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NON-SIMILAR SOLUTION OF UNSTEADY THERMAL BOUNDARY LAYER EQUATION

BY

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Abstract:

To obtain this present study we studied basic equations. We studied the equation of continuity and derived the Navier-Stockes (N-S) equations of motion for viscous compressible and incompressible fluid flow. Boundary layer and thermal boundary layer equations are also derived. Then we studied similar solution of boundary layer and thermal boundary layer equations. We also performed unsteady solutions of thermal boundary layer equations. We used some non-dimensional variable to non-dimensionalised thermal boundary layer equations. The non-dimensional boundary layer equations are non-linear partial differential equations. To find out the non-similar solutions of unsteady thermal boundary layer equation we used finite difference method. The effect on the velocity and temperature profiles for various parameters entering into the problems are separately discussed and shown graphically.

Keywords and phrases : the Navier-Stockes equations, viscous compressible and incompressible fluid, thermal boundary layer, finite difference method.

বিমূর্ত সার (*Bengali version of the Abstract*)

এই পত্রে আমরা সংনম্য ও অসংনম্য ফ্লুইড প্রবাহের জন্য সন্ততা সমীকরণ এবং নির্ণিত নেভিয়ার - স্টোক্স্ (Navier-Stockes)- এর গতির সমীকরণগুলিকে অনুসন্ধান করেছি । সীমা-স্তর এবং তাপ-সীমা স্তরের সমীকরণগুলিকেও নির্ণয় করেছি । এরপর অনুরূপ সীমা-স্তর এবং তাপ-সীমা স্তরের সমীকরণগুলিকে বিচার বিশ্লেষণ করেছি । আমরা তাপ-সীমা স্তরের অস্থায়ী সমাধান নির্ণয় করেছি । অ-মাত্রিক তাপ-সীমা স্তরের সমীকরণে অ-মাত্রিক চলকে ব্যবহার করেছি । অস্থায়ী তাপ-সীমা স্তরের সমীকরণের অসদৃশ সমাধান নির্ণয়ের ক্ষেত্রে সসীম -

অন্তর (finite difference) পদ্ধতিকে ব্যবহার করেছি । এই সমস্যার ক্ষেত্রে বিভিন্ন প্রাচলের উপস্থিতির জন্য গতিবেগ এবং তাপের তাপমাত্রা রূপের উপর প্রভাবকে আলাদা ভাবে আলোচনা করেছি এবং লেখচিত্রের সাহায্যে দেখিয়েছি ।

1. Introduction

An important application of finite differences is in numerical analysis, especially in numerical differential equations, which aim at the numerical solution of ordinary, partial differential and thermal Boundary Layer equations respectively. The idea is to replace the derivatives appearing in the differential equation by finite differences that approximate them. The resulting methods are called finite difference methods. Common applications of the finite difference method are in computational science and engineering disciplines, such as thermal engineering, fluid mechanics, etc.

The unsteady solution of thermal boundary layer equation is one of the most interesting choices to the researcher by using finite difference method. The analysis so produced infact arose out natural tendency to investigate a subject that may be said to relate to some academic types of problems of solution of the equations of fluid mechanics. Falkner and Skan (1931) have made a study on some approximate solutions of the boundary equations. Callahan and Marner (1976) studied a transient free convection flow with mass transfer past a semi-infinite plate. Recently G. Revathi et al. (2013) performed the Non-similar solution for unsteady water boundary layer flows over a sphere with non-uniform mass transfer. T. Javed, M. Sajid, Z. Abbas, N. Ali, (2011) observer the non-similar solution for rotating flow over an exponentially stretching surface.

Our aim is to make some numerical calculation on unsteady thermal boundary layer equation. The non-dimensional stream function is employed as the independent variable across the layer.

2. Mathematical Model of the Flow

Introducing Cartesian coordinate system, the $x-$axis is chosen along the plate in the direction of the flow and the $y-$axis is normal to it. Initially we consider that the plate as well as the fluid is at the same temperature. Also it is assumed that the fluid and the plate is at rest after that the plate is to be moving with a constant velocity. U_0 in its own plane and instantaneously at time t > 0 , the temperature of the plate raised to T_w $(> T_\infty)$ which is there after maintained constant, where T_w is temperature at the wall and T_∞ is the temperature of the species far away from the plate. The physical model of the study is furnished in the following figure.

