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METHOD OF SOLUTION TO THE OVER-DAMPED NONLINEAR VIBRATING SYSTEM WITH SLOWLY VARYING COEFFICIENTS UNDER SOME CONDITIONS

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Abstract:

A simple analytical solution extended to certain damped-oscillatory nonlinear systems with varying coefficients. The solution obtained for different initial conditions for a second order nonlinear system show a good coincidence with those obtained by numerical method. The method is illustrated by an example.

Keywords and phrases : damped nonlinear systems, KBM method, eigen-value.

বিমূর্ত সার *(Bengali version of the Abstract)*

পরিবর্তনশীল সহগ সহ কোন অবমন্দিত দোলনের অ-রৈখিক তন্ত্রের ক্ষেত্রে একটি সহজ বিশ্লেষণাত্মক সমাধানকে বর্ধিত করা হয়েছে। বিভিন্ন প্রারম্ভিক শর্তের জন্য দ্বিতীয় ক্রমের অ-রৈখিক তন্ত্রের নির্ণিত সমাধান দেখায় যে সাংখ্য পদ্ধতিতে নির্ণিত ইহার সমাধান একই। এই পদ্ধতিটিকে একটি উদাহরণের সাহায্যে ব্যাখ্য করা হয়েছে।

1. INTRODUCTION

The Krylov-Bogoliubov-Mitroplshkii (KBM) [1-3] method is a widely used technique to obtain approximate solutions of weakly nonlinear systems. Originally, the method was developed by Krylov and Bogoliubov [1] for obtaining periodic solution of a second order nonlinear differential equation. Letter, the method was amplified and justified mathematically by Bogoliubov and Mitropolishkii [2-3]. Popov [4] extended the method to a damped oscillatory process in which a strong linear damping force acts. Murty, Dekshatulu and Krisna [5] and Shamsul [6-7] extended the method to over-damped nonlinear system with constants coefficients in which one of the eigen-values is multiple of the other eigen-value. Recently Shamsul [7] has presented a unified method for solving an *n*-th order differential equation (autonomous) characterized by oscillatory, damped oscillatory and non-oscillatory processes with constant coefficients. The aim of this article is to find an approximate solution of over-damped nonlinear differential systems based on the extended KBM (by Popov [4]) method in which one of the eigen-values is multiple (Ten times; *i.e.*, Decuple) of the other eigen-value.

2. METERIALS AND METHOD

Let us consider the nonlinear differential system

$$\ddot{x} + 2k(\tau)\dot{x} + \omega^2(\tau)x = -\varepsilon\, f(x, \dot{x}, \tau), \qquad \tau = \varepsilon t \tag{1}$$

where the over-dots denote differentiation with respect to t, ε is a small parameter, $\tau = \varepsilon t$ is the slowly varying time, $k(\tau) \geq 0$, f is a given nonlinear function. $\omega(\tau)$ is known as frequency. The coefficients in Eq. (1) are slowly varying in that their time derivatives are proportional to ε.

Setting $\varepsilon = 0$ and $\tau = \tau_0 =$ constant, in Eq.(1), we obtain the unperturbed solution of (1) in the form

$$x(t,0) = a_{1,0}e^{-\lambda_1(\tau_0)t} + a_{2,0}e^{-\lambda_2(\tau_0)t}, \tag{2}$$

When $\varepsilon \neq 0$ we seek a solution in accordance with the KBM method, of the form

$$x(t,\varepsilon) = a_1(t,\tau) + a_2(t,\tau) + \varepsilon u_1(a_1, a_2, t,\ \tau) + \varepsilon^2 ..., \tag{3}$$

where a_1 and a_2 satisfy the differential equations

$$\begin{aligned} \dot{a}_1 &= -\lambda_1(\tau)a_1 + \varepsilon A_1(a_1, a_2, \tau) + \varepsilon^2 ..., \\ \dot{a}_2 &= -\lambda_2(\tau)a_2 + \varepsilon A_2(a_1, a_2, \tau) + \varepsilon^2 ..., \end{aligned} \tag{4}$$

Confining our attention to the first few term $1, 2, \ldots, m$ in the series expansion of (3) and (4), we evaluate functions $u_1, \ldots, A_1, A_2, \ldots,$ such that a_1 and a_2

appearing in (3) and (4) satisfy (1) with an accuracy of ε^{m+1}. In order to determine these unknown functions it was early assumed by Murty *et,al* [5], Shamsul [7] that the functions $u_1,\ldots$ exclude all fundamental terms, since these are included in the series expansion (3) at order ε^0. To obtain a special over-damped solution of (1), we impose a restriction that $u_1 \cdots$ exclude the terms $a_1^{i_1} a_2^{i_2}, i_1\lambda_1 + i_2\lambda_2 < (i_1+i_2)k(\tau_0), \quad i_1, i_2 = 0,1,2\cdots$. The assumption assures that $u_1 \cdots$ are free from secular type terms $te^{-\lambda_1 t}$ and $te^{-\lambda_2 t}$ (see [7]).

