right factors. We do not explain the standard notations and definitions on the theory of entire function because those are available in [10].

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [8] Let f and g be two entire functions with $g(0) = 0$. Then for all sufficiently large values of r,

$$\mu(r, f \circ g) \geq \frac{1}{2}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|, f\right).$$

Lemma 2 [1] If f and g are two entire functions then for all sufficiently large values of r,

$$M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0)|, f\right) \leq M(r, f \circ g) \leq M(M(r,g), f).$$

Lemma 3 [2] If f be any entire function of order zero. Then (i) $\rho_f^{*} = 1$ and (ii) $\lambda_f^{*} = 1$.

Lemma 4 [4] If f be an entire function then for $\delta(> 0)$ the function $r^{\rho_f+\delta-\rho_f(r)}$ is ultimately an increasing function of r.

Lemma 5 [5] Let f be an entire function. Then for $\delta(> 0)$ the function $r^{\lambda_f+\delta-\lambda_f(r)}$ is ultimately an increasing function of r.

Lemma 6 [3] Let g be an entire function with $\lambda_f < 1$ and assume that $a_i(i = 1,2,\ldots n;\ n \leq \infty)$ are entire functions satisfying $T(r,a_i) = o\{T(r,g)\}$. If $\sum_{i=1}^{n} \delta(a_i, g) = 1$, then

$$\lim_{r\to\infty} \frac{T(r,g)}{\log M(r,g)} = \frac{1}{\pi}.$$

3. Theorems.

In this section we present the main results of the paper.

Theorem 1 Let f and g be two entire functions with $\rho_f > 0$ and $0 < \rho_g < \infty$. Also let $0 < \bar{\sigma}_g \leq \sigma_g < \infty$. Then for any $n > 1$

$$\limsup_{r\to\infty} \frac{\log^{[2]}\mu(r, f\circ g)}{\log\mu(r,g)} \geq \frac{\bar{\sigma}_g\ \rho_f}{\sigma_g (4n)^{\rho_g}}.$$

Proof. In view of Lemma 1 we obtain for a sequence of values of r tending to infinity that

$$\log^{[2]}\mu(r, f\circ g) \geq \log^{[2]}\left\{\frac{1}{4}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right), f\right)\right\}$$

$$\text{i.e., } \log^{[2]}\mu(r, f\circ g) \geq (\rho_f - \varepsilon)\log\left\{\frac{1}{8}\mu\left(\frac{r}{4}, g\right)\right\} + O(1)$$

$$\text{i.e., } \log^{[2]}\mu(r, f\circ g) \geq \left(\rho_f - \varepsilon\right)\log\frac{1}{8} + \left(\rho_f - \varepsilon\right)\log\mu\left(\frac{r}{4}, g\right) + O(1). \qquad (2)$$

Again from the definition of lower type, we obtain from (1) for any $n > 1$ and for all sufficiently large values of r that

$$\log\mu\left(\frac{r}{4},g\right) \geq \log M\left(\frac{r}{4n},g\right) + O(1) \geq (\bar{\sigma}_g - \varepsilon)\left(\frac{r}{4n}\right)^{\rho_g} + O(1). \qquad (3)$$

Therefore from (2) and (3) it follows for a sequence of values of r tending to infinity,

$$\log^{[2]}\mu(r,f \circ g) \geq \left(\rho_f - \varepsilon\right)\log\frac{1}{8} + \left(\rho_f - \varepsilon\right)(\bar{\sigma}_g - \varepsilon)\left(\frac{r}{4n}\right)^{\rho_g} + O(1), \qquad (4)$$

where we choose $\varepsilon\ (> 0)$ in such a way that $0 < \varepsilon < min\left\{\rho_f, \bar{\sigma}_g\right\}$.
Also for all sufficiently large values of r,

$$\log\mu(r,g) \leq \log M(r,g) \leq (\sigma_g + \varepsilon)(r)^{\rho_g}. \qquad (5)$$