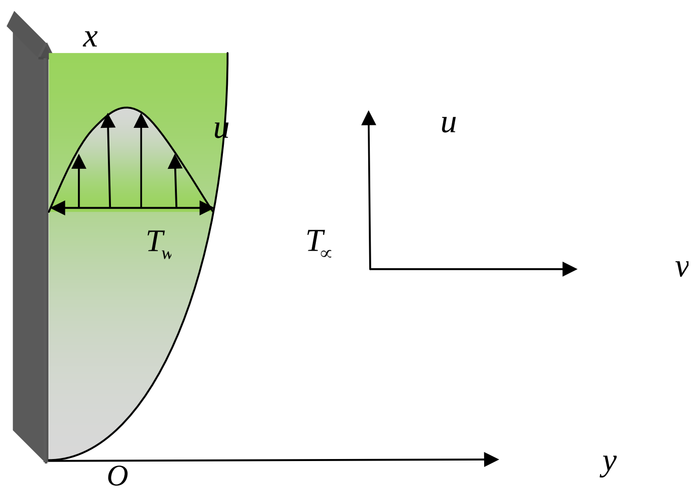


Fig. 1. The physical model and coordinate system

The equations relevant to the transient two dimensional problems are governed by the following system of coupled non-linear partial differential equations.

Continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ (1)

Momentum equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty)$ (2)

Energy equation $\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\kappa}{\rho c_p}\frac{\partial^2 T}{\partial y^2}$ (3)

With the corresponding initial and boundary conditions are

$$t \geq 0, u = 0, v = 0, T \to T_\infty, \quad \text{Everywhere} \tag{4}$$

$$t \geq 0, \begin{cases} u = 0, v = 0, T \to T_\infty \ \text{ at } \ x = 0 \\ u = 0, v = 0, T \to T_w \ \text{ at } \ y = 0 \\ u = 0, v = 0, T \to T_\infty \ \text{ as } \ y \to \infty \end{cases} \tag{5}$$

Where x, y is Cartesian coordinate system; u, v are x, y component of flow velocity

respectively is the local acceleration due to gravity ; υ is the kinematic viscosity; ρ is the density of the fluid ; κ is the thermal conductivity ; C_p is the specific heat at the constant pressure.

3. Mathematical Formulation

Since the solutions of the governing equations (1)-(3) under the initial (4) and boundary (5) conditions will be based on a finite difference method it is required to make the said equations dimensionless. For this purpose we now introduce the following dimensionless variables;

$$X = \frac{xU_0}{\upsilon}, Y = \frac{yU_0}{\upsilon}, U = \frac{u}{U_0}, V = \frac{v}{U_0}, \tau = \frac{tU_0^2}{\upsilon}, \overline{T} = \frac{T - T_\infty}{T_w - T_\infty}$$

Using these relations we have the following derivatives

$$\frac{\partial u}{\partial t} = \frac{U_0}{\upsilon}\frac{\partial U}{\partial \tau}, \quad \frac{\partial u}{\partial x} = \frac{U_0^2}{\upsilon}\frac{\partial U}{\partial X}, \quad \frac{\partial u}{\partial y} = \frac{U_0^2}{\upsilon}\frac{\partial U}{\partial Y},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_0^3}{\upsilon^2}\frac{\partial^2 U}{\partial Y^2}, \quad \frac{\partial v}{\partial y} = \frac{U_0^2}{\upsilon}\frac{\partial V}{\partial Y},$$

$$\frac{\partial T}{\partial t} = \frac{U_0^2 (T_w - T_\infty)}{\upsilon}\frac{\partial \overline{T}}{\partial \tau}, \quad \frac{\partial T}{\partial x} = \frac{U_0 (T_w - T_\infty)}{\upsilon}\frac{\partial \overline{T}}{\partial X}, \quad \frac{\partial T}{\partial y} = \frac{U_0 (T_w - T_\infty)}{\upsilon}\frac{\partial \overline{T}}{\partial Y},$$

$$\frac{\partial^2 T}{\partial y^2}=\frac{U_0^2(T_w-T_\infty)}{v^2}\frac{\partial^2 \overline{T}}{\partial Y^2}$$

Now we substitute the values of the above derivatives into the equations (1)-(3) and after simplification we obtain the following nonlinear coupled partial differential equations in terms of dimensionless variables

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0 \tag{6}$$

$$\frac{\partial U}{\partial \tau}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=\upsilon\frac{\partial^2 U}{\partial Y^2}+G_r\overline{T} \tag{7}$$

$$\frac{\partial \overline{T}}{\partial \tau}+U\frac{\partial \overline{T}}{\partial X}+V\frac{\partial \overline{T}}{\partial Y}=\frac{1}{P_r}\frac{\partial^2 \overline{T}}{\partial Y^2} \tag{8}$$

