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SOME RESULTS ON NORMAL MEET SEMILATTICES

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Abstract:

In this paper we introduce the concept of normal semilattices in presence of 0-distributivity and include a nice characterization of normal semilattices. We also study the p-ideals in pseudo complemented meet semilattices. Then we give the notion of S-semilattices and prove that every S-semilattice is comaximal, although its converse in not true. Finally, we prove that every S-semilattice is normal, but the converse need not be true.

Keywords and phrases : normal semilattices, 0-distributivity, ideals, meet semilattices

বিমূর্ত সার (Bengali version of the Abstract)

O - বন্টনের উপস্থিতিতে নর্মাল অর্ধ - ল্যাটিসের (Normal semilattices) *ধারণাকে এই পত্রে আমরা উপস্থাপন করেছি । ছদ্ম পূরক মিট্ অর্ধ - ল্যাটিসের* (pseudo complemented meet semilattices) *p - আইডিয়ালস্ - এর ও অনুসন্ধান করেছি । এরপর আমরা S - অর্ধ - ল্যাটিসের ধারণাকে দিয়েছি এবং প্রমান করেছি যে প্রত্যেক S - অর্ধ - ল্যাটিস একটি সহ - মহা অর্ধ - ল্যাটিস* (comaximal semilattices) *যদিও ইহার বিপরীত ক্রমটি সত্য নয় । শেষতঃ আমরা প্রমান করেছি যে প্রত্যেক S - অর্ধ - ল্যাটিস হচ্ছে একটি নর্মাল কিন্তু ইহার বিপরীত ক্রমের সত্য হওয়ার প্রয়োজন নেই ।*

1. Introduction :

In generalizing the notion of pseudo complemented lattices. J.C.Varlet [10] first introduced the concept of 0-distributive lattices. Then many authors including [1,2,5,7,8] studied them for lattices and semilattices. By [2], a meet semilattice S with 0 is called a 0-distributive meet semilattices if for all $a, b, c \in S$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge d = 0$ for some $d \geq b, c$. [9] introduced the concept of semi prime ideals of a lattice. Recently [5] have extended the Concept for meet semilattices. An ideal J of a meet semilattice S is called a semi prime ideal if for all $a,b,c \in S$ with $a \wedge b \in J$, $a \wedge c \in J$, imply $a \wedge d \in J$ for some $d \geq b,c$. Hence a meet semilattice S with 0 is called 0-distributive if (0] is a semiprime ideal of S. A meet semi lattice S is called *directed above* if for all $a,b \in S$, there exists $c \in S$ such that $c \geq a,b$. We know that every modular and distributive semilattices have the directed above property. Moreover [2] have shown that every 0-distributive meet semilattice is also directed above.

The following characterizations of 0-distributive semilattices is due to [2]

Theorem 1: *Let S be a directed above meet semilattice with 0. Then the following conditions are equivalent ;*

(i) *S is 0-distributive*

(ii) *For each* $a \in S$, $(a]^{\perp} = (a)^{\perp} = (a)^{0} = [a)^{0}$ *is an ideal.*

(iii) *Every maximal filter of S is prime.*

The following characterization of semi prime ideals is due to [5].

Theorem 2: *Suppose S is a directed above meet semi lattice with 0 and J be an ideal of S. The following conditions are equivalent ;*

(i) *J is semi prime.*

(ii) *For every* $a \in S, \{a\}^{\perp_J} = \{x \in S : x \wedge a \in J\}$ *is a semi prime ideal containing J.*

(iii) $A^{\perp_J} = \{x \in S : x \wedge a \in J \text{ for all } a \in A\}$ *is a semi prime ideal containing J, when A is finite.*

(iv) *Every maximal filter disjoint from J is prime.*

Normal semilattices

A semilattices *S* with 0 is called a normal semilattice if its every prime ideal contains a unique minimal prime ideal. For detailed literature on normal lattices, we refer the reader to see [3] where Cornish dealt with distributive lattices. But recently, Nag [6] studied the normality in 0-distributive lattices. In this section, we will study the normality of 0-distributive semilattices. Let *P* be a prime ideal of *S*. Define $O(P) = \{x \in S \mid x \wedge y = 0 \text{ for some } y \in S - P\}$.

