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CERTAIN FEATURES OF PARTIALLY α – COMPACT FUZZY SETS

By

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Abstract:

In this paper , we introduce the concept of partially α – shading (resp. partially α^* – shading), in ahort, $p\alpha$ – shading (resp. $p\alpha^*$ – shading) and partially α – compact (resp. partially α^* – compact), in short, $p\alpha$ – compact (resp. $p\alpha^*$ – compact) fuzzy sets and study their several features in fuzzy topological spaces.

Keywords and phrases : fuzzy sets, compact fuzzy sets, fuzzy topological spaces.

বিমূর্ত সার *(Bengali version of the Abstract)*

এই পত্রে আমরা আংশিক α - ঢাকনিযুক্তের (*resp. partially* α^* – *shading*) ধারণাকে, সংক্ষেপে $p\alpha$ - ঢাকনিযুক্তের এবং আংশিক α -সংহত (*resp. partially* α^* – *compact*), সংক্ষেপে $p\alpha$ - সংহত (*resp.* $p\alpha^*$ – *compact*) ফাজি সেটকে (*fuzzy sets*) উপস্থাপন করেছি এবং ফাজি টোপোলজীয় দেশে (*fuzzy topological spaces.*) ইহাদের বহুবিধ বৈশিষ্ট্যকে অনুসন্ধান করেছি ।

1. Introduction

The concept of α – compactness was first introduced by T. E. Gantner et al.[5] in 1978. α – compactness occupies a very important place in fuzzy topological spaces. The purpose of this paper is to introduce and study the concept of $p\alpha$ – compact (resp. $p\alpha^*$ – compact) fuzzy sets in more detail and to obtain several features of the concept. We find that this concept has many tangible flavors.

2. Preliminaries

In this section, we recall some fundamental definitions which are needed in the next section. These are essential in our study and can be found in the papers referred to.

Definition 2.1 [13] : Let X be a non-empty set and I is the closed unit interval [0, 1]. A fuzzy set in X is a function u : X $\rightarrow$ I which assigns to every element x $\in$ X. u(x) denotes a degree or the grade of membership of x. The set of all fuzzy sets in X is denoted by I^X. A member of I^X may also be called a fuzzy subset of X.

Definition 2.3 [10] : A fuzzy set is empty iff its grade of membership is identically zero . It is denoted by 0 or ϕ.

Definition 2.4 [10] : A fuzzy set is whole iff its grade of membership is identically one in X . It is denoted by 1 or X.

Definition 2.5 [3] : Let u and v be two fuzzy sets in X. Then we define

(i) u = v iff u(x) = v(x) for all x $\in$ X

(ii) u $\subseteq$ v iff u(x) $\leq$ v(x) for all x $\in$ X

(iii) λ = u $\cup$ v iff λ(x) = (u $\cup$ v) (x) = max [u(x) , v(x)] for all x $\in$ X

(iv) μ = u $\cap$ v iff μ(x) = (u $\cap$ v) (x) = min [u(x) , v(x)] for all x $\in$ X

(v) $\gamma = u^c$ iff γ(x) = 1 – u(x) for all x $\in$ X.

Remark : Two fuzzy sets u and v are disjoint iff u $\cap$ v = 0.

Definition 2.6 [3] : In general , if { u_i : i $\in$ J } is family of fuzzy sets in X , then union $\cup\, u_i$ and intersection $\cap\, u_i$ are defined by

$\cup\, u_i$(x) = sup { u_i(x) : i $\in$ J and x $\in$ X }

$\cap\, u_i$(x) = inf { u_i(x) : i $\in$ J and x $\in$ X } , where J is an index set.

Definition 2.7 [3] : Let f : X $\rightarrow$ Y be a mapping and u be a fuzzy set in X. Then the image of u, written $f(u)$, is a fuzzy set in Y whose membership function is given by

$$f(u)\,(y) = \begin{cases} \sup\{ u(x) : x \in f^{-1}(y)\} & if\ \ f^{-1}(y) \neq \phi \\ 0 & if\ \ f^{-1}(y) = \phi \end{cases} .$$

Definition 2.8 [3] : Let f : X $\rightarrow$ Y be a mapping and v be a fuzzy set in Y. Then the inverse of v, written f^{-1}(v) , is a fuzzy set in X whose membership function is given by f^{-1}(v) (x) = v(f(x)).

Distributive laws 2.9 [13] : Distributive laws remain valid for fuzzy sets in X i.e. if u , v and w are fuzzy sets in X , then

(i) u $\cup$ (v $\cap$ w) = (u $\cup$ v) $\cap$ (u $\cup$ w)

(ii) u $\cap$ (v $\cup$ w) = (u $\cap$ v) $\cup$ (u $\cap$ w).

Definition 2.10 [3] : Let X be a non-empty set and t $\subseteq$ I^X i.e. t is a collection of fuzzy set in X. Then t is called a fuzzy topology on X if

(i) 0 , 1 $\in$ t

(ii) $u_i \in$ t for each $i \in$J , then $\bigcup_i u_i \in$ t

(iii) u , v $\in$ t , then u $\cap$ v $\in$t

The pair $(X\ , t)$ is called a fuzzy topological space and in short, fts. Every member of t is called a t-open fuzzy set. A fuzzy set is t – closed iff its complements is t-open. In the sequel, when no confusion is likely to arise, we shall call a t – open (t – closed) fuzzy set simply an open (closed) fuzzy set.

Definition 2.11 [3] : Let $(X\ , t)$ and $(Y\ , s)$ be two fuzzy topological spaces. A mapping f : $(X\ , t)$ $\rightarrow$ $(Y\ , s)$ is called an fuzzy continuous iff the inverse of each s-open fuzzy set is t – open.