Where $P_r=\dfrac{\upsilon\rho C_p}{\kappa}$ (Prandtl number)

and $G_r=\mu g\beta\left(\dfrac{T_w-T_\infty}{U_o^3}\right)$ (Grashof number)

Also the associated initial and boundary conditions become

$$\tau\geq 0,\qquad U=0, V=0, \overline{T}=0 \quad \text{everywhere} \tag{9}$$

$$\tau\geq 0,\ \begin{cases} U=0, V=0, \overline{T}=0 \ \text{ at } \ X=0 \\ U=0, V=0, \overline{T}=1 \ \text{ at } \ Y=0 \\ U=0, V=0, \overline{T}=0 \ \text{ as } \ Y\to\infty \end{cases} \tag{10}$$

4. Numerical Solutions

In this section, we attempt to solve the governing second order nonlinear coupled dimensionless partial differential equations with the associated initial and boundary conditions. For solving a transient free convection flow with mass transfer past a semi infinite plate, *Callahan* and *Marner* (1976) used the difference method

From the concept of the above discussion, for simplicity the explicit finite difference method has been used to solve equations (5) - (7) subject to the conditions given by (8) and (9). To obtain the difference equations the region of the flow is divided into a grid or mesh of lines parallel to X and Y axes where X -axes is taken along the plate and Y - axes is normal to the plate. Here we consider that the plate of height $X_{\max}(=100)$ i.e. X varies from 0 to 100 and regard $Y_{\max}(=25)$ as corresponding to $Y \to \infty$ i.e. Y varies 0 to 25. There are $m = 125$ and $n = 125$ grid spacing in the X and Y directions respectively as shown in the Fig.2.

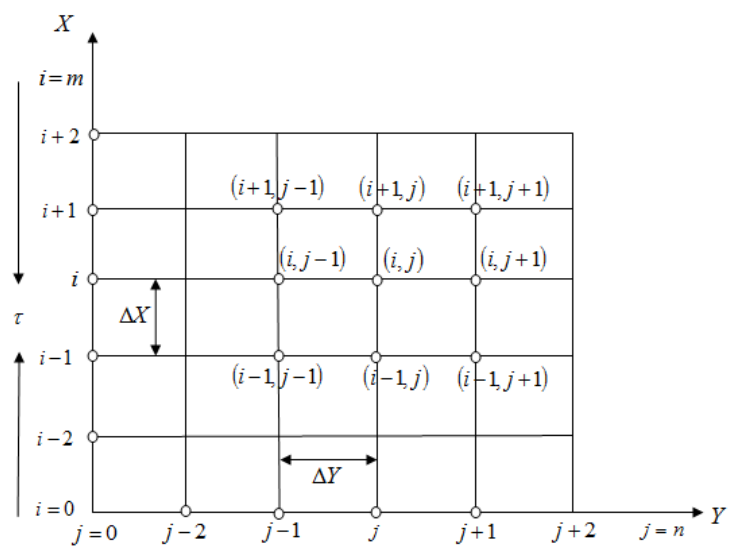


Fig. 2: The finite difference space grid

It is assumed that $\Delta X, \Delta Y$ are constant mesh sizes along X and Y directions respectively and taken as follows,

$$\Delta X = 0.8(0 \le x \le 100)$$
$$\Delta Y = 0.2(0 \le y \le 25)$$

With the smaller time step, $\Delta\tau = 0.005$

Now $U', V', \overline{T}'$ are denoted the values of $U, V, \overline{T}$ at the end of a time step respectively. Using the explicit finite difference approximation we have,

$$\left(\frac{\partial U}{\partial \tau}\right)_{i,j} = \frac{U'_{i,j} - U_{i,j}}{\partial \tau}, \quad \left(\frac{\partial U}{\partial X}\right)_{i,j} = \frac{U_{i,j} - U_{i-1,j}}{\Delta X}, \quad \left(\frac{\partial U}{\partial Y}\right)_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{\Delta Y}, \tag{11}$$

$$\left(\frac{\partial V}{\partial Y}\right)_{i,j} = \frac{V_{i,j} - V_{i,j-1}}{\Delta Y}, \ \left(\frac{\partial \overline{T}}{\partial \tau}\right)_{i,j} = \frac{\overline{T}'_{i,j} - \overline{T}_{i,j}}{\Delta \tau}, \ \left(\frac{\partial \overline{T}}{\partial X}\right)_{i,j} = \frac{\overline{T}_{i,j} - \overline{T}_{i-1,j}}{\Delta X}, \ \left(\frac{\partial \overline{T}}{\partial Y}\right)_{i,j} = \frac{\overline{T}_{i,j+1} - \overline{T}_{i,j}}{\Delta Y},$$

$$\left(\frac{\partial^2 U}{\partial Y^2}\right)_{i,j} = \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{(\Delta Y)^2}, \quad \left(\frac{\partial^2 \overline{T}}{\partial Y^2}\right)_{i,j} = \frac{\overline{T}_{i,j+1} - 2\overline{T}_{i,j} + \overline{T}_{i,j-1}}{(\Delta Y)^2} \tag{12}$$