Now from (4) and (5) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]}\mu(r,f \circ g)}{\log\mu(r,g)} \geq \frac{\left(\rho_f - \varepsilon\right)\log\frac{1}{8} + \left(\rho_f - \varepsilon\right)(\bar{\sigma}_g - \varepsilon)\left(\frac{r}{4n}\right)^{\rho_g} + O(1)}{(\sigma_g + \varepsilon)(r)^{\rho_g}}. \qquad (6)$$

As $\varepsilon\ (> 0)$ is arbitrary, we obtain from (6) that

$$\limsup_{r\to\infty}\frac{\log^{[2]}\mu(r,f \circ g)}{\log\mu(r,g)} \geq \frac{\bar{\sigma}_g\,\rho_f}{\sigma_g(4n)^{\rho_g}}.$$

This proves the theorem.

In the line of Theorem 1 the following corollary may be deduced:

Corollary 1 Let f and g be two entire functions with $\lambda_f > 0$ and $0 < \rho_g < \infty$. Also let $0 < \sigma_g < \infty$. Then for any $n > 1$

$$\limsup_{r\to\infty}\frac{\log^{[2]}\mu(r,f \circ g)}{\log\mu(r,g)} \geq \frac{\lambda_f}{(4n)^{\rho_g}}.$$

For an entire function f of order zero the following corollary can also proved with the help of λ_f^{**} :

Corollary 2 Let f be an entire function of order zero such that $\lambda_f^{**} > 0$ and g be an entire function of non-zero finite order with $0 < \sigma_g < \infty$. Then for any $n > 1$

$$\limsup_{r\to\infty}\frac{\log\mu(r,f\circ g)}{\log\mu(r,g)} \geq \frac{\lambda_f^{**}}{(4n)^{\rho_g}}.$$

Remark 1 If we take $0 < \lambda_g \leq \rho_g < \infty$ instead of "non-zero finite order with $0 < \sigma_g < \infty$" in Corollary 2 and the other conditions remain the same then with the help of ρ_f^{*} and Lemma 3 the following holds:

$$\limsup_{r\to\infty}\frac{\log^{[2]}\mu(r,f\circ g)}{\log^{[2]}\mu(r,g)} \geq 1.$$

Theorem 2 Let f and g be two entire functions with $\rho_f < \infty$ and $0 < \rho_g < \infty$. Also let $0 < \bar{\sigma}_g \leq \sigma_g < \infty$. Then for any $n > 1$

$$\limsup_{r\to\infty}\frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,g)} \leq \frac{n^{\rho_g}\rho_f\sigma_g}{\bar{\sigma}_g}.$$

Proof. Since for $0 \leq r < R$,

$$\mu(r,f\circ g) \leq M(r,f\circ g) \leq \frac{R}{R-r}\,\mu(R,f\circ g) \quad \{\text{cf. } [9]\},$$

by Lemma 2 it follows for all sufficiently large values of r that

$$\mu(r,f\circ g) \leq M(r,f\circ g) \leq M(M(r,g),f)$$

$$\text{i.e., } \log^{[2]}\mu(r,f\circ g) \leq \log^{[2]}M(M(r,g),f)$$

$$\text{i.e., } \log^{[2]}\mu(r,f\circ g) \leq (\rho_f - \varepsilon)\log M(r,g). \tag{7}$$

Therefore from (5) and (7) we have for all sufficiently large values of r,

$$\log^{[2]}\mu(r,f\circ g) \leq (\rho_f - \varepsilon)(\sigma_g + \varepsilon)(r)^{\rho_g}. \tag{8}$$

Again for any $n > 1$, we obtain from (1) for all sufficiently large values of r

$$\log\mu(r,g) \geq \log M\left(\frac{r}{n},g\right) \geq \left(\bar{\sigma}_g - \varepsilon\right)\left(\frac{r}{n}\right)^{\rho_g}. \tag{9}$$

Now from (8) and (9) we obtain for all sufficiently large values of r,

$$\frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,g)} \leq \frac{\left(\rho_f - \varepsilon\right)\left(\sigma_g + \varepsilon\right)(r)^{\rho_g}}{\left(\bar{\sigma}_g - \varepsilon\right)\left(\frac{r}{n}\right)^{\rho_g}}$$

$$\text{i.e. } \limsup_{r\to\infty}\frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,g)} \leq \frac{n^{\rho_g}\,\rho_f\,\sigma_g}{\bar{\sigma}_g}.$$

This completes the proof.