Definition 2.12 [10] : Let $(X\ , t)$ be an fts and A $\subseteq$ X. Then the collection t_A = { u|A : u $\in$ t } = { u $\cap$ A : u $\in$ t } is fuzzy topology on A, called the subspace fuzzy topology on A and the pair $(A\ , t_A)$ is referred to as a fuzzy subspace of $(X\ , t)$.

Definition 2.13 [4] : Let (A, t_A) and (B, s_B) be fuzzy subspaces of fuzzy topological spaces (X, t) and (Y, s) respectively and f is a mapping from (X, t) to (Y, s), then we say that f is a mapping from (A, t_A) to (B, s_B) if f(A) $\subseteq$ B.

Definition 2.14 [4] : Let (A, t_A) and (B, s_B) be fuzzy subspaces of fts's (X, t) and (Y, s) respectively. Then a mapping f : $(A, t_A) \rightarrow (B, s_B)$ is relatively fuzzy continuous iff for each v $\in$ s_B, the intersection f^{-1}(v) $\cap$ A $\in$ t_A.

Definition 2.15 [1] : Let λ be a fuzzy set in X , then the set { x $\in$ X : λ(x) > 0 } is called the support of λ and is denoted by λ_0 or suppλ .

Definition 2.16 [1] : Let (X, T) be a topological space. A function f : X $\rightarrow$ **R** (with usual topology) is called lower semi-continuous (l . s . c.) if for each a $\in$ **R** , the set f^{-1}(a , ∞) $\in$ T. For a topology T on a set X , let ω(T) be the set of all l . s . c. functions from (X, T) to I (with usual topology); thus ω(T) = { u $\in$ I^X : u^{-1}(a , 1] $\in$ T , a $\in$ I_1 } . It can be shown that ω(T) is a fuzzy topology on X.

Let P be a property of topological spaces and FP be its fuzzy topology analogue. Then FP is called a ' good extension' of P " iff the statement (X, T) has P iff $(X, \omega(T))$has FP" holds good for every topological space (X, T). Thus characteristic functions are l . s. c.

Definition 2.17 [11] : An fts (X, t) is said to be fuzzy – T_1 space iff for every x , y $\in$ X , x $\neq$ y, there exist u , v $\in$ t such that u(x) = 1, u(y) = 0 and v(x) = 0, v(y) = 1.

Definition 2.18 [12] : An fts (X, t) is said to be fuzzy – T_1 space iff for all x , y $\in$ X , x $\neq$ y, there exist u , v $\in$ t such that u (x) > 0, u (y) = 0 and v (x) = 0, v (y) > 0.

Definition 2.19 [5] : An fts (X, t) is said to be fuzzy Hausdorff iff for all x , y $\in$ X , x $\neq$ y , there exist u , v $\in$ t such that u(x) = 1 , v(y) = 1 and u $\cap$ v = 0.

Definition 2.20 [9] : An fts (X, t) is said to be fuzzy Hausdorff iff for all x , y $\in$ X , x $\neq$ y , there exist u , v $\in$ t such that u(x) = 1 , v(y) = 1 and u $\subseteq$ 1 – v.

Definition 2.21 [7] : An fts (X, t) is said to be fuzzy Hausdorff iff for all x , y $\in$ X , x $\neq$ y, there exist u , v $\in$ t such that u (x) > 0, v (y) > 0 and u $\cap$ v = 0.

Definition 2.22 [9] : An fts (X, t) is said to be fuzzy regular iff for each x $\in$ X and u $\in t^c$ with u(x) = 0 , there exist v , w $\in$ t such that v(x) = 1 , u $\subseteq$ w and v $\subseteq$ 1 – w.

Definition 2.23 [2] : Let $\lambda \in I^X$ and $\mu \in I^Y$. Then $(\lambda \times \mu)$ is a fuzzy set in X $\times$Y for which $(\lambda \times \mu)$ (x , y) = min { λ(x) , μ(y) } , for every (x , y) $\in$ X $\times$Y.

3. Characterizations of $p\alpha$ – compact (resp. $p\alpha^*$ – compact) fuzzy sets :

First we give two definitions:

Definition 3.1 : Let (X, t) be an fts and $\alpha \in$ I. A family M of fuzzy sets is called a $p\alpha$ – shading (resp. $p\alpha^*$ – shading) of a fuzzy set λ in X if for each x $\in \lambda_0$, ($\lambda_0 \neq$ X) there exists a u $\in$ M with u (x) > α (resp. u (x) $\geq \alpha$).

A subfamily of a $p\alpha$ – shading (resp. $p\alpha^*$ – shading) of λ which is also a $p\alpha$ – shading (resp. $p\alpha^*$ – shading) is called a $p\alpha$ – subshading (resp. $p\alpha^*$ – subshading) of λ.

If λ(x) $\neq$ 0 for all x $\in$ X i.e. λ_0 = X, then $p\alpha$ – shading (resp. $p\alpha^*$ – shading) and α – shading (resp. α – shading) will be same.

Definition 3.2 : Let (X, t) be an fts and $\alpha \in$ I. A fuzzy set λ in X is said to be $p\alpha$ – compact (resp. $p\alpha^*$ – compact) if every open $p\alpha$ – shading (resp. $p\alpha^*$ – shading) of λ has a finite $p\alpha$ – subshading (resp. $p\alpha^*$ – subshading).

Theorem 3.3 : Let (X, t) be an fts, A $\subseteq$ X and λ be a fuzzy set in X. Then

(i) If λ is $p\alpha$ – compact with respect to t, then λ is $p\alpha$ – compact with respect to t_A, where $0 \leq \alpha < 1$.

(ii) If λ is $p\alpha^*$ – compact with respect to t, then λ is $p\alpha^*$ – compact with respect to t_A, where $0 < \alpha \leq 1$.