From the system partial differential equations with substituting the above relations into the corresponding differential equation we obtain an appropriate set of finite difference equations,

$$\frac{U_{i,j} - U_{i-1,j}}{\Delta X} + \frac{V_{i,j} - V_{i,j-1}}{\Delta Y} = 0 \tag{13}$$

$$\frac{U'_{i,j} - U_{i,j}}{\Delta \tau} + U_{i,j}\frac{U_{i,j} - U_{i-1,j}}{\Delta X} + V_{i,j}\frac{U_{i,j+1} - U_{i,j}}{\Delta Y} = \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{(\Delta Y)^2} + G_r\overline{T}_{i,j} \tag{14}$$

$$\frac{\overline{T}' - \overline{T}_{i,j}}{\Delta \tau} + U_{i,j}\frac{\overline{T}_{i,j} - \overline{T}_{i-1,j}}{\Delta X} + V_{i,j}\frac{\overline{T}_{i,j+1} - \overline{T}_{i,j}}{\Delta Y} = \frac{1}{P_r}\frac{\overline{T}_{i,j+1} - 2\overline{T}_{i,j} + \overline{T}_{i,j-1}}{(\Delta Y)^2} \tag{15}$$

And the initial and boundary conditions with the finite difference scheme are

$$\left.\begin{aligned} U^0_{i,j} &= 0, V^0_{i,j} = 0, \overline{T}^0_{i,j} = 0 \\ U^n_{0.j} &= 0, V^n_{0,j} = 0, \overline{T}^n_{0,j} = 0 \\ U^n_{i,0} &= 0, V^n_{i,0} = 0, \overline{T}^n_{i,0} = 1 \\ U^n_{i,L} &= 0, V^n_{i,L} = 0, \overline{T}^n_{i,L} = 0 \end{aligned}\right\} \tag{16}$$

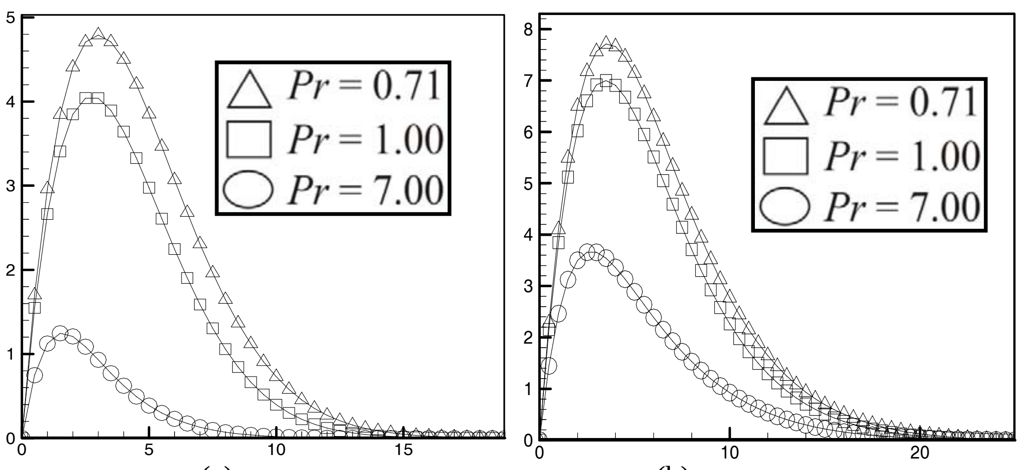
[where $L \to \infty$]

Here the subscripts i and j designate the grid points with x and y coordinates respectively and the superscript n represents a value of time, $\tau = n\Delta\tau$ where $n = 0,1,2$......... From the initial condition (9), the values of $U, \overline{T}$ are known at $\tau = 0$. During any one time step, the coefficients $U_{i,j}$ and $V_{i,j}$

