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ON SEMI PRIME IDEALS IN LATTICES

By

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Abstract :

Recently Yehuda Rav has given the concept of Semi-prime ideals in a general lattice by generalizing the notion of 0-distributive lattices. In this paper we study several properties of these ideals and include some of their characterizations. We give some results regarding maximal filters and include a number of Separation properties in a general lattice with respect to the annihilator ideals containing a semi-prime ideal.

Keywords and phrases : semi-prime ideals, 0-distributive lattices, annihilator ideals.

বিমূর্ত সার *(Bengali version of the Abstract)*

বর্তমানে Yehuda Rav সাধারণ ল্যাটিসের অর্ধ - মৌলিক আইডিয়েলস্ -এর (Semi - Prime Ideals) একটি ধারণা দিয়েছেন 0 - বন্টিত ল্যাটিসের (0-distributive lattices) ধারণাকে সাধারণীকরণের সাহায্যে । এই পত্রে আমরা এই সব আইডিয়েলস্ - এর বহুবিধ ধর্মকে বিচার বিশ্লেষণ করেছি এবং ইহাদের কিছু বৈশিষ্ট্যকে অর্ন্তভুক্ত করেছি । সর্বাধিক পরিমান ফিলটার প্রসঙ্গে কিছু ফলাফল প্রকাশ করেছি এবং অর্ধ - মৌলিক আইডিয়েলস্ - এর ধারণকারী এনিহিলেটর আইডিয়েলস্ -এর (annihilator ideals) সাপেক্ষে সাধারণ ল্যাটিসের বিচ্ছেদের ধর্মাবলীকে অর্ন্তভুক্ত করেছি ।

1. Introduction:

In generalizing the notion of pseudo complemented lattice, J. C. Varlet [4] introduced the notion of 0-distributive lattices. Then [1] have given several characterizations of these lattices. On the other hand, [2] have studied them in meet semi lattices. A lattice L with 0 is called a *0-distributive* lattice if for all $a,b,c \in L$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$. Of course every distributive lattice is 0-distributive. 0-distributive lattice L can be characterized by the fact that the set of all elements disjoint to $a \in L$ forms an ideal. So every pseudo complemented lattice is 0-distributive.

Recently, V. Rav [3] has generalized this concept and gave the definition of semi prime ideals in a lattice. For a non-empty subset I of L, I is called a *down set* if for $a \in I$ and $x \leq a$ imply $x \in I$. Moreover I is an *ideal* if $a \vee b \in I$ for all $a,b \in L$. Similarly, F is called a *filter* of L if for $a,b \in F$, $a \wedge b \in F$ and for $a \in F$ and $x \geq a$ imply $x \in F$. F is called a *maximal filter* if for any filter $M \supseteq F$ implies either $M = F$ or $M = L$. A proper ideal(down set) I is called a *prime ideal(down set)* if for $a,b \in L$, $a \wedge b \in I$ imply either $a \in I$ *or* $b \in I$. A prime ideal P is called a *minimal prime ideal* if it does not contain any other prime ideal. Similarly, a proper filter Q is called a *prime filter* if $a \vee b \in Q$ ($a,b \in L$) implies either $a \in Q$ or $b \in Q$. It is very easy to check that F is a filter of L if and only if L-F is a prime down set. Moreover, F is a prime filter if and only if L-F is a prime ideal.

An ideal I of a lattice L is called a *semi prime ideal* if for all $x, y, z \in L,$ $x \wedge y \in I$ and $x \wedge z \in I$ imply $x \wedge (y \vee z) \in I$. Thus, for lattice L with 0, L is called *0-distributive* if and only if $(0]$ is a semi prime ideal. In a distributive lattice L, every ideal is a semi prime ideal. Moreover, every prime ideal is semi prime. In a pentagonal lattice $\{0, a, b, c, 1; a < b\}$, is semi prime but not prime. Here (b] and (c] are prime, but (a] is not even semi prime. Again in $M_3 = \{0, a, b, c, 1; a \wedge b = b \wedge c = a \wedge c = 0; a \vee b = a \vee c = b \vee c = 1\}$, $(a]$, $(b]$, $(c]$ are not semi prime.

Lemma 1. *Non empty intersection of all prime(semi prime) ideals of a lattice is a semi-prime ideal.*

Proof: Let $a,b,c \in L$ and $I = \bigcap\{P : P$ is a prime ideal$\}$ and I is nonempty. Let $a \wedge b \in I$ and $a \wedge c \in I$. Then $a \wedge b \in P$ and $a \wedge c \in P$ for all P. Since each P is prime (semi prime), so $a \wedge (b \vee c) \in P$ for all P. Hence $a \wedge (b \vee c) \in I$, and so I is semi-prime.