Proof : (i) : Let M = { u_i : $i \in$ J } be an open $p\alpha$ – shading of λ with respect to t_A. By definition of subspace fuzzy topology, there exist $v_i \in$ t such that u_i = A $\cap$ v_i. Therefore, { A $\cap$ v_i : $i \in$ J } is an open $p\alpha$ – shading of λ with respect to t. As λ is $p\alpha$ – compact with respect to t, then λ has a finite $p\alpha$ – subshading, say { A $\cap$ v_{i_k} } (k = 1, 2, …… , n) such that (A $\cap$ v_{i_k}) (x) > α for each x $\in$ λ_0. But then { u_{i_k} } (k = 1, 2, …… , n) is a finite $p\alpha$ – subshading of M. Thus λ is $p\alpha$ compact with respect to t_A.

(ii) The proof is similar.

The following example will show that the $p\alpha$ – compact, $0 \leq \alpha < 1$ (resp. $p\alpha^*$ – compact, $0 < \alpha \leq 1$) fuzzy set in an fts need not be closed.

Example 3.4 : Let X = { a , b , c } and I = [0 , 1]. Let u_1 , u_2 , u_3 , u_4 $\in$ I^X defined by u_1(a) = 0.6, u_1(b) = 0.2, u_1(c) = 0.4; u_2(a) = 0.3, u_2(b) = 0.1, u_2(c) = 0.7; u_3(a) = 0.6, u_3(b) = 0.2, u_3(c) = 0.7; u_4(a) = 0.3, u_4(b) = 0.1, u_4(c) = 0.4. Now, take t = { 0 , 1 , u_1 , u_2 , u_3 , u_4 }, then we see that (X, t) is an fts. Let λ $\in$ I^X with λ(a) = 0.4, λ(b) = 0, λ(c) = 0.7. Take α = 0.5. Then by definition of $p\alpha$ – compact, λ is clearly $p\alpha$ – compact. But λ is not closed, as its complement λ^c is not open in (X, t).

This example is also applicable for $p\alpha^*$ – compactness.

Theorem 3.5 : Let (X, t) and (Y, s) be two fuzzy topological spaces and f : $(X, t) \rightarrow (Y, s)$ be fuzzy continuous and onto. Then

(i) If λ is $p\alpha$ – compact in (X, t), then f (λ) is $p\alpha$ – compact in (Y, s), where $0 \leq \alpha < 1$.

(ii)If λ is $p\alpha^*$ – compact in (X, t), then f (λ) is $p\alpha^*$ – compact in (Y, s), where $0 \leq \alpha < 1$.

Proof : (i) : Let M = { u_i : $i \in$ J } be an open $p\alpha$ – shading of f (λ) in (Y, s). Since f is fuzzy continuous, then f^{-1}(M) = { $f^{-1}(u_i)$: $u_i \in$ M } is an open $p\alpha$ – shading of λ in (X, t). For, if x $\in \lambda_0$, then f (x) $\in$ f (λ_0). So there exists $u_{i_0} \in$ M such that u_{i_0}(f (x)) > α which implies that $f^{-1}(u_{i_0})$ (x) > α. As λ is $p\alpha$ – compact in (X, t), then f^{-1}(M) has a finite $p\alpha$ – subshading, say { $f^{-1}(u_{i_1})$, $f^{-1}(u_{i_2})$, …… , $f^{-1}(u_{i_n})$ }. Now, if y $\in$ f (λ_0), then y = f (x) for some x $\in \lambda_0$. Then there exists k such that $f^{-1}(u_{i_k})$ (x) > α which implies that u_{i_k} (f (x)) > α or u_{i_k} (y) > α. Hence f (λ) is $p\alpha$ – compact in (Y, s).

(ii) The proof is similar.

Theorem 3.6 : Let (X, t) and (Y, s) be two fuzzy topological spaces and f : $(X, t) \rightarrow (Y, s)$ be bijective. Then

(i) If λ is $p\alpha$ – compact in (Y, s), then $f^{-1}(\lambda)$ is $p\alpha$ – compact in (X, t),where $0 \leq \alpha < 1$.

(ii) If λ is $p\alpha^*$ – compact in (Y, s), then $f^{-1}(\lambda)$ is $p\alpha^*$ – compact in (X, t), where $0 \leq \alpha < 1$.

Proof : (i) : Let M = { u_i : $i \in$ J } be an open $p\alpha$ – shading of $f^{-1}(\lambda)$ in (X, t). Then f (M) = { f (u_i) : $i \in$ J } is an open $p\alpha$ – shading of λ in (Y, s). For, if y $\in \lambda_0$, then f^{-1}(y) $\in f^{-1}(\lambda_0)$. So there exists $u_{i_0} \in$ M such that u_{i_0}(f^{-1}(y)) > α which implies that f (u_{i_0}) (y) > α.

Since λ is $p\alpha$ – compact in (Y , s), then f (M) has a finite $p\alpha$ – subshading, say { f (u_{i_1}), f (u_{i_2}), …… , f (u_{i_n}) }. For if x $\in$ $f^{-1}(\lambda_0)$, then x = f^{-1}(y) for some y $\in$ λ_0. Therefore, there exists k such that f (u_{i_k}) (y) > α which implies that u_{i_k} (f^{-1}(y)) > α or u_{i_k} (x) > α. Hence $f^{-1}(\lambda)$ is $p\alpha$ – compact in (X , t).

(ii) The proof is similar.

Theorem 3.7 : Let (A , t_A) be a fuzzy subspace of an fts(X , t) and f : (X , t) $\to$ (A , t_A) be fuzzy continuous and onto. Then

(i) If λ is $p\alpha$ – compact in (X , t), then f (λ) is $p\alpha$ – compact in (A , t_A), where 0 $\le \alpha$ < 1.

(ii) If λ is $p\alpha^*$ – compact in (X , t), then f(λ) is $p\alpha^*$ – compact in (A , t_A),where 0 $\le \alpha$ < 1.

