ISSN 0973-8975
J.Mech.Cont. & Math. Sci., Vol.-7, No.-2, Januaryy (2013) Pages 1067-1076

1- DISTRIBUTIVE JOIN – SEMI LATTICE

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Abstract

In this paper, we have studied some properties of ideals and filters of a join-semilattice. We have also introduced the notion of dual annihilator. We have discussed 1-distributive join-semilattice and given several characterizations of 1-distributive join-semilattices directed below. Finally we have included a generalization of prime separation theorem in terms of dual annihilators.

Keywords and phrases : ideals, join-semilattice, 1-distributive lattice, dual annihilator

বিমূর্ত সার ***(Bengali version of the Abstract)***

এই পত্রে আমরা যুক্ত - অর্ধ ল্যাটিসের (join-semilattice) আইডিয়েলস্ এবং ফিল্টারের (ideals and filters) ধর্মাবলীকে পুঙ্খানুপুঙ্খ ভাবে পরীক্ষা করেছি । 1 - বন্টিত যুক্ত - অর্ধ ল্যাটিসের এবং নিম্নাভিমুখী 1 - বন্টিত যুক্ত - অর্ধ ল্যাটিসের প্রদত্ত বিভিন্ন বিশিষ্টায়নগুলিকে আলোচনা করেছি । সর্বশেষে আমরা মুখ্য বিচ্ছেদ উপপাদ্যের সাধারণীকরণকে দ্বৈত এনিহিলেটর (dual annihilators.) হিসাবে অর্ন্তভুক্ত করেছি ।

1. Introduction

Varlet [7] have given the definition of a 1-distributive lattice.Then Balasubramani et al [1]have established some results on this topic. A lattice L with 1 is called a 1-distributive lattice if for all $a,b,c \in L$ with

$a \vee b = 1 = a \vee c$ impliy $a \vee (b \wedge c) = 1$. Any distributive lattice with 1 is 1-distributive. In this paper we will study the 1-distributive join-semilattices.

An ordered set $(S, \leq)$ is said to be a join-semilattice if sup{*a*,*b*}exists for all $a, b \in S$, we write $a \vee b$ in place of sup{*a*,*b*}.

A join-semilattice *S* is called distributive if $a \leq b_1 \vee b_2 (a, b_1, b_2 \in S)$ implies the existence of $a_1, a_2 \in S, a_1 \leq b_1, a_2 \leq b_2$ with $a = a_1 \vee a_2$. For detailed literature on join-semilattices, we refer the reader to consultTalukder et al. [5,6], Noor et al. [3] and Gratzer [2].

A join-semilattice *S* with 1 is said to be 1-distributive if for any $a, b, c \in S$ such that $a \vee b = 1 = a \vee c$ implies $a \vee d = 1$ for some $d \leq b, c$.

Consider the join-semilattices S_1 and S_2 given in the Figure 1.1. It can be easily seen that S_1 is not 1-distributive but S_2 is 1-distributive.

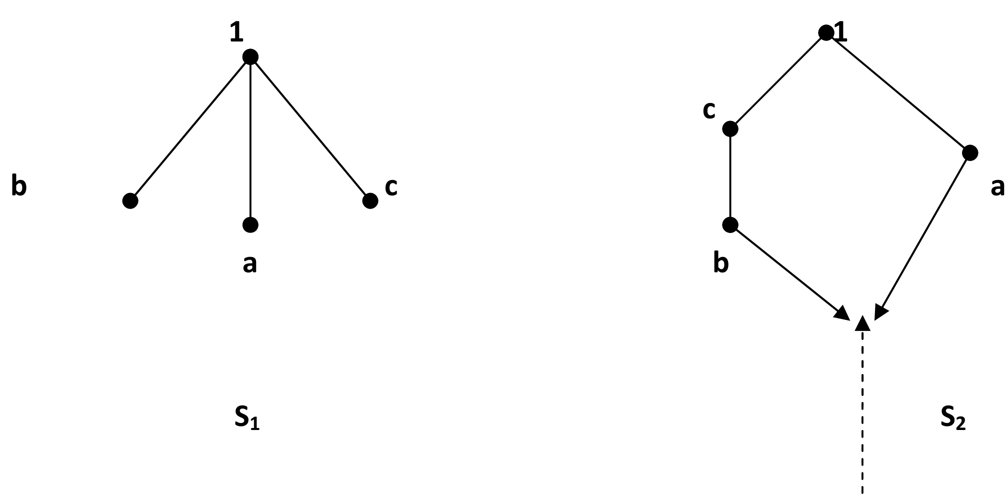

Figure 1.1

Both distributive and modular joinsemilatticesshare a common property " for all $a,b \in S$ there exists $c \in S$ such that $c \le a,b$". This property is known as the directed below property. Hence a join semilatticewith this property is known as a directed below semilattice. Observe that in Figure 1.1, S_1 is not directed below, but S_2 is directed below.