Corollary 2. *Intersection of two prime(semi prime) ideals is a semi-prime ideal.*

Lemma 3. *Every filter disjoint from an ideal I is contained in a maximal filter disjoint from I.*

Proof: Let F be a filter in L disjoint from I. Let F be the set of all filters containing F and disjoint from I. Then F is nonempty as $F \in$ F. Let C be a chain in F and let $M = \bigcup(X : X \in C)$. We claim that M is a filter. Let $x \in M$ and $y \geq x$. Then $x \in X$ for some $X \in C$. Hence $y \in X$ as X is a filter. Therefore, $y \in M$. Let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in C$. Since C is a chain, either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$. So $x, y \in Y.$ Then $x \wedge y \in Y$ and so $x \wedge y \in M$. Moreover, $M \supseteq F$. So M is a maximum element of C. Then by Zorn's Lemma, F has a maximal element, say $Q \supseteq F$.

Theorem 4. *Let A be a non-empty subset of a lattice L and J be an ideal of L. Then* $A^{\perp_J} = \bigcap(P : P \text{ is minimal prime down set containing } J \text{ but not containing } A)$.

Proof. Suppose $X = \bigcap(P : A \not\subset P, P \text{ is a minimal prime down set})$. Let $x \in A^{\perp_J}$. Then

$x \wedge a \in J$ for all $a \in A$. Choose any P of right hand expression. Since $A \not\subset P$, there exists $z \in A$ but $z \notin P$. Then $x \wedge z \in J \subseteq P.$ So $x \in P$, as P is prime. Hence $x \in X$.

Conversely, let $x \in X.$ If $x \notin A^{\perp_J}$, then $x \wedge b \notin J$ for some $b \in A$. Let D = [$x \wedge b$).
Hence D is a filter disjoint from J. Then by Lemma 3, there is a maximal filter $M \supseteq D$ but disjoint from J. Then L-M is a minimal prime down set containing J. Now $x \notin L - M$ as $x \in D$ implies $x \in M$. Moreover , A $\nsubseteq$ L-M as $b \in A$, but $b \in M$ implies $b \notin L - M$, which is a contradiction to $x \in X$. Hence $x \in A^{\perp_J}$.

Lemma 5. *Let I be an ideal of a lattice L. A filter M disjoint from I is a maximal filter disjoint from I if and only if for all $a \notin M$, there exists $b \in M$ such that $a \wedge b \in I$.*

Proof: Let M be maximal and disjoint from I and $a \notin M$. Let $a \wedge b \notin I$ for $b \in M$. Consider $M_1 = \{y \in L : y \geq a \wedge b, \quad b \in M\}$. Clearly M_1 is a filter. For any $b \in M$, $b \geq a \wedge b$ implies $b \in M_1$. So $M_1 \supseteq M$. Also $M_1 \cap I = \phi$. For if not, let $x \in M_1 \cap I$. This implies $x \in I$ and $x \geq a \wedge b$ for some $b \in M$. Hence $a \wedge b \in I$, which is a contradiction. Hence $M_1 \cap I \neq \phi$. Now $M \subset M_1$ because $a \notin M$ but $a \in M_1$. This contradicts the maximality of M. Hence there exists $b \in M$ such that $a \wedge b \in I$.

Conversely, if M is not maximal disjoint from I, then there exists a filter $N \supset M$ and disjoint with I. For any $a \in N - M$, there exists $b \in M$ such that $a \wedge b \in I$. Hence, $a, b \in N$ implies $a \wedge b \in I \cap N$, which is a contradiction. Hence M must be a maximal filter disjoint with I.

Let L be a lattice with 0. For $A \subseteq L$, We define $A^{\perp} = \{x \in L : x \wedge a = 0 \text{ for all } a \notin A\}$. $A^{\perp}$ is always down set of L. Moreover, it is convex but it is not necessarily an ideal.

Theorem 6. *Let L be a pseudo complemented lattice. Then for $A \subseteq L$, $A^{\perp} = \{x \in L : x \wedge a = 0 \text{ for all } a \notin A\}$ is a semi-prime ideal.*

Proof: We have already mentioned that $A^{\perp}$ is a down set of L. Since L is pseudo complemented if it is 0-distributive. Now let $x, y \in A^{\perp}$. Then $x \wedge a = 0 = y \wedge a$ for all $a \in L$. Hence $a \wedge (x \vee y) = 0$ for all $a \in A$. This implies $x \vee y \in A^{\perp}$ and so $A^{\perp}$ is an ideal.