Proof : (i) : Let M = { u_i : i $\in$ J } be an open $p\alpha$ – shading of f (λ) in (A , t_A). By definition of subspace fuzzy topology, there exists v_i $\in$ t such hat u_i = A $\cap$ v_i. Since f is fuzzy continuous, then f^{-1}(M) = { $f^{-1}(u_i)$: u_i $\in$ M } i.e f^{-1}(M) = { f^{-1}(A $\cap$ v_i) : i $\in$ J }is an open $p\alpha$ – shading of λ in (X , t). For, if x $\in$ λ_0, then f (x) $\in$ f (λ_0). So there exists u_{i_0} $\in$ M such that u_{i_0} (f (x)) > α which implies that $f^{-1}(u_{i_0})$ (x) > α i.e. f^{-1}(A $\cap$ v_{i_0}) (x) > α . As λ is $p\alpha$ – compact in (X , t), then f^{-1}(M) has a finite $p\alpha$ – subshading, say { f^{-1}(A $\cap$ v_{i_1}), f^{-1}(A $\cap$ v_{i_2}), …… , f^{-1}(A $\cap$ v_{i_n}) }. Now, if y $\in$ f (λ_0), then y = f (x) for some x $\in$ λ_0. Then there exists k such that f^{-1}(A $\cap$ v_{i_k}) (x) > α which implies that (A $\cap$ v_{i_k}) (f (x)) > α or u_{i_k} (y) > α. Hence f (λ) is $p\alpha$ – compact in (A , t_A).

(ii) The proof is similar.

Theorem 3.8 : Let (A , t_A) and (B , s_B) be fuzzy subspaces of fuzzy topological spaces (X , t) and (Y , s) respectively and f : $(A , t_A) \rightarrow (B , s_B)$ be relatively fuzzy continuous and onto. Then

(i) If λ is $p\alpha$ – compact in (A , t_A), then f(λ) is $p\alpha$ – compact in (B , s_B), where $0 \leq \alpha < 1$.

(ii) If λ is $p\alpha^*$ – compact in (A , t_A),then f(λ) is $p\alpha^*$ – compact in (B , s_B), where $0 \leq \alpha < 1$.

Proof : Let M = { v_i : $v_i \in s_B$ } be an open $p\alpha$ – shading of f (λ) in (B , s_B) for every $i \in$ J. Since f is fuzzy relatively continuous, then $f^{-1}(v_i) \cap$ A $\in t_A$ and hence f^{-1}(M) = { $f^{-1}(v_i) \cap$ A : $v_i \in s_B$ } is an open $p\alpha$ – shading of λ in (A , t_A). For, if x $\in \lambda_0$, then f (x) $\in$ f (λ_0). So there exists $v_{i_0} \in$ M such that v_{i_0} (f (x)) > α which implies that ($f^{-1}(v_{i_0}) \cap$ A) (x) > α. As λ is $p\alpha$ – compact in (A , t_A), then f^{-1}(M) has a finite $p\alpha$ – subshading, say { $f^{-1}(v_{i_1})$ $\cap$ A, $f^{-1}(v_{i_2}) \cap$ A, …… , $f^{-1}(v_{i_n}) \cap$ A }. Now, if y $\in$ f (λ_0), then y = f (x) for some x $\in \lambda_0$. Then there exists k such that ($f^{-1}(v_{i_k}) \cap$ A) (x) > α which implies that v_{i_k} (f (x)) > α or v_{i_k} (y) > α. Hence f (λ) is $p\alpha$ – compact in (B , s_B).

(ii) The proof is similar.

Theorem 3.9 : Let λ be a $p\alpha$ – compact, $0 \leq \alpha < 1$ (resp. $p\alpha^*$ – compact, $0 < \alpha \leq 1$) fuzzy set in a fuzzy – T_1 space (X , t) (as def. – 2.17) with $\lambda_0 \subset$ X (proper subset). Let x $\notin \lambda_0$ (λ (x) = 0) , then there exist u , v $\in$ t such that u (x) = 1 , $\lambda_0 \subseteq v^{-1}(0 , 1]$.

Proof : Suppose y $\in \lambda_0$. Then clearly x $\neq$ y. As (X, t) is fuzzy – T_1 space, there exist u_y , $v_y \in$ t such that u_y(x) = 1 , u_y(y) = 0 and v_y(x) = 0 , v_y(y) = 1. Let us take $\alpha \in I_1$. Then v_y(y) > α > 0 , as v_y(y) = 1. Hence we see that { v_y : y $\in \lambda_0$ } is an open $p\alpha$ – shading of λ in (X, t). Since λ is $p\alpha$ – compact, then λ has a finite $p\alpha$ – subshading, say { v_{y_k} : $y_k \in \lambda_0$ } (k = 1, 2, …… , n) such that v_{y_k}(y) > α for each y $\in \lambda_0$. Now, let v = $v_{y_1} \cup v_{y_2} \cup ...... \cup v_{y_n}$ and u = $u_{y_1} \cap u_{y_2} \cap ...... \cap u_{y_n}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, u $\in$ t. Moreover, $\lambda_0 \subseteq v^{-1}$(0 , 1] and u (x) = 1, as u_{y_k}(x) = 1 for each k.

Similar proof of $p\alpha^*$ – compact can be given.

Theorem 3.10: Let λ and μ be disjoint $p\alpha$ – compact, 0 ≤ α < 1 (resp. $p\alpha^*$ – compact, 0 < α ≤ 1) fuzzy sets in a fuzzy – T_1 space (X, t) (as def. – 2.17) with λ_0 , $\mu_0 \subset$ X (proper subsets). Then there exist u , v $\in$ t such that $\lambda_0 \subseteq u^{-1}$(0 , 1] and $\mu_0 \subseteq v^{-1}$(0 , 1].