A subset *I* of a join-semilattice*S* is called an upset if $x \in I$ and $y \in S$ with $x \le y$ implies $y \in I$.

Let *S* be a join-semilattice. A non-empty subset *F* of *S* is called a filter if

(i)F is an upset,

and(ii) $a,b \in F$ implies there exists $d \le a,b$ such that $d \in F$.

A filter *F* is called proper filterof a join-semilatttice *S*if $F \neq S$.

A proper filter (upset)*F* in *S* is called a prime filter (upset) if $a \vee b \in F$ implies either $a \in F$ or $b \in F$.

For $a \in S$, the filter $F = \{x \in S \mid x \ge a\}$is called the principal filter generated by *a*. It is denoted by [*a*).A prime upset (filter) is called a minimal prime upset (filter) if it does not contain any other prime upset (filter).

A subset *I* of*S* is called an ideal if

(i) $a, b \in I$ implies $a \vee b \in I$ (ii) $a \in S, i \in I$ with $a \le i$ implies $a \in I$.

An ideal *I* of a join-semilattice S is called prime ideal if $I \neq S$ and $S - I$ isa prime filter.

A maximal ideal *I*of *S*is a proper ideal which is not contained in any other proper ideal. That is, if there is a proper ideal *J* such that $I \subseteq J$, then $I = J$.

Let *S* be a join-semilattice with 1. For $A \subseteq S$,

set$A^{\perp^d} = \{x \in S \mid x \vee a = 1 \text{ for all } a \in A\}$. Then $A^{\perp^d}$ is called the dual annihilator of *A*.. This is always an upset but not necessarily a filter.

For $a \in S$, we denote $\{a\}^{\perp^d} = \{x \in S \mid x \vee a = 1\}$.

Moreover, $A^{\perp^d} = \cap_{a \in A} \{\{a\}^{\perp^d}\}$.

2. Some properties of ideals and filters of a join-semilattice

Lemma 2.1. Let S be a join-semilattice with 1. Then every prime upset contains a minimal prime upset.

Proof. Let F be a prime upset of S and let A denote the set of all prime upsets Q contained in F. Then A is non-empty as $F \in A$. Let C be a chain in A and let $M = \cap\{X \mid X \in C\}$. We claim that M is a prime upset. M is non-empty as $1 \in M$. Let $a \in M$ and $a \leq b$. Then $a \in X$ for all $X \in C$. Hence $b \in X$ for all $X \in C$ as X is an upset. Thus $b \in M$. Again, let $x \vee y \in M$ for some $x, y \in S$. Then $x \vee y \in X$ for all $X \in C$. Since X is prime upset, so either $x \in X$ or $y \in X$ this implies either $x \in M$ or $y \in M$. Hence M is a prime upset. Therefore, we can apply to A the dual form of Zorn's Lemma to conclude the existence of a minimal member of A.

Theorem 2.2. Let S be a directed below join-semilattice. Then the intersection of any two filters of S is also a filter.

Proof. Let F, Q be two filters of a directed below join-semilattice S. Let $a \in F \cap Q$ and $b \in S$ with $b \geq a$. Then $a \in F$ and $a \in Q$. Since both F and Q are filters, so $b \in F$ and $b \in Q$. Hence $b \in F \cap Q$.

Again let $a, b \in F \cap Q$. So $a, b \in F$ and $a, b \in Q$. Since F and Q are both filters, then there exists $f \in F$ and $q \in Q$ such that $f, q \leq a, b$. Let $c = f \wedge q$. Then $c \in F \cap Q$, where $c \leq a, b$. Hence $F \cap Q$ is a filter.

Lemma 2.3. Let I be a non-empty proper subset of a join-semilattice S. Then I is an ideal if and only if $S - I$ is a prime upset.

Proof. Let I be an ideal of a join-semilattice S. Now let $x \in S - I$ and $x \leq y$. Then $x \notin I$, so $y \notin I$ as I is an ideal. Hence $y \in S - I$. Thus $S - I$ is an upset. Since I is an ideal, so $S - I \neq S$. Therefore $S - I$ is a proper upset. Let $a, b \in S$ with

$a \vee b \in S - I$, then $a \vee b \notin I$. Therefore either $a \notin I$ or $b \notin I$ as I is an ideal. Hence either $a \in S - I$ or $b \in S - I$. Therefore $S - I$ is a prime upset.