Now let $x \wedge y \in A^{\perp}$ and $x \wedge z \in A^{\perp}$. Then $x \wedge y \wedge a = 0 = x \wedge z \wedge a$ for all $a \in A$. This implies $y \leq (x \wedge a)^*, z \leq (x \wedge a)^*$ and so $y \vee z \leq (x \wedge a)$ and this implies $x \wedge a \wedge (y \vee z) = 0$ for all $a \in L$. Hence $x \wedge (y \vee z) \in A^{\perp}$ and so $A^{\perp}$ is a semi prime ideal.

Let $A \subseteq L$ and J be an ideal of L. We define $A^{\perp_J} = \{x \in L : x \wedge a \in J \text{ for all } a \in A\}$. This is clearly a down set containing J. In presence of distributivity, this is an ideal. $A^{\perp_J}$ is called an annihilator of A relative to J. We denote $I_J(L)$, by the set of all ideals containing J. Of course, $I_J(L)$ is a bounded lattice with J and L as the smallest and the largest elements. If

$A \in I_J(L)$, and $A^{\perp_J}$ is an ideal, then $A^{\perp_J}$ is called an annihilator ideal and it is the pseudo complement of A in $I_J(L)$.

Following Theorem gives some nice characterizations semi prime ideals.

Theorem 7. *Let L be a lattice and J be an ideal of L. The following conditions are equivalent.*
(i) J is semi prime.
(ii) $\{a\}^{\perp_J} = \{x \in L : x \wedge a \in J\}$ is a semi prime ideal containing J.
(iii) $A^{\perp_J} = \{x \in L : x \wedge a \in J \text{ for all } a \in A\}$ is a semi prime ideal containing J.
(iv) $I_J(L)$ is pseudo complemented
(v) $I_J(L)$ is a 0 –distributive lattice.
(vi) Every maximal filter disjoint from J is prime.

Proof: (i)$\Rightarrow$(ii). $\{a\}^{\perp_J}$ is clearly a down set containing J. Now let $x, y \in \{a\}^{\perp_J}$. Then $x \wedge a \in J,\ y \wedge a \in J$. Since J is semi prime, so $a \wedge (x \vee y) \in J$. This implies $\{a\}^{\perp_J}$ is an ideal containing J. Now let $x \wedge y \in \{a\}^{\perp_J}$ and $x \wedge z \in \{a\}^{\perp_J}$. Then $x \wedge y \wedge a \in J$ and $x \wedge z \wedge a \in J$. Thus, $(x \wedge a) \wedge y \in J$ and $(x \wedge a) \wedge z \in J$. Then $(x \wedge a) \wedge (y \vee z) \in J$, as J is semi prime. This implies $x \wedge (y \vee z) \in \{a\}^{\perp_J}$, and so $\{a\}^{\perp_J}$ is semi prime.

(ii)$\Rightarrow$(iii). This is trivial by Lemma 1, as $A^{\perp_J} = \bigcap(\{a\}^{\perp_J}; a \in A)$.

(iii)$\Rightarrow$(iv). Since for any $A \in I_J(L),\ A^{\perp_J}$ is an ideal, it is the pseudo complement of A in $I_J(L)$, so $I_J(L)$ is pseudo complemented.

(iv)$\Rightarrow$(v). This is trivial as every pseudo complemented lattice is 0-distributive.

(v)$\Rightarrow$(vi). Let $I_J(L)$ is 0-distributive. Suppose F is a maximal filter disjoint from J. Suppose $f, g \notin F$. By Lemma 5, there exist $a, b \in F$ such that $a \wedge f \in J, b \wedge g \in J$. Then $f \wedge a \wedge b \in J,\ g \wedge a \wedge b \in J$. Hence $(f] \wedge (a \wedge b] \subseteq J \text{ and } (g] \wedge (a \wedge b] \subseteq J$. Then

$(f \vee g] \wedge (a \wedge b] = ((f] \vee (g]) \wedge (a \wedge b] \subseteq J$, by the 0-distributive property of $I_J(L)$. Hence, $(f \vee g) \wedge a \wedge b \in J$. This implies $f \vee g \notin F$ as $F \cap J = \varphi$, and so F is prime.

(vi) $\Rightarrow$ (i) Let (vi) holds. Suppose $a, b, c \in L$ with $a \wedge b \in J$, $a \wedge c \in J$. If $a \wedge (b \vee c) \notin J$, then $[a \wedge (b \vee c)) \cap J = \varphi$. Then by Lemma 3, there exists a maximal filter $F \supseteq [a \wedge (b \vee c))$ and disjoint from J. Then $a \in F, b \vee c \in F$. By (vi) is prime. Hence either $a \wedge b \in F$ or $a \wedge c \in F$. In any case $J \cap F \neq \phi$, which gives a contradiction. Hence $a \wedge (b \wedge c) \in J$, and so J is semi prime.