Proof : Suppose y $\in \lambda_0$. Then y $\notin \mu_0$, as λ and μ are disjoint. Since μ is $p\alpha$ – compact in (X, t), then by theorem 3.9, there exist u_y , $v_y \in$ t such that u_y(y) = 1 and $\mu_0 \subseteq v_y^{-1}$(0 , 1]. Let us take $\alpha \in I_1$ with u_y(y) > α > 0 , as u_y(y) = 1. Then we see that { u_y : y $\in \lambda_0$ } is an open $p\alpha$ – shading of λ in (X, t). Since λ is $p\alpha$ – compact, then λ has a finite $p\alpha$ – subshading, say { u_{y_k} : $y_k \in \lambda_0$ } (k = 1, 2, …… , n) such that u_{y_k}(y) > α for each y $\in \lambda_0$. Furthermore, as μ is $p\alpha$ – compact , so μ has a finite $p\alpha$ – subshading, say { v_{y_k} : $y_k \in \mu_0$ } (k = 1, 2, …… , n) such that v_{y_k}(x) > α for each x $\in \mu_0$, as $\mu_0 \subseteq v_{y_k}^{-1}$(0 , 1] for each k. Now, let u = $u_{y_1} \cup u_{y_2} \cup ...... \cup u_{y_n}$ and v = $v_{y_1} \cap v_{y_2} \cap ...... \cap v_{y_n}$. Hence we see that $\lambda_0 \subseteq u^{-1}$(0 , 1] and $\mu_0 \subseteq v^{-1}$(0 , 1]. Thus u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u, v $\in$ t.

Similar proof of $p\alpha^*$ – compact can be done.

The following example will show that the $p\alpha$ – compact, $0 \leq \alpha < 1$ (resp. $p\alpha^*$ – compact, $0 < \alpha \leq 1$) fuzzy set in a fuzzy – T_1 space (X, t) (as def. – 2.17) need not be closed.

Example 3.11 : Let X = { a , b , c } and I = [0 , 1]. Let u_1 , u_2 , u_3 , u_4 , u_5 , u_6 $\in I^X$ defined by u_1(a) = 1, u_1(b) = 0, u_1(c) = 0; u_2(a) = 0, u_2(b) = 1, u_2(c) = 0; u_3(a) = 0, u_3(b) = 0, u_3(c) = 1; u_4(a) = 1, u_4(b) = 1, u_4(c) = 0; u_5(a) = 1 , u_5(b) = 0 , u_5(c) = 1; u_6(a) = 0 , u_6(b) = 1 , u_6(c) = 1. Now, put t = { 0 , 1 , u_1 , u_2 , u_3 , u_4 , u_5 , u_6 }, then we have (X, t) is a fuzzy – T_1 space. Let $\lambda \in I^X$ with λ(a) = 0.2, λ(b) = 0.6, λ(c) = 0. Take α = 0.4. Then by definition of $p\alpha$ – compact, λ is clearly $p\alpha$ – compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

This example is also applicable for $p\alpha^*$ – compactness.

Theorem 3.12 : Let λ be a $p\alpha$ – compact, $0 \leq \alpha < 1$ (resp. $p\alpha^*$ – compact, $0 < \alpha \leq 1$) fuzzy set in a fuzzy – T_1 space (X, t) (as def. – 2.18) with $\lambda_0 \subset$ X (proper subset). Let x $\notin \lambda_0$ (λ(x) = 0) , then there exist u , v $\in$ t such that u (x) > 0 , $\lambda_0 \subseteq v^{-1}(0, 1]$.

Proof : Similar as Theorem 3.9.

Theorem 3.13 : Let λ and μ be disjoint $p\alpha$ – compact, $0 \leq \alpha < 1$ (resp. $p\alpha^*$ – compact, $0 < \alpha \leq 1$) fuzzy sets in a fuzzy – T_1 space (X, t) (as def. – 2.18) with λ_0 , $\mu_0 \subset$ X (proper subsets). Then there exist u , v $\in$ t such that $\lambda_0 \subseteq u^{-1}(0, 1]$ and $\mu_0 \subseteq v^{-1}(0, 1]$.

Proof : Similar as Theorem 3.10.

The following example will show that the $p\alpha$ – compact, $0 \le \alpha < 1$ (resp. $p\alpha^*$ – compact, $0 < \alpha \le 1$) fuzzy set in a fuzzy – T_1 space (as def. – 2.18) need not be closed.

Example 3.11 will serve the purpose.

Theorem 3.14 : Let λ be a $p\alpha$ – compact, $0 \le \alpha < 1$ (resp. $p\alpha^*$ – compact, $0 < \alpha \le 1$) fuzzy set in a fuzzy Hausdorff space (X , t) (as def. – 2.19) with $\lambda_0 \subset$ X (proper subset). Let x $\notin \lambda_0$ (λ(x) = 0) , then there exist u , v $\in$ t such that u (x) = 1 , $\lambda_0 \subseteq v^{-1}(0 , 1]$ and u $\cap$ v = 0.

Proof : Let y $\in \lambda_0$. Then clearly x $\neq$ y. Since (X , t) is fuzzy Hausdorff space, there exist u_y , $v_y \in$ t such that u_y(x) = 1 , v_y(y) = 1 and $u_y \cap v_y$ = 0. Let us take $\alpha \in I_1$ such that v_y(y) > α > 0 , as v_y(y) = 1. Hence we see that { v_y : y $\in \lambda_0$ } is an open $p\alpha$ – shading of λ in (X , t). As λ is $p\alpha$ – compact in (X , t), then λ has a finite $p\alpha$ – subshading, say { v_{y_k} : $y_k \in \lambda_0$ } (k = 1, 2, …… , n) such that v_{y_k}(y) > α for each y $\in \lambda_0$. Now, let v = $v_{y_1} \cup v_{y_2} \cup \ldots\ldots \cup v_{y_n}$ and u = $u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, u $\in$ t. Moreover, $\lambda_0 \subseteq v^{-1}(0 , 1]$ and u (x) = 1, as u_{y_k}(x) = 1 for each k.
Finally, we have to show that u $\cap$ v = 0. As $u_{y_k} \cap v_{y_k}$ = 0 implies that u $\cap\, v_{y_k}$ = 0 , by distributive law , we see that u $\cap$ v = u $\cap$($v_{y_1} \cup v_{y_2} \cup \ldots\ldots \cup v_{y_n}$) = 0 .
Similar proof of $p\alpha^*$ – compactness.