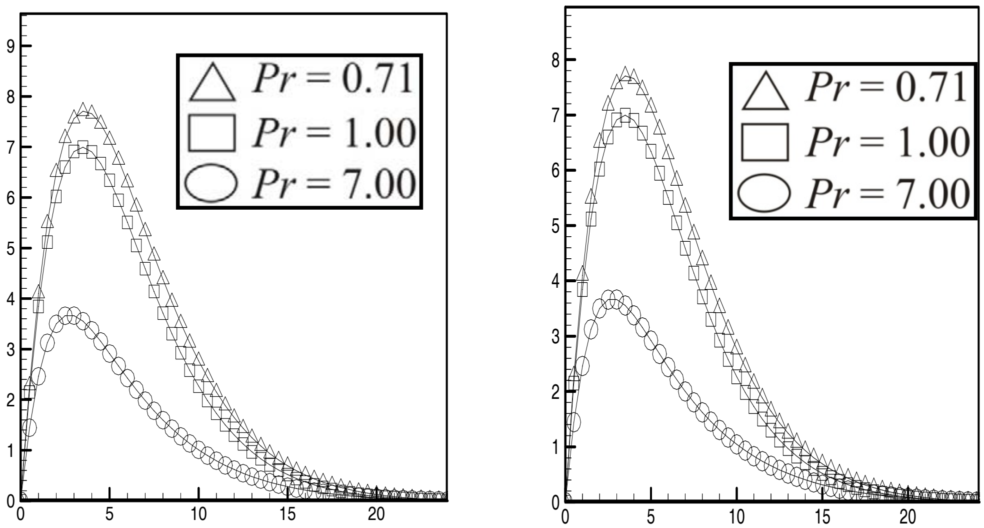
appearing in equations (11)-(12) are created as constants. Then at the end of any time-step $\Delta\tau$ the temperature $\overline{T}'$, the new velocity U', the new induced velocity field V' at all interior nodal points may be obtained by successive applications of equations (13), (14), (15) respectively. This process is repeated in time and provided the time-step is sufficiently small, $U, V, \overline{T}$ should eventually converge to values which approximate the steady-state solution of equations (13)-(15). These converged solutions are shown graphically in figures.

5. Results and Discussion

The main goal of the computation is to obtain the steady state solutions for the non-dimensional velocity U and temperature $\overline{T}$ for different values of Prandtl number (Pr) and Grashof number (Gr). For this purpose computations have been carried out up to dimensionless time τ = 10 to 80. The results of the computations, however, show graphical changes in the below mentioned quantities to time τ =40 have been reached and after this at $\tau = (50-80)$ graphical change negligible. Thus the solution for dimensionless time τ =80 is essentially steady state solutions. Along with the steady state solutions the solutions for the transient values of U versus Y , $\overline{T}$ versus Y are shown in below for different values of parameters. Three values of prandtl number are considered as 0.71, 1.0 and 7.0. Here, Pr = 0.71 represent air at 20^0, Pr =1.0 correspond to electrolyte solutions(such as salt water) and Pr =7.0 represents water. In this present study, we will discuss the graphical solution for different values of parameters at dimensional time τ = 10, 40, 50 and 80. From the graphical representation we observed interesting solutions.



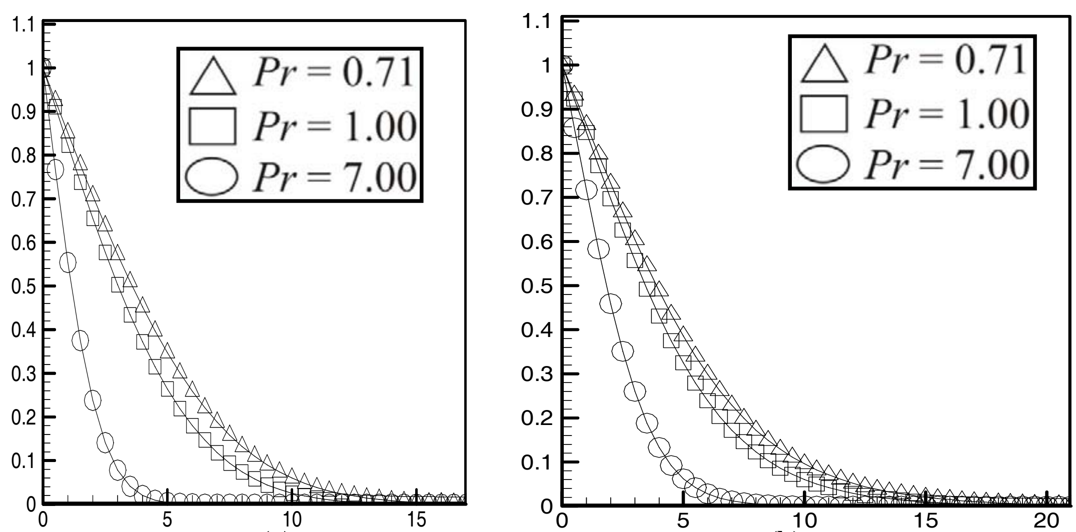
(a) (b)