The following theorem is a natural consequence of Theorem 1 and Theorem 2.

Theorem 3 Let f and g be two entire functions with $0 < \rho_f < \infty$ and $0 < \rho_g < \infty$. Also let $0 < \bar{\sigma}_g \leq \sigma_g < \infty$. Then for any $n > 1$

$$\frac{\bar{\sigma}_g\,\rho_f}{\sigma_g(4n)^{\rho_g}} \leq \limsup_{r\to\infty}\frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,g)} \leq \frac{n^{\rho_g}\,\rho_f\,\sigma_g}{\bar{\sigma}_g}.$$

The proof is omitted.

Now for an entire function of order zero the following theorem can be carried out in the line of Theorem 1 and Theorem 2.

Theorem 4 Let f be any entire function of order zero such that $0 < \rho_f^{**} < \infty$ and g be any entire function of non-zero finite order with $0 < \bar{\sigma}_g \leq \sigma_g < \infty$. Then for any $n > 1$

$$\frac{\bar{\sigma}_g\,\rho_f^{**}}{\sigma_g(4n)^{\rho_g}} \leq \limsup_{r\to\infty}\frac{\log\mu(r,f\circ g)}{\log\mu(r,g)} \leq \frac{n^{\rho_g}\,\rho_f^{**}\sigma_g}{\bar{\sigma}_g}.$$

Remark 2 If we take $0 < \lambda_g \leq \rho_g < \infty$ instead of "non-zero finite order with $0 < \bar{\sigma}_g \leq \sigma_g < \infty$" in Theorem 4 and the other conditions remain the same then with the help of ρ_f^* in terms of maximum term and Lemma 3 it can be carried out that

$$\frac{\lambda_g}{\rho_g} \leq \limsup_{r \to \infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r, g)} \leq \frac{\rho_g}{\lambda_g}.$$

Theorem 5 Let f and g be two entire functions with $\lambda_f > 0$ and $0 < \rho_g < \infty$. Also let $0 < \bar{\sigma}_g \leq \sigma_g < \infty$. Then for any $n > 1$

$$\limsup_{r \to \infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log \mu(r, g)} \geq \frac{\lambda_f \bar{\sigma}_g}{\sigma_g (4n)^{\rho_g}}.$$

Proof. By Lemma 1 we obtain for all sufficiently large values of r that

$$\log^{[2]} \mu(r, f \circ g) \geq \log^{[2]} \left\{ \frac{1}{4} \mu\left(\frac{1}{8} \mu\left(\frac{r}{4}, g \right), f \right) \right\}$$

$$\text{i.e., } \log^{[2]} \mu(r, f \circ g) \geq (\lambda_f - \varepsilon) \log \left\{ \frac{1}{8} \mu\left(\frac{r}{4}, g \right) \right\} + O(1)$$

$$\text{i.e., } \log^{[2]} \mu(r, f \circ g) \geq (\lambda_f - \varepsilon) \log \frac{1}{8} + (\lambda_f - \varepsilon) \log \mu\left(\frac{r}{4}, g \right) + O(1). \qquad (10)$$

Therefore from (3) and (10) it follows for all sufficiently large values of r ,

$$\log^{[2]} \mu(r, f \circ g) \geq (\lambda_f - \varepsilon) \log \frac{1}{8} + (\lambda_f - \varepsilon)(\bar{\sigma}_g - \varepsilon) \left(\frac{r}{4n} \right)^{\rho_g} + O(1). \qquad (11)$$