Proposition 3 : *If S is a 0-distributive semilattice, then $O(P)$ is an ideal of S and $O(P) \subseteq P$.*

Proof: Clearly, $O(P)$ is a downset. Let $x, y \in O(P)$. Then $x \wedge p = y \wedge q = 0$ for some $p, q \notin P$. This implies $x \wedge p \wedge q = y \wedge p \wedge q = 0$. Since S is 0-distributive, so there exists $t \geq x, y$ such that $t \wedge p \wedge q = 0$. Now since P is prime, so $p \wedge q \notin P$. This implies $t \in O(P)$. Thus $O(P)$ is an ideal. Obviously, $O(P) \subseteq P$.

Proposition 4 : *Let S be a 0-distributive semilattice. Then $O(P)$ is semiprime.*

Proof: Let $x, y, z \in S$ such that $x \wedge y \in O(P)$ and $x \wedge z \in O(P)$. Then $x \wedge y \wedge a = x \wedge z \wedge b = 0$ for some $a, b \in S - P$. This implies $(x \wedge a \wedge b) \wedge y = (x \wedge a \wedge b) \wedge z = 0$. So there exists $t \geq y, z$, such that $(x \wedge a \wedge b) \wedge t = 0$ as S is 0-distributive . Now since $a \wedge b \in S - P$, we have $x \wedge t \in O(P)$. Thus $O(P)$ is semiprime.

The following result is a generalization of a result due to [6]

Lemma 5: *Let S be a 0-distributive semilattice and P be a prime ideal of S. If Q is a minimal prime ideal containing $O(P)$ such that $Q \not\subseteq P$, then for any $y \in Q - P$, there exists $z \notin Q$ such that $y \wedge z \in O(P)$.*

Proof: Suppose the condition does not hold. Let $y \in Q - P$ and for all $z \notin Q$, we have $y \wedge z \notin O(P)$. Set $D = (S - Q) \vee [y)$. We claim that $O(P) \cap D = \varphi$. If not, let $p \in O(P) \cap D$. Then $p \in O(P)$ and $p \geq a \wedge y$ for some $a \notin Q$. Now $a \wedge y \leq p$ implies $a \wedge y \in O(P)$, which is a contradiction. Thus, $O(P) \cap D = \varphi$. Let M be the set of all proper filters of S containing D and disjoint from $O(P)$. Then clearly, M is non-empty as $D \subseteq M$. Let C be a chain in M and let $M = \bigcup\{X \mid X \in C\}$. We claim that M is a filter with $D \subseteq M$ and $M \cap O(P) = \varphi$. Let $x \in M$ and $y \geq x$. Then $x \in X$ for some $X \in C$. This implies $y \in X$ as X is a filter and hence $y \in M$. Now let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in C$. Since C is a chain, we have either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$. Then $x, y \in Y$ and hence $x \wedge y \in Y$ as Y is a filter. Hence $x \wedge y \in M$. Thus M is a filter containing D. Clearly, $M \cap O(P) = \varphi$.

Hence by Zorn's lemma, there is a maximal filter, say, R such that $D \subseteq R$ and $R \cap O(P) = \varphi$. Since $O(P)$ is semiprime, so by Theorem 2, R is a prime filter of S. Therefore $S - R$ is a minimal prime ideal containing $O(P)$. Moreover $S - R \subseteq Q$ and $S - R \neq Q$ as $y \in Q$ but $y \notin S - Q$. This contradicts the minimality of Q. Therefore there must exist $z \notin Q$ such that $y \wedge z \in O(P)$. □

Lemma 6 : *Let P be a prime ideal of a 0-distributive semilattice S. Then each minimal prime ideal containing $O(P)$ is contained in P.*

Proof: Let Q be a minimal prime ideal containing $O(P)$. If $Q \not\subseteq P$ then choose $y \in Q - P$. Then by Lemma 5, there exists $z \notin Q$ such that $y \wedge z \in O(P)$. This implies $x \wedge y \wedge z = 0$ for some $x \notin P$. As P is prime, we have $x \wedge y \notin P$. This implies $z \in O(P) \subseteq Q$, which is a contradiction.