Conversely, let $S - I$ is a prime upset and $x, y \in I$, then $x, y \notin S - I$. Thus $x \vee y \notin S - I$ as $S - I$ is a prime upset. Hence $x \vee y \in I$. Again, let $x \in I$ and $y \leq x$. Then $x \notin S - I$. Therefore $y \notin S - I$ as $S - I$ is an upset. Hence $y \in I$ and thus I is an ideal.

Thus we have the following result.

Corollary 2.4. Let I be a non-empty subset of a join-semilattice S. Then I is a maximal ideal if and only if $S - I$ is a minimal prime upset.

Theorem 2.5. Every proper ideal of a join-semilattice S with 1 is contained in a maximal ideal.

Proof. Let I be a proper ideal in S with 1. Let P be the set of all proper ideals containing I. Then P is non-empty as $I \in P$. Let C be a chain in P and let $M = \cup \{X | X \in C\}$.

We claim that M is an ideal with $I \subseteq M$. Let $x \in M$ and $y \leq x$. Then $x \in X$ for some $X \in C$. Hence $y \in X$ as X is an ideal. Therefore $y \in M$. Again let $x, y \in M$, then $x \in X$ and $y \in Y$ for some $X, Y \in C$. Since C is a chain, so either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$, so $x, y \in Y$. Then $x \vee y \in Y$ as Y is an ideal. Hence $x \vee y \in M$. Moreover, M contains I, so M is maximal element of C. Then by Zorn's Lemma, P has a maximal element, say Q with $I \subseteq Q$. □

Now we give a characterization of maximal ideals of a join semilattice.

Theorem 2.6. Let S be a join-semilattice with 1. A proper ideal M in S is maximal if and only if for any element $a \in S - M$ there exists an element $b \in M$ such that $a \vee b = 1$.

Proof. Suppose M is maximal and $a \notin M$. Let $a \vee b \neq 1$ for all $b \in M$. Consider $M_1 = \{y \in S |\ y \leq a \vee b, \text{for some } b \in M\}$. Clearly M_1 is an ideal and is proper as $1 \notin M_1$. For every $b \in M$, we have $b \leq a \vee b$ and so $b \in M_1$. Thus

$M \subseteq M_1$.Also $a \notin M$ but $a \in M_1$. So $M \subset M_1$ which contradicts the maximality of *M*. Hence there must exists some $b \in M$ such that $a \vee b = 1$.

Conversely, if the proper ideal *M*is not maximal, then as $1 \in S$, there exists a maximal ideal *N* such that $M \subset N$. For any element $a \in N - M$ there exists an element $b \in M$ such that $a \vee b = 1$.Hence$a, b \in N$ imply $1 = a \vee b \in N$ which is a contradiction. Thus *M* must be a maximal ideal .

3. Some characterizations of 1-distributive join-semilattices.

In this section ,we prove our main results of this paper.

Theorem 3.1. Every 1-distributive join-semilattice is directed below.

Proof. Let *S* be a 1-distributive join-semilatticeand $b, c \in S$. Then $1 \vee b = 1 \vee c = 1$ which implies there exists $d \in S$ with $d \leq b, c$ such that $1 \vee d = 1$. Thus *d* is lower bound of *b*,*c*. Hence *S* is directed below.

The converse of the above theorem is not true by S_2 of Figure 1.1.

Theorem 3.2. Let $a, a_1, a_2, \cdots, a_n$ be elements of a 1-distributive join-semilattice *S* such that $a \vee a_1 = a \vee a_2 = \cdots = a \vee a_n = 1$. Then $a \vee b = 1$ for some $b \leq a_1, a_2, \cdots, a_n$.

Proof.We want to prove this theorem using mathematical induction method. Let $a \vee a_1 = a \vee a_2 = 1$. Since *S* is 1-distributive so, $a \vee b_1 = 1$ for some $b_1 \leq a_1, a_2$, that is, the statement is true for a_1 and a_2.Let $a \vee a_1 = a \vee a_2 = \cdots = a \vee a_{k-1}$=1. Then for the 1-distributivity of *S*,$a \vee b_2 = 1$ for some $b_2 \leq a_1, a_2, \cdots, a_{k-1}$. Now, suppose $a \vee a_k = 1$. Hence $a \vee b = 1$ for some $b \leq b_2, a_k$ as *S* is 1-distributive.This implies that $a \vee b = 1$ for some $b \leq a_1, a_2, \cdots, a_k$. Hence by the method of mathematical induction the theorem is true for some $b \leq a_1, a_2, \cdots, a_n$.□

Following result gives some nice characterizations of 1-distributive join-semilattices.