Combining (5) and (11) we obtain for all sufficiently large values of r

$$\frac{\log^{[2]} \mu(r, f \circ g)}{\log \mu(r, g)} \geq \frac{(\lambda_f - \varepsilon) \log \frac{1}{8} + (\lambda_f - \varepsilon)(\bar{\sigma}_g - \varepsilon) \left(\frac{r}{4n} \right)^{\rho_g} + O(1)}{(\sigma_g + \varepsilon)(r)^{\rho_g}}.$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,g)}\geq\frac{\lambda_f\,\bar{\sigma}_g}{\sigma_g(4n)^{\rho_g}}.$$

Thus the theorem follows.

Theorem 6 Let f and g be two entire functions with $\lambda_f<\infty$ and $0<\rho_g<\infty$. Also let $0<\bar{\sigma}_g\leq\sigma_g<\infty$. Then for any $n>1$

$$\liminf_{r\to\infty}\frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,g)}\leq\frac{n^{\rho_g}\,\lambda_f\sigma_g}{\bar{\sigma}_g}.$$

Proof. Since for $0\leq r<R$,

$$\mu(r,f\circ g)\leq M(r,f\circ g)\leq\frac{R}{R-r}\,\mu(R,f\circ g)\quad \{\text{cf. } [9]\},$$

by Lemma 2 and the above inequality it follows for a sequence of values of r tending to infinity,

$$\mu(r,f\circ g)\leq M(r,f\circ g)\leq M(M(r,g),f)$$

$$\text{i.e., }\log^{[2]}\mu(r,f\circ g)\leq\log^{[2]}M(M(r,g),f)$$

$$\text{i.e., }\log^{[2]}\mu(r,f\circ g)\leq(\lambda_f+\varepsilon)\log M(r,g). \tag{12}$$

Therefore from (5) and (12) we have for a sequence of values of r tending to infinity,

$$\log^{[2]}\mu(r,f\circ g)\leq(\lambda_f+\varepsilon)(\sigma_g+\varepsilon)(r)^{\rho_g}. \tag{13}$$

Now from (9) and (13) we obtain for a sequence of values of r tending to infinity

$$\frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,g)}\leq\frac{(\lambda_f+\varepsilon)(\sigma_g+\varepsilon)(r)^{\rho_g}}{(\bar{\sigma}_g-\varepsilon)\left(\frac{r}{n}\right)^{\rho_g}}$$

$$\text{i.e. }\liminf_{r\to\infty}\frac{\log^{[2]}\mu(r,f\circ g)}{\log\mu(r,g)}\leq\frac{n^{\rho_g}\,\rho_f\sigma_g}{\bar{\sigma}_g}.$$

This completes the proof.

The following theorem is a natural consequence of Theorem 5 and Theorem 6.

Theorem 7 Let f and g be two entire functions with $0 < \lambda_f < \infty$ and $0 < \rho_g < \infty$. Also let $0 < \bar{\sigma}_g \le \sigma_g < \infty$. Then for any $n > 1$

$$\frac{\lambda_f \, \bar{\sigma}_g}{\sigma_g (4n)^{\rho_g}} \le \liminf_{r\to\infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log \mu(r, g)} \le \frac{n^{\rho_g} \lambda_f \, \sigma_g}{\bar{\sigma}_g}.$$

The proof is omitted.

For an entire function f of order zero the following theorem can also be carried out in the line of Theorem 5 and Theorem 6.

Theorem 8 Let f be any entire function of order zero such that $0 < \lambda_f^{**} < \infty$ and g be any entire function of non -zero finite order with $0 < \bar{\sigma}_g \le \sigma_g < \infty$. Then for any $n > 1$

$$\frac{\lambda_f^{**} \, \bar{\sigma}_g}{\sigma_g (4n)^{\rho_g}} \le \liminf_{r\to\infty} \frac{\log \mu(r, f \circ g)}{\log \mu(r, g)} \le \frac{n^{\rho_g} \lambda_f^{**} \, \sigma_g}{\bar{\sigma}_g}.$$

