ISSN 0973-8975

J.Mech.Cont.& Math. Sci., Vol.-7, No.-1, January (2012) Pages 1025-1030

ON PAIRWISE ALMOST NORMALITY

By

[1]Ajoy Mukharjee and [2]Madhusudhan Paul

[1]Department of Mathematics, St. Joseph's College, Darjeeling, W. Bengal, India,

[2]Department of Mathematics, University of North Bengal,W. Bengal, India,

Abstract.

In this paper, we introduce the notion of pairwise almost normality which is a generalization of almost normality.

Keywords and phrases *: bitopological space, pairwise normal, pairwise almost normal*

বিমূর্ত সার (*Bengali version of the Abstract*)

এই পত্রে আমরা যুগল প্রায় স্বাভাবিকতার (almost normality) ধারণাকে প্রায় স্বাভাবিকতার সাধারণীকরণ (generalization) হিসাবে উপস্থাপন করেছি ।

1. Introduction

The systematic study of bitopological spaces was initiated by Kelly [5]. A bitopological space (X,P_1,P_2) is a set X equipped with two topologies P_1 and P_2. Several authors have contributed a lot to enrich the bitopological setting by introducing remarkable theories and ideas. Kelly [5] introduced the notions of separation axioms in bitopological spaces by generalizing the notion of separation axioms of topological spaces. In this paper, we introduce some separation axioms in bitopological spaces weaker than the separation axioms due to Kelly [5].

Unless or otherwise mentioned, X stands for the bitopological space (X,P ,P).

2. Preliminaries

We recall the following known definitions.

Definition 1 (Singal and Singal [6]). A set $A\subset X$ is said to be (P_i,P_j)regularly open if $A=(P_i)\text{int}((P_j)\text{cl}A)$.

A subset of X is said to be (P_i,P_j)regularly closed if its complement is (P_i,P_j)regularly open. In other words, a set $A\subset X$ is (P_i,P_J)regularly closed iff $A=(P_i)\mathrm{cl}((P_j)\mathrm{int}A)$.

Definition 2 (Singal and Singal). The bitopological space X is said to be pairwise almost regular if for each $x\in X$ and each (P_i,P_J)regularly closed set A with $x\notin A$, there exist a (P_j)open set U and a (P_j)open set V such that $x\in U, B\subset V$ and $U\cap V=\emptyset$.

Definition 3 (Kelly [5]). X is said to be pairwise normal if for any pair of a (P_i)closed set A and a (P_j)closed set B with $A\cap B=\emptyset$, $i\neq j$, there exist $U\in P_j$ and $V\in P_i$ such that $A\subset U, B\subset V$ and $U\cap V=\emptyset$.

Definition 4 (Fletcher, Hoyle III and Patty [4]). A cover U of X is pairwise open if $U\subset P_1\cup P_2$ and for each $i\in\{1,2\}$, $U\cap P_i$ contains a nonempty set.

Definition 5 (Datta [3]). A pairwise open cover V of X is said to be a parallel refinement of a pairwise open cover U of X if every (P_i)open set of V is contained in some (P_i)open set of U.

Definition 6 (Bose et al. [1]). A subcollection C of a refinement V of a pairwise open cover U of X is U-locally finite if for each $x\in X$, there exists a neighbourhood of x intersecting a finite number of members of C, the neighbourhood being (P_i)open if x belongs to a (P_i)open set of U.

Definition 7 (Bose et al. [1]). The bitopological space X is pairwise paracompact if every pairwise open cover U of X has a U-locally finite parallel refinement.

In the sequel, we use the following theorems.

Theorem 8 (Kelly [5]). If E is (P_i)closed and F is (P_j)closed with $E\cap F=\square$ in the pairwise normal space X, then there exists a real valued function g on X such that

(i) $g(x)=0$ for $x\square E$, $g(x)=1$ for $x\square F$ and $0\leq g(x)\leq 1$ for all $x\square X$.

(ii) g is (P_j)upper semicontinuous and (P_i)lower semicontinuous.

We now introduce the following definitions.

Definition 9. The bitopological space X is said to be pairwise seminormal if for any pair of a (P_i)closed set A and a (P_j)open set B with $A\subset B$, there exists a (P_i)open set U such that $A\square U\square$ $(P_j)(\mathrm{int})((P_i)\mathrm{cl}U)\square B$.