Theorem 3.3. For a directed below join-semilattice*S* with 1, the following conditions are equivalent: (i)*S* is 1-distributive.(ii)$\{a\}^{\perp^d}$is a filter for all $a \in S$.(iii)$A^{\perp^d}$is a filter for all finite subsets*A* of *S*(iv)Every maximal ideal is prime.

Proof.$(i) \Leftrightarrow (ii)$. Let $x \in \{a\}^{\perp^d}$ and $y \geq x$.Since $x \in \{a\}^{\perp^d}$, so we get $a \vee x = 1$ implies $a \vee y = 1$ as $y \geq x$.Hence $y \in \{a\}^{\perp^d}$, and so $\{a\}^{\perp^d}$ is an upset. Again, let $x, y \in \{a\}^{\perp^d}$. Thus $a \vee x = 1 = a \vee y$. By 1-distributivity of *S*, there exists z with $z \leq x, y$ such that $a \vee z = 1$. Therefore $z \in \{a\}^{\perp^d}$, and so $\{a\}^{\perp^d}$ is a filter.

Conversely let $x, y, z \in S$ with $x \vee y = 1 = x \vee z$. Then $y, z \in \{x\}^{\perp^d}$. Since $\{x\}^{\perp^d}$ is a filter, so there exists $t \leq y, z$ such that $t \in \{x\}^{\perp^d}$, and so $t \vee x = 1$. This implies S is 1-distributive.

$(ii) \Leftrightarrow (iii)$is trivial by Theorem 2.2 as $A^{\perp^d} = \bigcap_{a \in A} \{a\}^{\perp^d}$.

$(i) \Rightarrow (iv)$.Let I be a maximal ideal of S. Then by Corollary 2.4, S-I is a minimal prime upset. Now suppose $x, y \in S - I$. Then $x, y \notin I$, and so by the maximality of I, $I \vee (x] = S$, $I \vee (y] = S$. This implies $a \vee x = 1 = b \vee y$ for some $a, b \in I$. Thus $a \vee b \vee x = a \vee b \vee y = 1$. Since S is 1-distributive, there exists $d \leq x, y$ such that $a \vee b \vee d = 1$.

Now $a \vee b \in I$ implies $a \vee b \notin S - I$, and S-I is prime implies $d \in S - I$. Therefore S-I is a prime filter and so I is a prime ideal.

$(iv) \Rightarrow (i)$. Let S be not 1-distributive. Then there are $a, b, c \in S$ such that $a \vee b = a \vee c = 1$ and $a \vee d \neq 1$ for all $d \leq b, c$. Now, set $I = \{x \in S \mid x \leq a \vee y, y \leq b, c\}$.Clearly I is an ideal and it is proper as $1 \notin I$.By

Theorem 2.5, $I \subseteq J$ for some maximal ideal *J*. Now we claim that either $b \in J$ or $c \in J$. If $b, c \notin J$, then $b, c \in S - J$. As J is a prime ideal, then we have $S - J$ is a prime filter and $b, c \in S - J$. Since $S - J$ is a filter, there is $e \in S - J$ such that

$a \leq b, c$. Hence $a \vee e \in S - J$ gives a contradiction. Hence, $b \notin J$ or $c \notin J$ this implies, either $a \vee b \notin J$ or $a \vee c \notin J$. Thus, $1 \notin J$ which contradicts the maximality of *J*. Therefore, $a \vee d = 1$ for some $d \leq b, c$ and hence *S* is a 1-distributive.□

Note that in case of a 1-distributive lattice L, For any $A \subseteq L$, $A^{\perp^d}$ is a filter. But this is not true in a directed below join-semilattice S with 1, as the intersection of infinite number of filters in S is not necessarily a filter.

Corollary 3.4. In a 1-distributive join-semilattice, every proper ideal is contained in a prime ideal.

Proof. This immediately follows by Theorem 2.5 and Theorem 3.3.□

Theorem 3.5. In a 1-distributive join-semilattice*S* if $\{1\} \neq A$ is the intersection of all filters of S not equal to {1}, then $A^{\perp^d} = \left\{x \in S \,\middle|\, \{x\}^{\perp^d} \neq \{1\}\right\}$.