Remark 3 If we take $0 < \lambda_g \le \rho_g < \infty$ instead of "non-zero finite order with $0 < \bar{\sigma}_g \le \sigma_g < \infty$" in Theorem 8 and the other conditions remain the same then with the help of λ_f^{*} in terms of maximum term and Lemma 3 it can be carried out that

$$\frac{\lambda_g}{\rho_g} \le \liminf_{r\to\infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r, g)} \le \frac{\rho_g}{\lambda_g}.$$

In the line of Theorem 6 the following two corollaries may be deduced:

Corollary 3 Let f and g be two entire functions with $0 < \rho_f < \infty$ and $0 < \rho_g < \infty$. Also let $0 < \sigma_g < \infty$. Then for any $n > 1$

$$\liminf_{r\to\infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log \mu(r, g)} \le n^{\rho_g} \rho_f.$$

Corollary 4 Let f be an entire function of order zero such that $0 < \rho_f^{**} < \infty$ and g be any entire function of non-zero finite order with $0 < \sigma_g < \infty$. Then for any $n > 1$

$$\liminf_{r\to\infty} \frac{\log \mu(r, f \circ g)}{\log \mu(r, g)} \le n^{\rho_g} \rho_f^{**}.$$

Remark 4 If we take $0 < \lambda_g \leq \rho_g < \infty$ instead of "non -zero finite order with $0 < \sigma_g < \infty$" in Corollary 4 and the other conditions remain the same then with the help of μ_f^* and Lemma 3 the following holds:

$$\liminf_{r\to\infty} \frac{\log^{[2]}\mu(r, f\circ g)}{\log^{[2]}\mu(r,g)} \leq 1.$$

Theorem 9 Let f be an entire function of order zero and g be entire such that ρ_g is finite. Also suppose that there exist entire functions $a_i (i = 1, 2, \ldots n; n \leq \infty)$ satisfying (A) $T(r, a_i) = o\{T(r,g)\}$ as $r \to \infty$ and (B) $\sum_{i=1}^{n} \delta(a_i, g) = 1$. Then for $0 \leq r < R$,

$$\liminf_{r\to\infty} \frac{\log\mu(r, f\circ g)}{\log\mu(R,g)} \leq \frac{3}{\pi}\rho_f^{**} 2^{\rho_g}.$$

Proof. If $\rho_f^{**} = \infty$, then the result is obvious. So we suppose that $\rho_f^{**} < \infty$.
Since for $0 \leq r < R$,

$$\mu(r, f\circ g) \leq M(r, f\circ g) \leq \frac{R}{R-r}\,\mu(R, f\circ g) \quad \{\text{cf. } [9]\},$$

we obtain by Lemma 2 for $\varepsilon\,(> 0)$ and for all sufficiently large values of r,

$$\log\mu(r, f\circ g) \leq \left(\rho_f^{**} + \varepsilon\right)\log M(r,g). \qquad (14)$$

Since $\limsup_{r\to\infty} \frac{T(r,g)}{r^{\rho_g(r)}} = 1$, for given $\varepsilon\ (0 < \varepsilon < 1)$ we get for all sufficiently large values of r,

$$T(r,g) < (1+\varepsilon) r^{\rho_g(r)} \qquad (15)$$

and for a sequence of values of r tending to infinity

$$T(r,g) > (1-\varepsilon) r^{\rho_g(r)}. \qquad (16)$$

Since $\log M(r,g) \leq 3T(2r, g)$, for a sequence of values of r tending to infinity we get for any $\delta\,(> 0)$,

$$\frac{\log M(r,g)}{T(r,g)} \leq \frac{3(1+\varepsilon)}{(1-\varepsilon)} \cdot \frac{(2r)^{\rho_g+\delta}}{(2r)^{\rho_g+\delta-\rho_g(2r)}} \cdot \frac{1}{(r)^{\rho_g(r)}} \leq \frac{3(1+\varepsilon)}{(1-\varepsilon)}(2)^{\rho_g+\delta},$$

because $(r)^{\rho_g+\delta-\varepsilon_g(r)}$ is ultimately an increasing function of r.
Now from above we have for a sequence of values of r tending to infinity,

$$\log M(r,g) \le \left\{\frac{3(1+\varepsilon)}{(1-\varepsilon)}(2)^{\rho_g+\delta}\right\} \mathrm{T}(r,g). \tag{17}$$