Differentiating $x(t,\varepsilon)$ two times with respect to ***t***, substituting for the derivatives $\ddot{x}, \dot{x}$ and x in the original equation (1) and equating the coefficient of ε, we obtain

$$\begin{aligned}
&-\lambda_1' a_1 - \lambda_2' a_2 + \lambda_2 A_1 + \lambda_1 A_2 - \left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2}\right)(A_1 + A_2) \\
&+\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right) u_1 \\
&= -f^{(0)}(a_1, a_2, \tau),
\end{aligned} \tag{5}$$

where $\lambda_1' = \dfrac{d\lambda_1}{d\tau},\ \lambda_2' = \dfrac{d\lambda_2}{d\tau},\ f^{(0)} = f(x_0, \dot{x}_0, \tau)$ and $x_0 = a_1(t,\tau) + a_2(t,\tau)$.

It is assumed that both $f^{(0)}$ can be expanded in Taylor's series [6-7]

$$f^{(0)} = \sum_{i_1,i_2=0}^{\infty} F_{i_1,i_2}(\tau) a_1^{i_1} a_2^{i_2}, \quad u_1 = \sum_{i_1,i_2=0}^{\infty} U_{i_1,i_2}(\tau) a_1^{i_1} a_2^{i_2} \tag{6}$$

It is assumed that both $f^{(0)}$ and u_1 can be expanded in Taylor's series (see[7]), subject to the condition that u_1 $u_1 \cdots$ exclude the terms $a_1^{i_1} a_2^{i_2}, i_1\lambda_1 + i_2\lambda_2 \langle (i_1+i_2)k(\tau_0)$ (already mentioned above). Moreover, we assume that A_1 and A_2 respectively contains terms a_1 and a_2.

Example: As example of the above procedure, let us consider differential equation with a large linear damping force, $-2k(\tau)\dot{x}$

$$\ddot{x}+2k(\tau)\dot{x}+\omega^2(\tau)x=-\varepsilon x^3, \tag{7}$$

Here, $f=x^3$ and non-zero coefficient of $f^{(0)}$ are $F_{3,0}=F_{0,3}=1$ and $F_{2,1}=F_{1,2}=3$. Therefore (5) becomes

$$\begin{aligned}&-\lambda_1' a_1-\lambda_2' a_2+\lambda_2 A_1+\lambda_1 A_2-\left(\lambda_1 a_1\frac{\partial}{\partial a_1}+\lambda_2 a_2\frac{\partial}{\partial a_2}\right)(A_1+A_2)\\&+\left(\lambda_1 a_1\frac{\partial}{\partial a_1}+\lambda_2 a_2\frac{\partial}{\partial a_2}-\lambda_1\right)\left(\lambda_1 a_1\frac{\partial}{\partial a_1}+\lambda_2 a_2\frac{\partial}{\partial a_2}-\lambda_2\right)u_1\\&=-(a_1^3+a_2^3+3a_1^2a_2+3a_1a_2^2)\end{aligned} \tag{8}$$

$$-\lambda_1' a_1-\lambda_2' a_2+\lambda_2 A_1+\lambda_1 A_2-\left(\lambda_1 a_1\frac{\partial}{\partial a_1}+\lambda_2 a_2\frac{\partial}{\partial a_2}\right)(A_1+A_2)=-(3a_1a_2^2+a_2^3) \tag{9}$$

and

$$\left(\lambda_1 a_1\frac{\partial}{\partial a_1}+\lambda_2 a_2\frac{\partial}{\partial a_2}-\lambda_1\right)\left(\lambda_1 a_1\frac{\partial}{\partial a_1}+\lambda_2 a_2\frac{\partial}{\partial a_2}-\lambda_2\right)u_1=-(a_1^3+3a_1^2a_2) \tag{10}$$

The particular solution of (10) is

$$u_1=-\frac{a_1^3}{2\lambda_1(3\lambda_1-\lambda_2)}-\frac{3a_1^2a_2}{2\lambda_1(\lambda_1+\lambda_2)} \tag{11}$$

Now we have to solve (9) for two functions A_1 and A_2. Consider the situation that $\lambda_1\approx 10\lambda_2$. In this case $a_2^3\to a_{2,0}^3e^{-\lambda_1 t}$, so A_1 as well as the equation of A_1 contain a_2^3, i.e., A_1 satisfies the equation

$$-\left(\lambda_1 a_1\frac{\partial}{\partial a_1}+\lambda_2 a_2\frac{\partial}{\partial a_2}-\lambda_2\right)A_1-\lambda_1' a_1=-a_2^3 \tag{12}$$