Theorem 3.15 : Let λ and μ be disjoint $p\alpha$ – compact, $0 \le \alpha < 1$ (resp. $p\alpha^*$ – compact, $0 < \alpha \le 1$) fuzzy sets in a fuzzy Hausdorff space (X , t) (as def. – 2.19) with λ_0 , $\mu_0 \subset$ X (proper subsets). Then there exist u , v $\in$ t such that $\lambda_0 \subseteq u^{-1}(0 , 1]$, $\mu_0 \subseteq v^{-1}(0 , 1]$ and u $\cap$ v = 0.

Proof : Suppose y $\in \lambda_0$. Then y $\notin \mu_0$, as λ and μ are disjoint. Since μ is $p\alpha$ – compact fuzzy set in (X, t), then by theorem 3.14, there exist u_y , $v_y \in$ t such that u_y(y) = 1, $\mu_0 \subseteq v_y^{-1}$(0 , 1] and $u_y \cap v_y$ = 0. Let us take $\alpha \in I_1$ such that u_y(y) > α > 0 , as u_y(y) = 1. Then we see that { u_y : y $\in \lambda_0$ } is an open $p\alpha$ – shading of λ in (X, t). Since λ is $p\alpha$ – compact in (X, t), then λ has a finite $p\alpha$ – subshading, say { u_{y_k} : $y_k \in \lambda_0$ } (k = 1, 2, …… , n) such that u_{y_k} (y) > α for each y $\in \lambda_0$. Furthermore, as μ is $p\alpha$ – compact , then μ has a finite $p\alpha$ – subshading, say { v_{y_k} : $y_k \in \mu_0$ } (k = 1, 2, …… , n) such that v_{y_k} (x) > α for each x $\in \mu_0$, as $\mu_0 \subseteq v_{y_k}^{-1}$(0 , 1] for each k. Now, let u = $u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and v = $v_{y_1} \cap v_{y_2} \cap \ldots\ldots \cap v_{y_n}$. Hence we see that $\lambda_0 \subseteq u^{-1}$(0 , 1] and $\mu_0 \subseteq v^{-1}$(0 , 1]. Thus u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u, v $\in$ t.

Finally, we have to show that u $\cap$ v = 0. We observe that $u_{y_k} \cap v_{y_k}$ = 0 for each k implies that $u_{y_k} \cap$ v = 0 for each k, by distributive law , we see that u $\cap$ v = ($u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$)$\cap$ v = 0 .

Similar proof of $p\alpha^*$ – compact can be done.

The following example will show that the $p\alpha$ – compact, $0 \le \alpha < 1$ (resp. $p\alpha^*$ – compact, $0 < \alpha \le 1$) fuzzy set in a fuzzy Hausdorff space (as def. – 2.19) need not be closed.

Example 3.11 will work for the same.

Theorem 3.16 : Let λ be a $p\alpha$ – compact, $0 \le \alpha < 1$ (resp. $p\alpha^*$ – compact, $0 < \alpha \le 1$) fuzzy set in a fuzzy Hausdorff space (X, t) (in the sense of Def. 2.20) with $\lambda_0 \subset$ X (proper subset). Let x $\notin \lambda_0$ (λ(x) = 0) , then there exist u , v $\in$ t such that u (x) = 1 , $\lambda_0 \subseteq v^{-1}$(0 , 1] and u $\subseteq$ 1 – v.

Proof : Suppose y $\in \lambda_0$. Then clearly x $\neq$ y. Since (X, t) is fuzzy Hausdorff space, there exist u_y , $v_y \in$ t such that u_y(x) = 1 , v_y(y) = 1 and $u_y \subseteq 1 - v_y$. Let us take $\alpha \in I_1$ such that v_y(y) > α > 0 , as v_y(y) = 1. Thus we see that { v_y : y $\in \lambda_0$ } is an open $p\alpha$ – shading of λ in (X, t). Since λ is $p\alpha$ – compact in (X, t), then λ has a finite $p\alpha$ – subshading, say { v_{y_k} : $y_k \in \lambda_0$ } (k = 1, 2, …… , n) such that v_{y_k} (y) > α for each y $\in \lambda_0$.

Now, let v = $v_{y_1} \cup v_{y_2} \cup ...... \cup v_{y_n}$ and u = $u_{y_1} \cap u_{y_2} \cap ...... \cap u_{y_n}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, u $\in$ t. Moreover, $\lambda_0 \subseteq v^{-1}(0 , 1]$ and u (x) = 1, as u_{y_k} (x) = 1 for each k.

Finally, we have to show that u $\subseteq$ 1 – v . Since $u_y \subseteq 1 - v_y$ implies that u $\subseteq$ 1 – v_y . As u_{y_k} (x) $\leq 1 - v_{y_k}$ (x) for all x $\in$ X and for each k, then u $\subseteq$ 1 – v . If not , then there exist x $\in$ X , such that u_y(x) >1 $-v_y$(x) . We have u_y(x) $\leq u_{y_k}$ (x) for each k. Then for some k , u_{y_k} (x) >1 $-v_{y_k}$ (x) . But this is a contradiction, as u_{y_k} (x) $\leq 1 - v_{y_k}$ (x) for each k . Hence u $\subseteq$ 1 – v .

Similar proof of $p\alpha^*$ – compactness.