Hence $Q \subseteq P$.

Proposition 7 : *If P is a prime ideal in a 0-distributive semilattice S, then the ideal $O(P)$ is the intersection of all the minimal prime ideals contained in P.*

Proof: Suppose $X = \bigcap\{Q \mid Q \textit{ is a } \min \textit{imal prime ideal and } Q \subseteq P\}$. We shall show that $O(P) = X$. Let Q be a minimal prime ideal such that $Q \subseteq P$. Suppose $x \in O(P)$. Then $x \wedge y = 0$ for some $y \notin P$. This implies $y \notin Q$ and hence $x \in Q$ as Q is a prime ideal. Thus $x \in X$ and hence $O(P) \subseteq X$. If $O(P) \neq X$, then there exists $x \in X$ such that $x \notin O(P)$. Then $[x) \cap O(P) = \varphi$. So by Zorn's lemma there exists a maximal filter F such that $[x) \subseteq F$ and $F \cap O(P) = \varphi$. Hence by Theorem 2, F is a prime filter as $O(P)$ is semiprime. Therefore, $S - F$ is a minimal prime ideal containing $O(P)$. Now $x \notin S - F$ implies $x \notin X$ which gives a contradiction. Hence $O(P) = X$.

Cornish [3] has given nice characterizations of normal lattices in presence of distributivity. Recently [6] generalizes a part of Cornish's result. Now we extend the result in case of a 0-distributive semilattice.

Theorem 8: *A 0-distributive semilattice S is a normal semilattice if and only if $O(P)$ is a prime ideal for every prime ideal P of S.*

Proof: Suppose S is a normal semilattice and P is a prime ideal of S. By Proposition 7, we have $O(P)$ is the intersection of all the minimal prime ideals contained in P. Since P contains a unique minimal prime ideal, so, $O(P)$ is the minimal prime ideal.

Conversely, suppose the condition holds. Let P be a prime ideal of S. Then by Proposition 7, $O(P)$ is the intersection of all minimal prime ideals contained in P. By the assumption $O(P)$ is prime, so $O(P)$ is the only minimal prime ideal contained in P. Thus S is normal.

Two ideals P and Q of a semilattice S are called comaximal if $P \vee Q = S$. A semilattice S with 0 is said to be a comaximal semilattice if any two minimal prime ideals of S are comaximal. By [6] we know that a distributive lattice with 0 is normal if and only if it is comaximal. [6] showed that this is not true for 0-distributive lattices. In case of meet semilattices, clearly, every comaximal semilattice with 0 is normal, but its converse is not necessarily true.

Every normal semilattice is not necessarity comaximal. For example, consider the meet semilattice S_1 in Figure 1 is 0-distributive.

Here $(a]$, $(b]$ are the only prime ideals of S_1. This shows that S_1 is normal. But $(a] \vee (b] \neq S_1$ implies that S_1 is not comaximal.

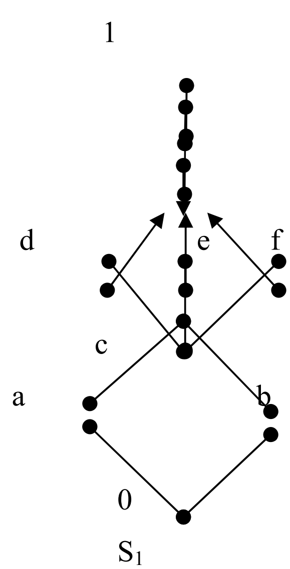


FIGURE 1

Now we turn our attention to pseudo complemented meet semilattices. It is evident that the underlying semilattice with 0 of a pseudo complemented meet semilattice is 0-distributive and the following rules hold in any pseudo complemented meet semilattice.