Clearly, a pairwise normal space is pairwise seminormal.

Definition 10. The bitopological space X is said to be pairwise almost normal if for any pair of a (P_i)closed set A and a (P_j, P_i)regularly closed set B with $A \cap B = \emptyset$, there exist a (P_j)open set U and a (P_i)open set V such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

It readily follows, X is pairwise almost normal iff for any pair of a (P_i)closed set A and a $(P_j\ P_i)$regularly open set B with $A \subset B$, there exists a (P_i)open set U such that $A \square U \square (P_i)clU \square B$.

Obviously, a pairwise normal space is pairwise almost normal.

Example 11 . (Bose and Mukharjee [2]) For any $a \square R$, we define

$P_1 = \{\emptyset, R, (-\infty, a], (a, \infty)\}$

$P_2 = \{\emptyset, R, (-\infty, a), [a, \infty)\}$.

Here the bitopological space (R, P_1, P_2) is pairwise almost normal but the space is not pairwise normal.

Definition 12. Let (X, P_1, P_2) and (Y, Q_1, Q_2) be two bitopological spaces. A function $f:(X, P_1, P_2) \rightarrow (Y, Q_1, Q_2)$ is said to be a pairwise homeomorphism if $f:(X, P_\square) \rightarrow (Y, Q_\square)$ and $f:(X, P_\square) \rightarrow (Y, Q_\square)$ are homeomorphism.

3. Results

Theorem 13. In the bitopological space X, following statements are equivalent:

(a) X is pairwise almost normal.

(b) For each (P_i)closed set A and each (P_j, P_i)regularly open set B with $A \subset B$, there exists a (P_j)open set U such that $A \subset U \subset (P_i)clU \subset B$.

(c) For each (P_j, P_i)regularly closed set A and each (P_i)open set B with $A \subset B$, there exists a (P_i)open set U such that $A \subset U \subset (P_j)clU \subset B$.

(d) For a (P_i)closed set A and a (P_i, P_j)regularly closed set B with $A \cap B = \emptyset$, there exist a (P_j)open set U and a (P_i)open set V such that $A \subset U, B \subset V$ and $(P_i)clU \cap ((P_j))clV = \emptyset$.

Proof. (a)$\square$(b): Straightforward and hence omitted.

(b)□(c): A is a (P_i,P_i)regularly closed set and B is a (P_i)open set with A□B. Then X-B is (P_i)closed and X-A is (P_j, P_i)regularly open with $X\text{-}B\subset X\text{-}A$. So by ($b$), there exists a (P_j)open set G such that $X\text{-}B\subset G\subset(P_i)\mathrm{cl}G\subset X\text{-}A$. Hence we have $A\subset X\text{-}(P_i)\mathrm{cl}G\subset X\text{-}G\subset B$. We put $X\text{-}(P_i)\mathrm{cl}G=U$. Thus $(P_j)\mathrm{cl}U\subset X\text{-}G$. So we obtain $A\subset U\subset(P_j)\mathrm{cl}U\subset B$.

(c)⇒(d): A is a (P_j,P_i)regularly closed set and B is a (P_i)closed set with $A\cap B=\emptyset$. Then X-B is (P_i)open and $A\subset X\text{-}B$. So by (c), there exist a (P_i)open set G such that $A\subset G\subset(P_j)\mathrm{cl}G\subset X\text{-}B$. Now considering the pair (A,G), we obtain a (P_i)open set U such that $A\subset U\subset(P_j)\mathrm{cl}U\subset G$. Then we have $B\subset X\text{-}(P_i)\mathrm{cl}\,G\subset X\text{-}G$. Hence on putting $V=X\text{-}(P_j)\mathrm{cl}G$, we see A□U,B□V with $(P_j)\mathrm{cl}U \cap (P_i)\mathrm{cl}V=$□.

(d)□(a): Obvious.

Theorem 14. A pairwise almost regular and pairwise paracompact space X is pairwise almost normal.