On the contrary, in accordance with unified KBM method A_2 contains the term $3a_1a_2^2$ (see [7]). Therefore, the equation of A_2 becomes

$$-\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)A_2 - \lambda_2' a_2 = -3a_1a_2^2 \tag{13}$$

The particular solutions of (12) and (13) are

$$A_1 = -\frac{\lambda_1' a_1}{\lambda_1 - \lambda_2} + \frac{a_2^3}{2\lambda_2}, \; A_2 = \frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} + \frac{3a_1a_2^2}{2\lambda_2} \tag{14}$$

Substituting the functional values of A_1, A_2 from (14) into (4) and rearranging, we obtain

$$\dot{a}_1 = -\lambda_1 a_1 - \frac{\varepsilon\lambda_1' a_1}{\lambda_1 - \lambda_2} + \frac{\varepsilon a_2^3}{2\lambda_2}, \qquad \dot{a}_2 = -\lambda_2 a_2 + \frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} + \frac{3a_1a_2^2}{2\lambda_2} \tag{15}$$

In general equations (15) have no exact solution. Usually, a numerical procedure is used to solve them. In this paper, we have used the *Runge-Kutta* (fourth order) method. Numerically, it is advantageous to solve the transformed equation (15) instead of the original equations (7), because a large step size can be used in the integration (see [8] for detail).

Thus for $\lambda_1 \approx 10\lambda_2$, we obtain a first order solution of (7) of the form

$$x(t,\varepsilon) = a_1 + a_2 + \varepsilon u_1, \tag{16}$$

where a_1 and a_2 are given by (15), and u_1 is given by (11).

3. RESULTS AND DISCUSSIONS

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, one compares the approximate solution to the numerical solution (considered to be exact). With regard to such a comparison concerning the presented KBM method of this article, we refer to the works of Murty *et,.al* [5] (who found an over-damped solution of a second order nonlinear system with constant coefficients), and Shamsul [6-7]. In our present paper, for different initial conditions, w*e* have compared the perturbation solutions (16) of *Duffing's* equations (7) to those obtained by Runge-Kutta Fourth-order procedure.

First of all, x is calculated by (16) with initial conditions $[x(0)=1.028571$ $\dot{x}=-.542857]$ or $a_1=1.0000$, $a_2=0.0000$ for $\lambda_1=-.5$, $\lambda_2=-5$, $\varepsilon=.1$. The solutions are various values of t are presented in the second column of Table 1.The corresponding numerical solutions is also computed by Runge-Kutta fourth-order method and are given in the third column of the Table 1. All the results are shown in Table 1. Percentage errors have also been calculated and given in the fourth column of the Table 1.

Secondly, we have computed by (16) for another sets of initial conditions (i) $[x(0)=.555844$ $\dot{x}=1.799757]$ or $a_1=1.0000$, $a_2=-.50000$ for $\lambda_1=-.5$, $\lambda_2=-5$, $\varepsilon=.1$ and (ii) $[x(0)=0.083117$ $\dot{x}=4.149870]$ or $a_1=1.0000$, $a_2=-1.0000$ for $\lambda_1=-.5$, $\lambda_2=-5$, $\varepsilon=.1$. The solutions are various values of t are presented in the second column of Table 2 and Table 3. The corresponding numerical solutions are also computed by Runge-Kutta fourth-order method and are given in the third column of the Table 2 and Table 3. Percentage errors have also been calculated and given in the fourth column of the Table 2 and Table 3.

Finally, we have computed by (16) for another sets of initial conditions (i) $[x(0)=1.057143$ $\dot{x}=-.585714]$ or $a_1=1.0000$, $a_2=0.0000$ for $\lambda_1=-.5$, $\lambda_2=-5$, $\varepsilon=.2$ and (ii) $[x(0)=1.085714$ $\dot{x}=-..628571]$ or $a_1=1.0000$, $a_2=0.0000$ for $\lambda_1=-.5$, $\lambda_2=-5$, $\varepsilon=.3$. The solutions are various values of t are presented in the second column of Table 4 and Table 5. The corresponding numerical solutions are also computed by Runge-Kutta fourth-order method and are given in the third column of the Table 4 and Table 5. Percentage errors have also been calculated and given in the fourth column of the Table 4 and Table 5.