Proof. Let $x \in A^{\perp^d}$. Then $x \vee a = 1$ for all$a \in A$. Since $A \neq \{1\}$, so$\{x\}^{\perp^d} \neq \{1\}$.Thus $x \in R.H.S.$ That is,$A^{\perp^d} \subseteq R.H.S.$

Conversely, let $x \in R.H.S.$ so$\{x\}^{\perp^d} \neq \{1\}$. Also since *S* is 1-distributive then $\{x\}^{\perp^d}$ is a filter of *S*. Hence$A \subseteq \{x\}^{\perp^d}$ and so $A^{\perp^d} \supseteq \{x\}^{\perp^d \perp^d}$. This implies $x \in A^{\perp^d}$, Thus $R.H.S \subseteq A^{\perp^d}$which completes the proof.

Finally we give a necessary and sufficient condition for a join-semilattice*S* with 1, to be 1-distributive which is a generalization of Pawar and et al. [4;Theorem 7].

Theorem 3.6 Let *S* be a join-semilattice with 1. Then *S* is 1-distributive if and only if for any ideal *I* disjoint with $\{x\}^{\perp^d} (x \in S)$, there exists a prime ideal containing *I* and disjoint with $\{x\}^{\perp^d}$.

Proof. Suppose *S* is 1-distributive join-semilattice. Let *P* be the set of all ideals containing *I*, but disjoint from $\{x\}^{\perp^d}$. Clearly *P* is non-empty as $I \in P$. Let *C* be a chain in P and let $M = \cup\{X \mid X \in C\}$. First we claim that M is an ideal with $I \subseteq M$ and $M \cap \{x\}^{\perp^d} = \phi$. Let $x \in M$ and $y \leq x$. Then $x \in X$ for some $X \in C$. Hence $y \in X$ as X is an ideal. Thus $y \in M$. Again, let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in C$. Since C is a chain, so either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$, so $x, y \in Y$. Then $x \vee y \in Y$ as Y is an ideal. Hence $x \vee y \in M$. Thus M is an ideal. Moreover, M contains I and $M \cap \{x\}^{\perp^d} = \phi$. Then by Zorn's lemma, there exists a maximal element Q in P. Hence by Zorn's lemma as in Theorem 2.5, there exists a maximal ideal P containing I and disjoint from $\{x\}^{\perp^d}$. We claim that $x \in P$. If not , then $P \vee (x]$ is an ideal containing P . By the maximality of P, $(P \vee (x]) \cap \{x\}^{\perp^d} \neq \varphi$. Let $t \in (P \vee (x]) \cap \{x\}^{\perp^d}$.Then $t \leq p \vee x$ for some $p \in P$ and $t \vee x = 1$.This implies $p \vee x = 1$ and so $p \in \{x\}^{\perp^d}$, which is a contradiction. Now suppose $y \notin P$. Then $(P \vee (y]) \cap \{x\}^{\perp^d} \neq \varphi$ by the maximality of P. Let $s \in (P \vee (y]) \cap \{x\}^{\perp^d}$.Then $s \leq p_1 \vee y$ for some $p_1 \in P$ and $s \vee x = 1$. This implies $(p_1 \vee x) \vee y = 1$. Since $p_1 \vee x \in P$, so by Theorem 2.6, P is a maximal ideal of S. Therefore by Theorem 3.3, P is a prime ideal.

Conversely, let $x, y, z \in S$ such that $x \vee y = 1$, $x \vee z = 1$. Suppose for all $d \leq y, z$ we have $x \vee d \neq 1$. Then $d \notin \{x\}^{\perp^d}$. Set $I = \{a \in S \mid a \leq x \vee e, \text{for all } e \leq y, z\}$. First we claim that I is a proper ideal. Clearly I is non-empty as $x \in I$. Let $p \in I$ and $q \leq p$. Then $p \leq x \vee e$ and so $q \leq x \vee e$. Thus $p \vee q \leq x \vee e$. Hence $p \vee q \in I$. Therefore I is an ideal and I is a proper ideal as $1 \notin I$. Again $x \in I$ and $e \in I$ for all $e \leq y, z$. Then $\{x\}^{\perp^d} \cap I = \phi$ and hence there is a prime ideal J such that $I \subseteq J$ and $\{x\}^{\perp^d} \cap J = \phi$. Thus $x \in J$ and $e \in J$ for all $e \leq y, z$. Now we claim that either $y \in J$ or $z \in J$. If $y, z \notin J$ then $y, z \in S - J$. As J is a prime ideal, then $S - J$ is a prime filter and $y, z \in S - J$. Since $S - J$ is a filter, there is $f \in S - J$ such that $f \leq y, z$ which is a contradiction. Hence either $y \in J$ or $z \in J$. This implies either $x \vee y \in J$ or $x \vee z \in J$. Thus $1 \in J$ which contradicts the primeness of J. Hence $x \vee d = 1$. Thus S is 1-distributive.

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