Proof. Let A be (P_i)closed and B be (P_j,P_i)regularly closed with $A\cap B=\emptyset$. For $x\in A$, we have $(P_i)\mathrm{cl}\{x\}\subset X\text{-}B$. Now using pairwise almost regularity of X, we obtain a (P_j)open set G_x and a (P_i)open set H_x such that $x\in G_x, B\subset H_x$ and $G_x\cap H_x=\emptyset$ which in turn implies $(P_i)\mathrm{cl}G_x\cap B=\emptyset$. Also $G_x\subset (P_i)\mathrm{cl}G_x\subset X\text{-}B$. So $\mathrm{U}=\{G_x|x\in A\}\cup\{X\text{-}A\}$ is a pairwise open cover of X. So using pairwise paracompactness of X, we get a U-locally finite parallel refinement V of U. We put $G=\bigcup\{V\in \mathrm{V}|V\cap A\neq\emptyset\}$. So G is (P_j)open with $A\subset G$. Now we consider a point $y\in B$. Then y belongs to the (P_i)open set X-A of the cover U and so there exists a (P_i)open nbd D_y of y intersecting a finite number of elements $V_1(y),V_2(y),\ldots,V_m(y)$ of V such that for $k=1,2,\ldots,m$, $A\cap V_k(y)\neq\emptyset$. As $A\cap V_k(y)\neq\emptyset$ and V is a parallel refinement of U, it follows that $V_k(y)\subset G_{xk}$ for some $x_k\in A$. Thus $(P_i)\mathrm{cl}V_k(y)\subset(P_i)\mathrm{cl}G_{xk}\subset X\text{-}B$. Hence $B\subset X\text{-}(P_i)\mathrm{cl}V_k(y)$ for each $k=1,2,\ldots, m$. Now on putting $H=\bigcap_{k=1}^{m}(X-(P_i)\mathrm{cl}V_k(y))$, we obtain $B\subset H$ and $G\cap H=$□. Therefore X is pairwise almost normal.

Theorem 15. If $f:X\rightarrow Y$ is a homeomorphism and X is pairwise almost normal, then Y is also pairwise almost normal.

Proof. Suppose A is (Q_i)closed and B is (Q_j,Q_i)open with $A\subset B$. So $f^{-1}(A)$ is (P_i)closed and $f^{-1}(B)$ is (P_j)open with $f^{-1}(A)\subset f^{-1}(B)$. Since $f^{-1}(B)$ is (P_j)open, we have $f^{-1}(A)\subset(P_j)\mathrm{int}((P_i)\mathrm{cl}(f^{-1}(B)))$ and $(P_j)\mathrm{int}((P_i)\mathrm{cl}(f^{-1}(B)))$ is (P_j,P_i)regularly open. Now using the pairwise almost normality of X, we get a (P_j)open set U such that $f^{-1}(A)\subset U\subset(P_i)\mathrm{cl}U\subset(P_j)\mathrm{int}((P_i)\mathrm{cl}(f^{-1}(B)))$ which in turns implies $A\subset f(U)\subset f((P_i)\mathrm{cl}U)\subset f((P_j)\mathrm{int}((P_i)\mathrm{cl}(f^{-1}(B))))$. Since f is a homeomorphism, we have the following results:

(i) $f(U)$ is (Q_j)open,

(ii) $f((P_i)\mathrm{cl}U)=(P_i)\mathrm{cl}(f(U))$,

(iii) $f((P_j)\mathrm{int}((P_i)\mathrm{cl}(f^{-1}(B)))=B$.

Thus $A\subset f(U)\subset(P_i)\mathrm{cl}(f(U))\subset B$. So Y is almost pairwise normal.

Theorem 16. The pairwise almost normal space X is pairwise normal iff it is pairwise seminormal.

Proof. We need only to prove `if' part of the theorem. Let A be (P_j)closed and B be (P_i)open with $A\subset B$. Then pairwise seminormality of X ensure the existence of a (P_i)open set U such that $A\subset U\subset(P_i)\mathrm{int}((P_j)\mathrm{cl}U)\subset B$. Now it follows, $(P_i)\mathrm{int}((P_j)\mathrm{cl}U)$ is a (P_i,P_j)regularly open set. So by pairwise almost normality we obtain a (P_i)open set V with $A\subset V\subset(P_i)\mathrm{cl}V\subset(P_i)\mathrm{int}((P_i)\mathrm{cl}U)$. Hence pairwise normality of X follows.

Theorem 17. Let X be pairwise almost normal and A be (P_i, P_j)regularly closed $(P_j)G_\delta$ subset of X. Then for $x\notin A$, there exists a functions f on X into $[0,1]$ such that $f^{-1}(0)=A$, $f(x)=1$ and f is (P_j)upper semicontinuous and (P_i)lower semicontinuous.