Now combining (14) and (17) it follows for a sequence of values of r tending to infinity that

$$\log \mu(r, f \circ g) \le \left(\rho_f^{**} + \varepsilon\right)\left\{\frac{3(1+\varepsilon)}{(1-\varepsilon)}(2)^{\rho_g+\delta}\right\} \mathrm{T}(r,g). \tag{18}$$

As $M(r,g) \le \frac{R}{R-r}\mu(R,g)$, it follows from (18) that

$$\frac{\log \mu(r, f \circ g)}{\log \mu(R,g)} \le \left(\rho_f^{**} + \varepsilon\right)\left\{\frac{3(1+\varepsilon)}{(1-\varepsilon)}(2)^{\rho_g+\delta}\right\} \frac{\mathrm{T}(r,g)}{\log M(r,g)}.$$

Since $\varepsilon\,(>0)$ and $\delta\,(>0)$ are both arbitrary we get from above that

$$\liminf_{r\to\infty} \frac{\log \mu(r, f \circ g)}{\log \mu(R,g)} \le 3.\rho_f^{**}.2^{\rho_g}.\liminf_{r\to\infty} \frac{\mathrm{T}(r,g)}{\log M(r,g)}. \tag{19}$$

Thus in view of Lemma 6 theorem follows from (19):

Corollary 5 Let f be an entire function of order zero and g be entire such that ρ_g is finite. Then for any $n > 1$

$$\liminf_{r\to\infty} \frac{\log \mu(r, f \circ g)}{\log \mu(r,g)} \le 3.\rho_f^{**}(2n)^{\rho_g}.$$

Proof. Putting $R = nr$ in the inequality $\mu(r,f) \le M(r,f) \le \frac{R}{R-r}\,\mu(R,f)$ {cf.[9]} and in view of the inequality $T(r,g) \le \log M(r,g)$ we get that

$$\log \mu(nr,g) + O(1) \ge \log M(r,g) \ge \mathrm{T}(r,g)$$

$$\text{i.e.,}\ \ \log \mu(r,g) + O(1) \ge \mathrm{T}\left(\frac{r}{n},g\right). \tag{20}$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain from (14) and (20) for all sufficiently large values of r

$$\liminf_{r\to\infty} \frac{\log \mu(r, f \circ g)}{\log \mu(r,g) + O(1)} \le \left(\rho_f^{**} + \varepsilon\right) \liminf_{r\to\infty} \frac{\log M(r,g)}{\mathrm{T}(\frac{r}{n},g)}$$

$$\text{i.e.,}\quad \liminf_{r\to\infty}\frac{\log\mu(r,f\circ g)}{\log\mu(r,g)}\le\rho_f^{**}\liminf_{r\to\infty}\frac{\log M(r,g)}{T(\frac{r}{n},g)}. \qquad (21)$$

Since $\log M(r,g)\le 3T(2r,g)$, in view of (15) and (16) we get for a sequence of values of r tending to infinity and for any $\delta\,(>0)$,

$$\frac{\log M(r,g)}{T\left(\frac{r}{n},g\right)}\le\frac{3(1+\varepsilon)}{(1-\varepsilon)}\cdot\frac{(2r)^{\rho_g+\delta}}{(2r)^{\rho_g+\delta-\rho_g(2r)}}\cdot\frac{1}{\left(\frac{r}{n}\right)^{\rho_g\left(\frac{r}{n}\right)}}$$

$$\le\frac{3(1+\varepsilon)}{(1-\varepsilon)}\frac{\left(\frac{2nr}{n}\right)^{\rho_g+\delta}}{\left(\frac{2nr}{n}\right)^{\rho_g+\delta-\rho_g\left(\frac{2nr}{n}\right)}}\cdot\frac{1}{\left(\frac{r}{n}\right)^{\rho_g\left(\frac{r}{n}\right)}}$$

$$\le\frac{3(1+\varepsilon)}{(1-\varepsilon)}(2n)^{\rho_g+\delta},$$

because $\left(\frac{2nr}{n}\right)^{\rho_g+\delta-\rho_g\left(\frac{2nr}{n}\right)}$ is ultimately an increasing function of r.