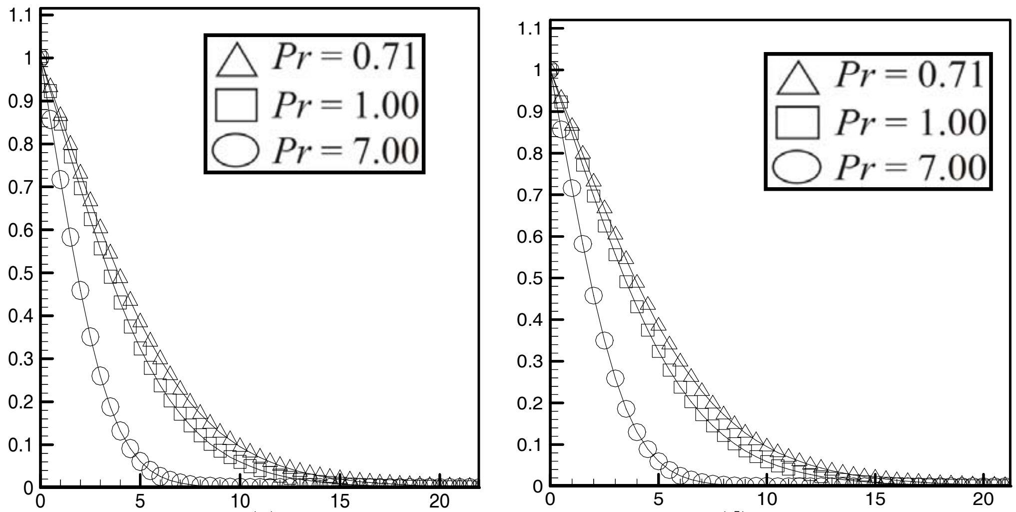
(a) (b)

Fig.3 Velocity Profile for different values of Prandtl number (Pr) and Grashof number $Gr = 2.0$ at time (a) $\tau = 10$, (b) $\tau = 40$, (c) $\tau = 50$ and (d) $\tau = 80$.

From Fig. 3(a) we observe that the velocity profile decreases with the increase of Pr at time $\tau = 10$. We observe that velocity profile decreases dramatically at $\mathrm{Pr} = 7.0$ and with the increasing of time the velocity profile remain unchanged which is shown by Fig. 3(b), Fig.3(c) and Fig. 3(d).



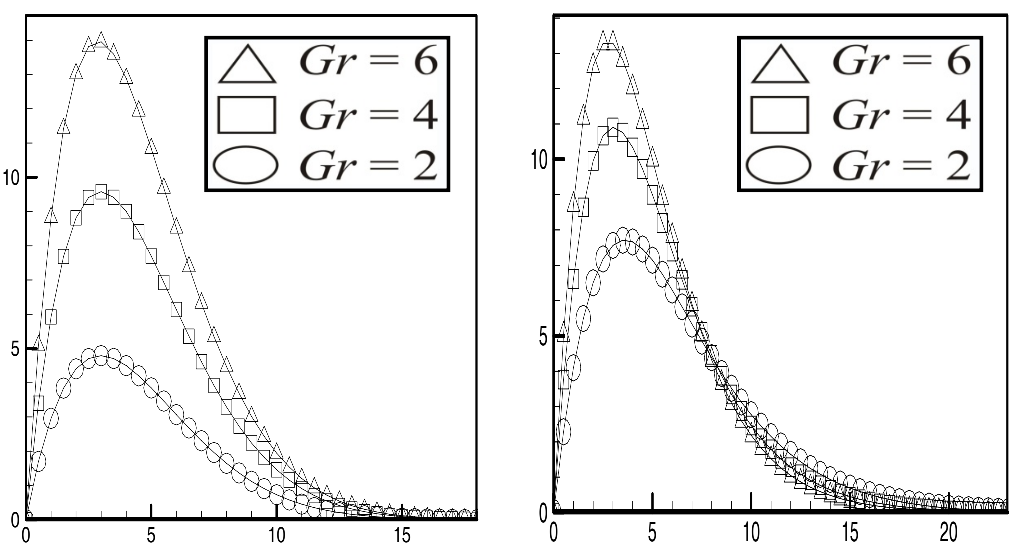
(a) (b)