Table 1

t	x_{nu}	x	$E(\%)$
0.0	1.028571	1.028571	0.00000
1.0	0.612460	0.312906	0.07282
2.0	0.368965	0.369302	0.09133
3.0	0.223239	0.223448	0.09362
4.0	0.135279	0.135406	0.09388
5.0	0.082024	0.082101	0.09387
6.0	0.049744	0.049790	0.09247
7.0	0.030170	0.030198	0.09280
8.0	0.018299	0.018316	0.09290
9.0	0.011.99	0.011109	0.09009
10.0	0.006732	0.006738	0.08910

Initial conditions $[x(0) = 1.028571 \;\; \dot{x} = -.542857\,]$ or $a_1 = 1.0000, \;\; a_2 = 0.0000$ for $\lambda_1 = -.5, \;\; \lambda_2 = -5\,, \;\varepsilon = .1$.

Table 2

t	x_{nu}	x	$E(\%)$
0.0	0.555844	0.555844	0.00000
1.0	0.609156	0.609559	0.06615
2.0	0.368944	0.369248	0.08239
3.0	0.223239	0.223428	0.08466
4.0	0.135280	0.135395	0.08500
5.0	0.082024	0.082094	0.08534
6.0	0.049744	0.049786	0.08443
7.0	0.030170	0.030195	0.08286
8.0	0.018299	0.018314	0.08197
9.0	0.011099	0.011108	0.08108
10.0	0.006732	0.006737	0.07427

Initial conditions $[x(0) = .555844 \;\; \dot{x} = 1.799757\,]$ or $a_1 = 1.0000, \;\; a_2 = -.50000$ for $\lambda_1 = -.5, \;\; \lambda_2 = -5\,, \;\varepsilon = .1$

Table 3

t	x_{nu}	x	$E(\%)$
0.0	0.083117	0.083117	0.00000.
1.0	0.605606	0.605908	0.04986
2.0	0.368765	0.369001	0.06399
3.0	0.223145	0.003293	0.06632
4.0	0.135223	0.135313	0.06655
5.0	0.081990	0.082044	0.06586
6.0	0.049723	0.049756	0.06636
7.0	0.030157	0.030177	0.06631
8.0	0.018291	0.018303	0.06560
9.0	0.011094	0.011101	0.06309
10.0	0.006729	0.006733	0.05944

Initial conditions $[x(0) = 0.083117 \;\; \dot{x} = 4.149870]$ or $a_1 = 1.0000, \;\; a_2 = -1.0000$ for $\lambda_1 = -.5, \;\; \lambda_2 = -5$, $\varepsilon = .1$.

Table 4

t	x_{nu}	x	$E(\%)$
0.0	1.057143	1.057143	0.00000
1.0	0.617485	0.619281	0.29085
2.0	0.369390	0.370725	0.36140
3.0	0.222943	0.223765	0.36870
4.0	0.134979	0.135477	0.36894
5.0	0.081815	0.082117	0.37369
6.0	0.049611	0.049794	0.36886
7.0	0.030088	0.030199	0.36891
8.0	0.018249	0.018316	0.36714
9.0	0.011068	0.011109	0.37043
10.0	0.006713	0.006738	0.37241

Initial conditions $[x(0) = 1.057143 \;\; \dot{x} = -.585714]$ or $a_1 = 1.0000, \;\; a_2 = 0.0000$ for $\lambda_1 = -.5, \;\; \lambda_2 = -5$, $\varepsilon = .2$

Table 5

t	x_{nu}	x	$E(\%)$
0.0	1.085714	1.085714	0.00000
1.0	0.621588	0.625656	0.65445
2.0	0.369175	0.372147	0.80503
3.0	0.222264	0.224083	0.81839
4.0	0.134450	0.135548	0.81666
5.0	0.081468	0.082132	0.81504
6.0	0.049395	0.049798	0.81587
7.0	0.029956	0.03020	0.81452
8.0	0.018168	0.018316	0.814619
9.0	0.011019	0.011109	0.81677
10.0	0.006684	0.006738	0.807899

Initial conditions $[x(0) = 1.085714 \quad \dot{x} = -..628571]$ or $a_1 = 1.0000, \quad a_2 = 0.0000$ for $\lambda_1 = -.5, \quad \lambda_2 = -5$, $\varepsilon = .3$.

From Table 1, Table 2, Table 3, Table 4 and Table 5 it is clear that percentage errors are much smaller than 1% and thus (16) show a good coincidence with the numerical solution.

4. Conclusion:

A new perturbation solution of a second order over-damped nonlinear system with slowly varying coefficients is found. The solution is simpler than classical KBM method. The solution gives better result when one of the eigen-value of the unperturbed is multiple (Ten times; *i.e.*, Decuple) of the other eigen-values. The method can be preceded to higher order nonlinear systems.

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