Corollary 8: *In a lattice L, every filter disjoint to a semi-prime ideal J is contained in a prime filter.*

Proof: This immediately follows from Lemma 3 and theorem 7.

Theorem 9: I*f J is a semi-prime ideal of a lattice L and $J \neq A = \bigcap\{J_\lambda : J_\lambda$ is an ideal containing $J\}$, Then $A^{\perp_J} = \{x \in L : \{x\}^{\perp_J} \neq J\}$.*

Proof: Let $x \in A^{\perp_J}$. Then $x \wedge a \in J$ for all $a \in A$. So $a \in \{x\}^{\perp_J}$ for all $a \in A$. Then $A \subseteq \{x\}^{\perp_J}$ and so $\{x\}^{\perp_J} \neq J$. Conversely, let $x \in L$ such that $\{x\}^{\perp_J} \neq J$. Since J is semi-prime, so $\{x\}^{\perp_J}$ is an ideal containing J. Then $A \subseteq \{x\}^{\perp_J}$, and so $A^{\perp_J} \supseteq \{x\}^{\perp_J \perp_J}$. This implies $x \in A^{\perp_J}$, which completes the proof.

[1] have provided a series of characterizations of 0-distributive lattices. Here we give some results on semi prime ideals related to their results.

Theorem 10. *Let L be a lattice and J be an ideal. Then the following conditions are equivalent.*

(i) J is semi-prime.

(ii) Every maximal filter of L disjoint with J is prime

(iii) Every minimal prime down set containing J is a minimal prime ideal containing J

(iv) Every filter disjoint with J is disjoint from a minimal prime ideal containing J.

(v) For each element $a \notin J$, there is a minimal prime ideal containing J but not containing a.

(vi) *Each* $a \notin J$ *is contained in a prime filter disjoint to J.*

Proof. (i)$\Leftrightarrow$(ii) follows from Theorem 7.
$(ii) \Rightarrow (iii)$. Let A be a minimal prime down set containing J. Then S-A is a maximal filter disjoint with J . Then by (ii) S-A is prime and so A is a minimal prime ideal.
$(iii) \Rightarrow (ii)$. Let F be a maximal filter disjoint with J. Then S-F is a minimal prime down set containing J. Thus by (iii), S-F is a minimal prime ideal and so F is a prime filter.
$(i) \Rightarrow (iv)$. Let F a filter of S disjoint from J. Then by Corollary 8, there is a prime filter $Q \supseteq F$ and disjoint from F.
$(iv) \Rightarrow (v)$. Let $a \in L$, $a \notin J$. Then $[a) \cap J = \varphi$. Then by (iv) there exists a minimal prime ideal A disjoint from $[a)$. Thus $a \notin A$.
$(v) \Rightarrow (vi)$. Let $a \in L$, $a \notin J$. Then by (v) there exists a minimal prime ideal P such that $a \notin P$. Implies $a \in L - P$ and L-P is a prime filter.
$(vi) \Rightarrow (i)$. Suppose J is not semi-prime . Then there exists $a, b, c \in L$ such that $a \wedge b \in J$, $a \wedge c \in J$ but $a \wedge (b \vee c) \notin J$. Then by (vi) there exists a prime filter Q disjoint from J and $a \wedge (b \vee c) \in Q$. Let $F = [a \wedge (b \vee c))$. Then $J \cap F = \varphi$ and $F \subseteq Q$. Now $a \wedge (b \vee c) \in Q$ implies $a \in Q$, $b \vee c \in Q$. Since Q is prime so either $a \wedge b \in Q$ or $a \wedge c \in Q$. Which gives a contradiction to the fact that $Q \cap J = \varphi$. Therefore , $a \wedge (b \vee c) \in J$ and so J is semi-prime.

Now we give another characterization of semi-prime ideals with the help of Prime Separation Theorem using annihilator ideals.

Theorem 11: *Let* J *be an ideal in a lattice* L. J *is semi- prime if and only if for all filter* F *disjoint to* $\{x\}^{\perp_J}$*, there is a prime filter containing F disjoint to* $\{x\}^{\perp_J}$.