Proof. Let A be (P_i, P_j)regularly closed $(P_j)G_\delta$ subset of the pairwise almost normal space X. Let $x\notin A$. Since A is $(P_j)G_\delta$ subset of X, there exists a countable collection $\{G_n|n\in N\}$ of (P_j)open sets such that $A=\bigcap_n G_n$. Since $A=\bigcap_n G_n$, we have $A\subset G_n$ for all $n\in N$. Again $x\notin A\Rightarrow x\notin G_m$ for some $m\in N$. We write, $V_1=\bigcap_{i=1}^{m} G_i$. So $x\notin V_1\Rightarrow V_1\subset X-\{x\}$. Also, $A\subset V_1$ and V_1 is (P_j)open. So by almost pairwise normality of X, we obtain a (P_j)open set H_1 such that $A\subset H_1\subset(P_i)\mathrm{cl}H_1\subset V_1$. Now, on putting $V_2=G_{m+1}\cap H_1$, we obtain $A\subset V_2\subset G_{m+1}$. Also, $V_2\subset H_1\subset(P_i)\mathrm{cl}H_1\subset V_1\Rightarrow V_2\subset(P_i)\mathrm{cl}V_2\subset V_1$ and hence $V_2\subset(P_i)\mathrm{cl}V_2\subset V_1\subset X-\{x\}$. In this way, we can obtain, (P_j)open sets $V_1,V_2,\ldots,V_r$ and (P_j)open sets $H_1,H_2,\ldots,H_{r-1}$ such that $A\subset V_l\subset G_{m+l-1}$, $l\in\{1,2,\ldots,r\}$ and $V_l\subset H_{l-1}\subset(P_i)\mathrm{cl}H_{l-1}\subset V_{l-1}\Rightarrow V_l\subset(P_i)\mathrm{cl}V_l\subset V_{l-1}\subset\ldots\subset V_1\subset X-\{x\}$. Now, considering $A\subset V_r$, we obtain a (P_j)open set H_r such that $A\subset H_r\subset(P_i)\mathrm{cl}H_r\subset V_r$. We now write, $V_{r+1}=G_{m+r}\cap H_r$. Then $A\subset V_{r+1}\subset G_{m+r}$. Also, $V_{r+1}\subset H_r\subset(P_i)\mathrm{cl}H_r\subset V_r\Rightarrow V_{r+1}\subset(P_i)\mathrm{cl}V_{r+1}\subset V_r$ So we get, $V_{r+1}\subset(P_i)\mathrm{cl}V_{r+1}\subset V_r\subset\ldots\subset V_1\subset X-\{x\}$. Thus by induction, we obtain a sequence $\{V_n|n\in N\}$

of (P_j)open sets with $A \subset V_n \subset G_{m+n-1}$ and $(P_i)\mathrm{cl}V_n \subset (P_i)\mathrm{cl}H_{r-1} \subset V_n$. Therefore, $A \subset \bigcap_n V_n \subset \bigcap_n G_m \Rightarrow A = \bigcap_n V_n$. Now following the prove of Theorem 8, we may obtain the function f.

Acknowledgements.

The authors are grateful to Prof. M. K. Bose, Department of Mathematics, University of North Bengal for his kind help and valuable suggestions towards the preparation of this article.

Reference

1) M. K. Bose, A. Roy Choudhury and A. Mukharjee, On bitopological paracompactness, Mat.Vesnik **60** (2008), 255-259.

2) M. K. Bose and Ajoy Mukharjee, On bitopological full normality, Mat. Vesnik **60** (2008), 11-18.

3) M. C. Datta, Paracompactness in bitopological spaces and an application to quasi-metric spaces, Indian J. Pure Appl. Math. (6) **8** (1977), 685-690.

4) P. Fletcher, H. B. Hoyle III and C. W. Patty, The comparison of topologies, Duke Math. J. **36** (1969), 325-331.

5) J. C. Kelly, Bitopological spaces, J. Lond. Math. Soc. (3) **13** (1963), 71-89.

6) M. K. Singal and Asha Rani Singal, Some more separation axioms in bitopological spaces, Ann. Soc. Sci. Brux., **84** (1970), 207-230.