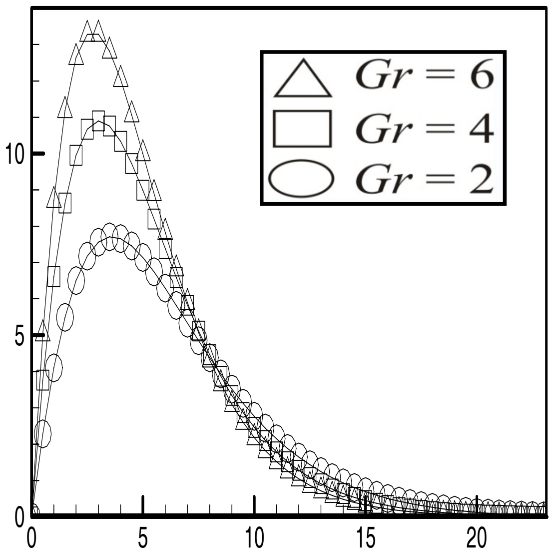
(c) (d)

Fig.4 Temperature Profile for different values of Prandtl number (Pr) and Grashof number $Gr = 2.0$ at time (a) $\tau = 10$, (b) $\tau = 40$, (c) $\tau = 50$ and (d) $\tau = 80$.

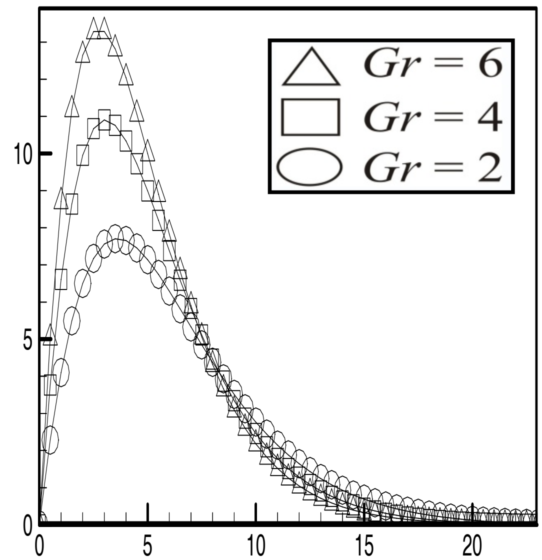
From Fig. 4(a) we observe that the temperature profile decreases with the increase of *Pr* at time $\tau = 10$. We observe that temperature profile decreases dramatically when $\mathrm{Pr} = 7.0$ and with the increasing of time the temperature profile remain unchanged which is shown by Fig. 4(b), Fig.4(c) and Fig. 4(d).



(a) (b)



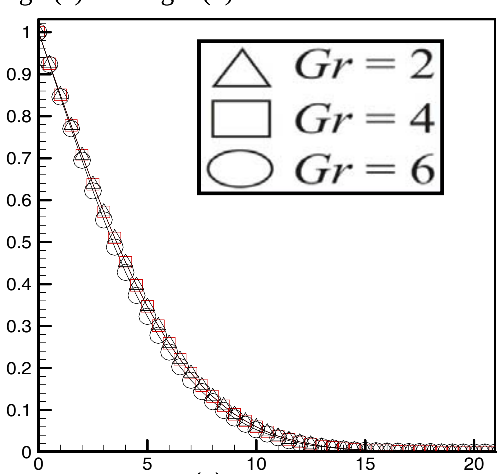
(c)



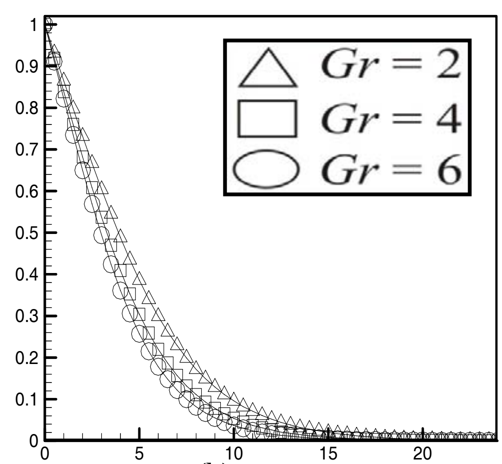
(d)

Fig. 5 Velocity Profile for different values of Grashof number (Gr) and Prandtl number $\Pr = 0.71$ at time (a) $\tau = 10$, (b) $\tau = 40$, (c) $\tau = 50$ and (d) $\tau = 80$.