Proof: Using Zorn's Lemma we can easily find a maximal filter Q containing F and disjoint to $\{x\}^{\perp_J}$. We claim that $x \in Q$. If not, then $Q \vee [x) \supset Q$. By maximality of Q, $(Q \vee [x)) \cap \{x^{\perp_J}\} \neq \varphi$. If $t \in (Q \vee [x)) \cap \{x\}^{\perp_J}$, then $t \geq q \wedge x$ for some $q \in Q$ and

$t \wedge x \in J$. This implies $q \wedge x \in J$ and so $q \in \{x\}^{\perp_J}$ gives a contradiction. Hence $x \in Q$.

Now let $z \notin Q$. Then $(Q \vee [z)) \cap \{x\}^{\perp_J} \neq \varphi$. Suppose $y \in (Q \vee [z)) \cap \{x\}^{\perp_J}$ then $y \geq q_1 \wedge z$ & $y \wedge z \in J$ for some $q_1 \in Q$. This implies $q_1 \wedge x \wedge z \in J$ and $q_1 \wedge x \in Q$. Hence by Lemma 5, Q is a maximal filter disjoint to $\{x\}^{\perp_J}$. Then by Theorem 7, Q is prime.

Conversely, let $x \wedge y \in J, x \wedge z \in J$. If $x \wedge (y \vee z) \notin J$, then $y \vee z \notin \{x\}^{\perp_J}$. Thus $[y \vee z) \cap \{x\}^{\perp_J} = \varphi$. So there exists a prime filter Q containing $[y \vee z)$ and disjoint from $\{x\}^{\perp_J}$. As $y, z \in \{x\}^{\perp_J}$, *so* $y, z \notin Q$. Thus $y \vee z \notin Q$, as Q is prime. This implies $[y \vee z) \not\subset Q$, a contradiction. Hence $x \wedge (y \vee z) \in J$, and so J is semi-prime.

We conclude the paper with the following characterization of semi- prime ideals.

Theorem 12. *Let J be a semi-prime ideal of a lattice L and $x \in L$. Then a prime ideal P containing $\{x\}^{\perp_J}$ is a minimal prime ideal containing $\{x\}^{\perp_J}$ if and only if for $p \in P$, there exists $q \in L - P$ such that $p \wedge q \in \{x\}^{\perp_J}$.*

Proof: Let P be a prime ideal containing $\{x\}^{\perp_J}$ such that the given condition holds. Let K be a prime ideal containing $\{x\}^{\perp_J}$ such that $K \subseteq P$. Let $p \in P$. Then there is $q \in L - P$ such that $p \wedge q \in \{x\}^{\perp_J}$. Hence $p \wedge q \in K$. Since K is prime and $q \notin K$, so
$p \in K$. Thus, $P \subseteq K$ and so $K = P$. Therefore, P must be a minimal prime ideal containing $\{x\}^{\perp_J}$.
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Conversely, let P be a minimal prime ideal containing $\{x\}^{\perp_J}$. Let $p \in P$. Suppose for all $q \in L - P$, $p \wedge q \notin \{x\}^{\perp_J}$. Let $D = (L - P) \vee [p)$. We claim that $\{x\}^{\perp_J} \cap D = \varphi$. If not, let $y \in \{x\}^{\perp_J} \cap D$. Then $p \wedge q \leq y \in \{x\}^{\perp_J}$, which is a contradiction to the assumption. Then by Theorem 11, there exists a maximal (prime) filter $Q \supseteq D$ and disjoint to $\{x\}^{\perp_J}$. By the proof of Theorem 11, $x \in Q$. Let $M = S - Q$. Then M is a prime ideal. Since $x \in Q$,so $t \wedge x \in J \subseteq M$ implies $t \in M$ as M is prime. Thus $\{x\}^{\perp_J} \subseteq M$. Now $M \cap D = \varphi$. This implies $M \cap (L - P) = \varphi$ and hence $M \subseteq P$. Also $M \neq P$, because $p \in D$ implies $p \notin M$ but $p \in P$. Hence M is a prime ideal containing $\{x\}^{\perp_J}$ which is properly contained in P. This gives a contradiction to the minimal property of P. Therefore the given condition holds.

References.

1) Balasubramani P. and Venkatanarasimhan P.V., Characterizations of the 0-Distributive Lattices, Indian J. pure appl.Math. 32(3) 315-324, (2001).

2) Powar Y.S. and Thakare N. K., 0-Distributive semilattices, Canad. Math. Bull. Vol.21(4) (1978), 469-475.

3) Rav Y., Semi prime ideals in general lattices, Journal of pure and Applied Algebra, 56(1989) 105- 118.

4) Varlet J. C., A generalization of the notion of pseudo-complementedness, Bull. Soc. Sci. Liege, 37(1968), 149-158.