Since $\varepsilon\,(>0)$ and $\delta\,(>0)$ are both arbitrary, we obtain from above that

$$\liminf_{r\to\infty}\frac{\log M(r,g)}{T\left(\frac{r}{n},g\right)}\le 3.(2n)^{\rho_g}. \qquad (22)$$

Thus from (21) and (22) it follows that

$$\liminf_{r\to\infty}\frac{\log\mu(r,f\circ g)}{\log\mu(R,g)}\le 3.\rho_f^{**}(2n)^{\rho_g}.$$

This proves the Corollary.

Using the notion of lower proximate order the following theorem can be proved in the line of Theorem 9:

Theorem 10 Let f be an entire function of order zero and g be entire with $\lambda_g<\infty$. Also suppose that there exist entire functions $a_i\,(i=1,2,\ldots n;\ n\le\infty)$ satisfying (A) $T(r,a_i)=o\{T(r,g)\}$ as $r\to\infty$ and (B) $\sum_{i=1}^{n}\delta(a_i,g)=1$. Then for $0\le r<R$,

$$\liminf_{r\to\infty}\frac{\log\mu(r,f\circ g)}{\log\mu(R,g)}\le\frac{3}{\pi}\rho_f^{**}2^{\lambda_g}.$$

Corollary 6 Let f be an entire function of order zero and g be entire with $\lambda_g < \infty$. Then for any $n > 1$

$$\liminf_{r\to\infty}\frac{\log\mu(r, f\circ g)}{\log\mu(r,g)} \leq 3.\rho_f^{**}(2n)^{\lambda_g}.$$

The proof of Corollary 6 is omitted as it can be carried out in the line of Corollary 5.

Theorem 11 Let f and g be two non constant entire functions such that f is of lower order zero and λ_f^{**} and λ_g are finite. Then for $0 \leq r < R$,

$$\limsup_{r\to\infty}\frac{\log\mu(R, f\circ g)}{\log\mu(r,g)} \geq \frac{\lambda_f^{**}}{3.8^{\lambda_g}}.$$

Proof. If $\lambda_f^{**} = 0$ then the result is obvious. So we suppose that $\lambda_f^{**} = 0$> 0. Since for $0 \leq r < R$,

$$\mu(r,f) < M(r,f) < \frac{R}{R-r}\,\mu(R,f) \quad \{\text{cf. [9]}\}.$$

With the help of the above inequality and Lemma 2 and $\varepsilon\ (0 < \varepsilon < \min\{\lambda_f^{**}, 1\})$ we get for all sufficiently large values of r,

$$\log\mu(R, f\circ g) + O(1) \geq \log M(r, f\circ g) \geq \log M\left(\frac{1}{8}M\left(\frac{r}{2},g\right) - |g(0)|, f\right)$$

$$\text{i.e.,}\quad \log\mu(R, f\circ g) + O(1) \geq (\lambda_f^{**} - \varepsilon)\log\left\{\frac{1}{8}M\left(\frac{r}{2},g\right) - |g(0)|\right\}$$

$$\text{i.e.,}\quad \log\mu(R, f\circ g) + O(1) \geq (\lambda_f^{**} - \varepsilon)\log\left\{\frac{1}{9}M\left(\frac{r}{4},g\right)\right\}$$

$$\text{i.e.,}\quad \log\mu(R, f\circ g) + O(1) \geq (\lambda_f^{**} - \varepsilon)\log M\left(\frac{r}{4},g\right) + (\lambda_f^{**} - \varepsilon)\log\frac{1}{9}$$

$$\text{i.e.,}\quad \log\mu(R, f\circ g) + O(1) \geq (\lambda_f^{**} - \varepsilon)\,\mathrm{T}\left(\frac{r}{4},g\right) + O(1). \tag{23}$$