Theorem 3.17 : Let λ and μ be disjoint $p\alpha$ – compact, $0 \leq \alpha < 1$ (resp. $p\alpha^*$ – compact, $0 < \alpha \leq 1$) fuzzy sets in a fuzzy Hausdorff space (X, t) (in the sense of Def. 2.20) with λ_0 , $\mu_0 \subset$ X (proper subsets). Then there exist u , v $\in$ t such that $\lambda_0 \subseteq u^{-1}(0 , 1]$, $\mu_0 \subseteq v^{-1}(0 , 1]$ and u $\subseteq$ 1 – v.

Proof : Suppose y $\in \lambda_0$. Then y $\notin \mu_0$, as λ and μ are disjoint. Since μ is $p\alpha$ – compact fuzzy set in (X, t), then by theorem 3.16, there exist u_y , $v_y \in$ t such that u_y(y) = 1, $\mu_0 \subseteq v_y^{-1}(0 , 1]$ and $u_y \subseteq 1 - v_y$. Let us assume that $\alpha \in I_1$ such that u_y(y) > α > 0 , as u_y(y) = 1. Then we see that { u_y : y $\in \lambda_0$ } is an open $p\alpha$ – shading of λ in (X, t). Since λ is $p\alpha$ – compact in (X, t), then λ has a finite $p\alpha$ – subshading, say { u_{y_k} : $y_k \in \lambda_0$ } (k = 1, 2, …… , n) such that u_{y_k} (y) > α for each y $\in$ λ_0. Furthermore, as μ is $p\alpha$ – compact , then μ has a finite $p\alpha$ – subshading, say { v_{y_k} : $y_k \in \mu_0$ } (k = 1, 2, …… , n) such that v_{y_k} (x) > α for each x $\in$ μ_0, as $\mu_0 \subseteq v_{y_k}^{-1}$(0 , 1] for each k. Now, let u = $u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and v = $v_{y_1} \cap v_{y_2} \cap \ldots\ldots \cap v_{y_n}$. Hence we see that $\lambda_0 \subseteq u^{-1}$(0 , 1] and $\mu_0 \subseteq v^{-1}$(0 , 1]. Thus u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u, v $\in$ t.

Finally, we have to show that u $\subseteq$ 1 – v. Since $u_{y_k} \subseteq 1 - v_{y_k}$ for each k implies that $u_{y_k} \subseteq$ 1 – v for each k and hence it is clear that u $\subseteq$ 1 – v.

Similar proof of $p\alpha^*$ – compactness.

The following example will show that the $p\alpha$ – compact, $0 \le \alpha < 1$ (resp. $p\alpha^*$ – compact, $0 < \alpha \le 1$) fuzzy set in a fuzzy Hausdorff space (in the sense of Def. 2.20) need not be closed.

Example 3.11 will surve the purpose.

Theorem 3.18 : Let λ be a $p\alpha$ – compact, $0 \le \alpha < 1$ (resp. $p\alpha^*$ – compact, $0 < \alpha \le 1$) fuzzy set in a fuzzy Hausdorff space (X, t) (in the sense of Def. – 2.21) with $\lambda_0 \subset$ X (proper subset). Let x $\notin \lambda_0$ (λ (x) = 0) , then there exist u , v $\in$ t such that u (x) > 0, $\lambda_0 \subseteq v^{-1}$(0 , 1] and u $\cap$ v = 0.

Proof : Similar as Theorem 3.14.

Theorem 3.19 : Let λ and μ be disjoint $p\alpha$ – compact, $0 \le \alpha < 1$ (resp. $p\alpha^*$ – compact, $0 < \alpha \le 1$) fuzzy sets in a fuzzy Hausdorff space (X, t) (in the sense of Def. – 2.21) with λ_0 , $\mu_0 \subset$ X (proper subsets). Then there exist u , v $\in$ t such that $\lambda_0 \subseteq u^{-1}$(0 , 1] , $\mu_0 \subseteq v^{-1}$(0 , 1] and u $\cap$ v = 0.

Proof : Similar as Theorem 3.15.

The following example will show that the $p\alpha$ – compact, $0 \le \alpha < 1$ (resp. $p\alpha^*$ – compact, $0 < \alpha \le 1$) fuzzy set in a fuzzy Hausdorff space, in the sense of Def. – 2.21 need not be closed.

Example 3.11 will surve the purpose.

Theorem 3.20 : Let λ be a $p\alpha$ – compact, $0 \le \alpha < 1$ (resp. $p\alpha^*$ – compact, $0 < \alpha \le 1$) fuzzy set in a fuzzy regular space $\left(X , t \right)$ with $\lambda_0 \subset$ X (proper subset). If for each x $\in \lambda_0$, there exists u $\in t^c$ with u(x) = 0, we have v , w $\in$ t such that v (x) = 1 , u $\subseteq$ w, $\lambda_0 \subseteq v^{-1}(0, 1]$ and v $\subseteq$ 1 – w.

Proof : Let $\left(X , t \right)$ be a fuzzy regular space and λ be a $p\alpha$ – compact fuzzy set in $\left(X , t \right)$. Then for each x $\in \lambda_0$, there exists u $\in t^c$ with u(x) = 0. As $\left(X , t \right)$ is fuzzy regular, we have v_x, w_x $\in$ t such that v_x(x) = 1, $u_x \subseteq w_x$ and $v_x \subseteq 1 - w_x$. Let us assume that $\alpha \in I_1$, then v_x (x) > α > 0, as v_x(x) = 1. Hence we see that { v_x : x $\in \lambda_0$ } is an open $p\alpha$ – shading of λ. Since λ is $p\alpha$ – compact, then λ has a finite $p\alpha$ – subshading, say { v_{x_k} : $x_k \in \lambda_0$ } (k = 1, 2, …… , n) such that v_{x_k}(x) > α for each x $\in \lambda_0$. Now, let v = $v_{x_1} \cup v_{x_2} \cup ...... \cup v_{x_n}$ and w = $w_{x_1} \cap w_{x_2} \cap ...... \cap w_{x_n}$. Thus we see that v and w are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, w $\in$ t. Moreover, $\lambda_0 \subseteq v^{-1}(0, 1]$, v (x) = 1 and u $\subseteq$ w, as u $\subseteq w_{x_k}$ for each k.