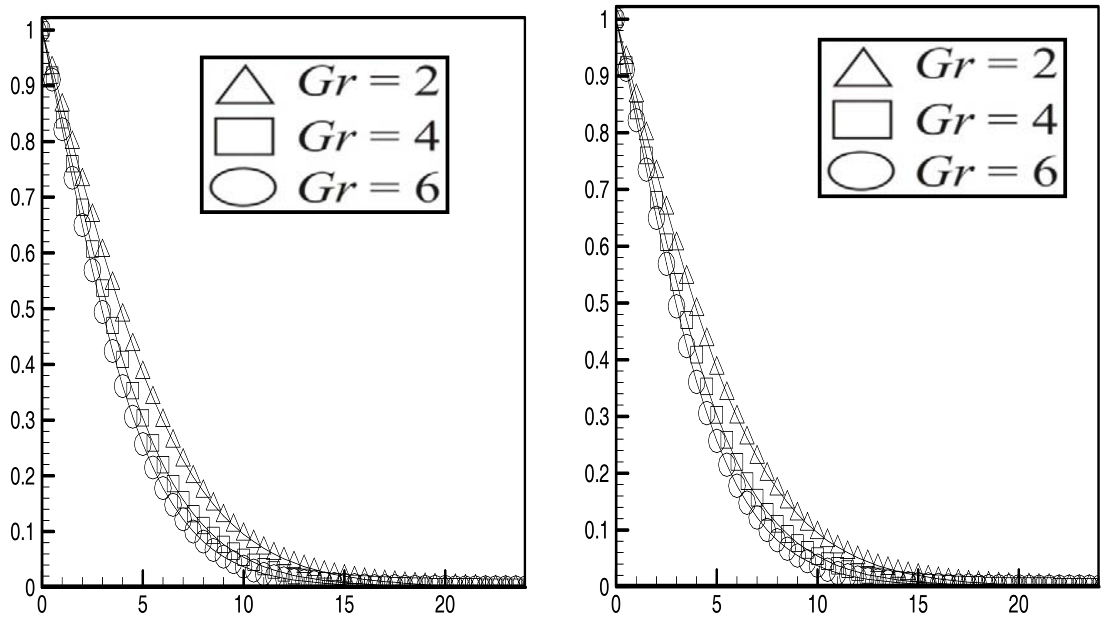
From Fig. 5(a) we observe that the velocity profile decreases with the increase of *Gr* at time $\tau = 10$. We observe that velocity profile decreases dramatically when $Gr = 2.0,\ 4.0,\ 6.0$. From the graphical representation we also found that with the increasing of time the velocity profile remain unchanged which is shown by Fig. 5(b), Fig.5(c) and Fig. 5(d).



(a)



(b)



(c) (d)

Fig.6 Temperature Profile for different values of Grashof number $Gr = 2.0$ and Prandtl number $\Pr = 0.71$ at time (a) $\tau = 10$, (b) $\tau = 40$, (c) $\tau = 50$ and (d) $\tau = 80$.

From Fig. 6(a) we observe that the temperature profile decreases slightly with the increase of Gr at time $\tau = 10$. At time $\tau = 40$, we observe that temperature profile decreases sharply as shown in Fig. 6(b) and with the increasing of time the temperature profile remain unchanged which is shown by Fig.6(c) and Fig. 6(d).

6. Conclusion

In this present study, we studied equation of continuity and derived the Navier-Stokes (N-S) equations of motion for viscous compressible and incompressible fluid flow. Then we studied the boundary layer equation in two-dimensional flow, energy equation and thermal boundary layer equation.

Finally, the thermal boundary layer equations have been derived from Navier-Stokes equation by boundary layer technique. Boundary layer equations have been non-dimensionalised by using non-dimensional variable. The non-dimensional boundary layer equations are non-linear partial differential equations. These equations are solved by using finite difference method. Finite difference solution of heat and mass transfer flow is studied to examine the velocity and temperature distribution characteristics. The effect on the velocity and temperature for the various parameters entering into the problems are separately discussed with the help of graphs. Then the results in the form of velocity and temperature distribution are shown graphically.

To obtain the steady-state solutions for the non-dimensional velocity U, temperature $\overline{T}$ and we use different values of Prandtl number $(\Pr)$, Grashof number (Gr). For this purpose, computations have been carried out up to dimensionless time $\tau = 10$ to 80. Along with the steady state solutions, the solutions for the transient values of U versus Y and $\overline{T}$ versus Y are obtained. The results of the computations, however, show graphical changes in the mentioned quantities to time $\tau = 40$ have been reached and after this at $\tau = (50-80)$ graphical change are negligible. Thus the solution for dimensionless time $\tau = 80$ are essentially steady-state solutions.

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