Since $\liminf_{r\to\infty}\frac{T(r,g)}{r^{\lambda_g(r)}} = 1$, for given $\varepsilon\ (> 0)$ we get for all sufficiently large values of r,

$$T(r,g) > (1-\varepsilon)r^{\lambda_g(r)} \tag{24}$$

and for a sequence of values of r tending to infinity

$$T(r,g) < (1+\varepsilon)r^{\lambda_g(r)}. \tag{25}$$

Now from (23) and (24) we get for $\delta\ (> 0)$ and for all sufficiently large values of r

$$\log\mu(R, f\circ g) + O(1) \geq (\lambda_f^{**} - \varepsilon)(1-\varepsilon)\frac{\left(\frac{2r}{8}\right)^{\lambda_g+\delta}}{\left(\frac{2r}{8}\right)^{\lambda_g+\delta-\lambda_g\left(\frac{2r}{8}\right)}}.$$

Since $(r)^{\lambda_g+\delta-\lambda_g(r)}$ is ultimately an increasing function of r it follows from above for all sufficiently large values of r that

$$\log\mu(R,f\circ g)+O(1)\geq\left(\lambda_f^{**}-\varepsilon\right)(1-\varepsilon)\frac{(2r)^{\lambda_g(2r)}}{(8)^{\lambda_g+\delta}}. \qquad (26)$$

So from (25) and (26) we get for a sequence of values of r tending to infinity that

$$\log\mu(R,f\circ g)+O(1)\geq\left(\lambda_f^{**}-\varepsilon\right)\frac{(1-\varepsilon)}{(1+\varepsilon)}\frac{T(2r,g)}{(8)^{\lambda_g+\delta}}.$$

Since $\log M(r,g)\leq 3T(2r,g)$ and $\mu(r,f)\leq M(r,f)\leq\frac{R}{R-r}\mu(R,f)$, we get from above for a sequence of values of r tending to infinity that

$$\log\mu(R,f\circ g)+O(1)\geq\left(\lambda_f^{**}-\varepsilon\right)\frac{1}{3}\frac{(1-\varepsilon)}{(1+\varepsilon)}\frac{\log M(r,g)}{(8)^{\lambda_g+\delta}}$$

$$\text{i.e.,}\quad \log\mu(R,f\circ g)+O(1)\geq\left(\lambda_f^{**}-\varepsilon\right)\frac{1}{3}\frac{(1-\varepsilon)}{(1+\varepsilon)}\frac{\log\mu(r,g)}{(8)^{\lambda_g+\delta}}.$$

Since $\varepsilon\ (>0)$ and $\delta\ (>0)$ are arbitrary it follows from above that

$$\limsup_{r\to\infty}\frac{\log\mu(R,f\circ g)+O(1)}{\log\mu(r,g)}\geq\frac{\lambda_f^{**}}{3.8^{\lambda_g}}$$

$$\text{i.e.,}\qquad \limsup_{r\to\infty}\frac{\log\mu(R,f\circ g)}{\log\mu(r,g)}\geq\frac{\lambda_f^{**}}{3.8^{\lambda_g}}.$$

Thus the theorem is proved.

In the line of Theorem 11 and Corollary 5 on can easily proof the following corollary.

Corollary 7 Let f and g be two non constant entire functions such that f is of lower order zero and λ_f^{**} and λ_g are finite. Then for any $n>1$

$$\limsup_{r\to\infty}\frac{\log\mu(r,f\circ g)}{\log\mu(r,g)}\geq\frac{\lambda_f^{**}}{3.(8n)^{\lambda_g}}.$$

The proof is omitted.

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