Finally, we have to show that v $\subseteq$ 1 – w. First, we observe that $v_{x_k} \subseteq 1 - w_{x_k} = 0$ for each k implies that $v_{x_k} \subseteq$ 1 – w for each k and hence it is clear that v $\subseteq$ 1 – w.

Similar proof of $p\alpha^*$ – compactness.

Theorem 3.21 : Let $\left(X , T \right)$ be a topological space and $\left(X , \omega(T) \right)$ be an fts. Let λ be a fuzzy set in X.

(i) If $0 \le \alpha < 1$, then λ is $p\alpha$ – compact in $\left(X , \omega(T) \right)$ iff $\lambda^{-1}(0, 1]$ is compact in $\left(X , T \right)$.

(ii) If 0 < α ≤ 1, then λ is $p\alpha^*$ – compact in $(X, \omega(T))$ iff $\lambda^{-1}(0, 1]$ is compact in (X, T).

Proof : (i) : Suppose λ is $p\alpha$ – compact fuzzy set in $(X, \omega(T))$. Let W = { V_i : i ∈ J } be an open cover of $\lambda^{-1}(0, 1]$ in (X, T). Then, since for each V_i, there exists a $u_i \in \omega$ (T) such that $V_i = u_i^{-1}(0, 1]$, we have W = { $u_i^{-1}(0, 1]$: i ∈ J }. Then the family G = { u_i : i ∈ J } is an open $p\alpha$ – shading of λ in $(X, \omega(T))$. To see this, let x ∈ λ_0. Since W is an open cover of $\lambda^{-1}(0, 1]$, then there exists a V_{i_0} ∈ W such that x ∈ V_{i_0}. But $V_{i_0} = u_{i_0}^{-1}(0, 1]$ for some u_{i_0} ∈ ω (T). Therefore x ∈ $u_{i_0}^{-1}(0, 1]$ which implies that u_{i_0}(x) > α. By $p\alpha$ – compactness of λ, G has a finite $p\alpha$ – subshading, say { u_{i_k} } (k = 1, 2, …… , n). Then { $u_{i_k}^{-1}(0, 1]$ } (k = 1, 2, …… , n) forms a finite subcover of W and hence $\lambda^{-1}(0, 1]$ in (X, T).

Conversely, suppose $\lambda^{-1}(0, 1]$ is compact in (X, T). Let M = { v_j : j ∈ J } be an open $p\alpha$ – shading of λ in $(X, \omega(T))$. Then the family H = { $v_j^{-1}(0, 1]$: j ∈ J } is an open cover of $\lambda^{-1}(0, 1]$ in (X, T). Now, for let x ∈ λ_0. Then there exists v_{j_0} ∈ M such that v_{j_0}(x) > α. Therefore x ∈ $v_{j_0}^{-1}(0, 1]$ and hence $v_{j_0}^{-1}(0, 1]$ ∈ H. By compactness of $\lambda^{-1}(0, 1]$, H has a finite subcover, say { $v_{j_k}^{-1}(0, 1]$ } (k = 1, 2, …… , n). Then the family { v_{j_k} } (k = 1, 2, …… , n) forms a finite $p\alpha$ – subshading of M and hence λ is $p\alpha$ – compact in $(X, \omega(T))$.

(ii) The proof is similar.

Theorem 3.22 : Let λ and μ be fuzzy sets in (X, t) and (Y, s) respectively. Then

(i) If λ and μ are $p\alpha$ – compact fuzzy sets in (X, t) and (Y, s) respectively, then ($\lambda \times \mu$) is $p\alpha$ – compact in (X×Y , t×s), where $0 \le \alpha < 1$.

(ii) If λ and μ are $p\alpha^{*}-$ compact fuzzy sets in (X, t) and (Y, s) respectively, then ($\lambda \times \mu$) is $p\alpha^{*}-$ compact in (X×Y , t×s), where 0 < α ≤ 1.

Proof : (i) : Suppose { u_i : i ∈ J } is an open $p\alpha-$ shading of λ in (X, t) i.e. u_i(x) > α for each x ∈ λ_0 and { v_j : j ∈ J } is an open $p\alpha-$ shading of μ in (Y, s) i.e. v_j(y) > α for each y ∈ μ_0. Now, let δ = { $u_i \times v_j$: u_i ∈ t and v_j ∈ s } be an open $p\alpha-$ shading of ($\lambda \times \mu$) in (X×Y , t×s). Thus we see that ($u_i \times v_j$) (x , y) = min (u_i(x) , v_j(y)) > α , for each (x , y) ∈ ($\lambda_0 \times \mu_0$). As λ is $p\alpha-$ compact in (X, t) and μ is $p\alpha-$ compact in (Y, s), then λ and μ have finite $p\alpha-$ subshadings in (X, t) and (Y, s) respectively, say u_{i_k} ∈ { u_i } and v_{j_k} ∈ { v_j } such that u_{i_k}(x) > α and v_{j_k}(y) > α for each x ∈ λ_0 and y ∈ μ_0 respectively. Hence we have δ has a finite $p\alpha-$ subshading, say ($u_{i_k} \times v_{j_k}$)∈ { $u_i \times v_j$ } such that ($u_{i_k} \times v_{j_k}$) (x , y) = min (u_{i_k}(x) , v_{j_k}(y)) > α for each (x , y) ∈ ($\lambda_0 \times \mu_0$). Therefore ($\lambda \times \mu$) is $p\alpha-$ compact in (X×Y , t×s).

(ii) The proof is similar.

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