Lemma 9: Let S be a pseudo complemented meet semilattice. Then for any $a, b \in S$ the following conditions hold;

i) $a \leq b$ implies $b^* \leq a^*$;

ii) $a \leq a^{**}$;

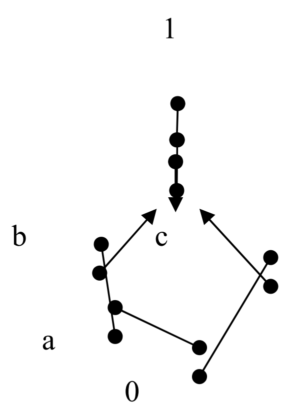
iii) $a = a^{***}$;

iv) $a^* \wedge b^* = d^*$ for some $d \geq a, b$.

v) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

Let S be a pseudo complemented meet semilattice. An ideal I of S called a p-ideal if $x \in I \quad \Rightarrow x^{**} \in I$.

Observe that not every ideal of a pseudo complemented meet semilattice is a p-ideal. For example consider the pseudo complemented meet semilattice of Figure-2. Here $a \in (a]$, but $a^{**} = b \notin (a]$. Hence $(a]$ is not a p-ideal.



S_2

Figure-2

Lemma 10: Let S be a pseudo complemented meet semilattice. Then an ideal I of S is a p-ideal if and only if for any $i, j \in I$, $(i^* \wedge j^*)^* \in I$.

Proof: Let I be a p-ideal. Then for any $i, j \in I$, we have $i^{**}, j^{**} \in I$. So there exists $d \geq i^{**}, j^{**}$ such that $d \in I$. Thus, $d^{**} \in I$ as I is a p-ideal.

Now, $i^{**} \leq d$ implies $i^{***} = i^{*} \geq d^{*}$. Similarly, $j^{*} \geq d^{*}$. Hence $d^{*} \leq i^{*} \wedge j^{*}$, and so $(i^{*} \wedge j^{*})^{*} \leq d^{**} \in I$.

Let S be meet semilattice with 0. Let $A \subseteq S$. Define $A^{*} = \{x \in S \mid x \wedge a = 0 \quad \textit{for all } a \in A\}$. An ideal I of S is called an α-ideal of S if for any $x \in I$, $(x]^{**} \subset I$.

The following Stone base Separation Theorem is due to [7].

Theorem11: Let S be a 0-distributive meet semilattice, I be an α-ideal and F be a meet subsemilattice of S such that $I \cap F = \varphi$. Then there is a prime α-ideal P such that $I \subseteq P$ and $P \cap F = \varphi$.

Let S be pseudo complemented meet semilattice. Then for any $x \in S$ we can easily show that $(x]^{*} = (x^{*}]$. Thus in a pseudo complemented meet semilattice an ideal I is an α-ideal if and only if it is a p-ideal. Thus the following result follows from the above Theorem 8.

Theorem 12: Let S be a pseudo complemented meet semilattice, I be a p-ideal and F be a filter of S such that $I \cap F = \varphi$. Then there is a prime p-ideal P such that $I \subseteq P$ and $P \cap F = \varphi$.

Recently [2] have proved that in a pseudo complemented meet semilattice, a prime ideal P is minimal if and only if $x \in P$ implies $x^{*} \notin P$. Following result is an extension of this result which is due to [7]. This also gives a characterization of prime p-ideals.

Theorem 13: *Let S be a pseudo complemented meet semilattice and let P be a prime ideal of S . Then the following conditions are equivalent;*

(i) *P is minimal*

(ii) $x \in P$ *implies that* $x^{*} \notin P$.

(iii) $x \in P$ *implies* $x^{**} \in P$, *that is* P *is a p-ideal.*

(iv) $P \cap D(S) = \varphi$.

S-Semilattice

For $A \subseteq S$, we defined $U(A) = \{x \in S \mid x \geq a \quad for\ all \ \ a \in A\}$.

A pseudo complemented meet semilattice *S₁* is called a S-semilattice if it satisfies the following Stone identity for all $a \in S_1$, $U\{a^*, a^{**}\} = \{1\}$. In other words, 1 is the only common upper bound of a^* and a^{**}. We have the following characterization of an S-semilattice.

Proposition 14: *Let S_1 be a pseudo complemented meet semilattice . Then S_1 is an S-semilattice if and only if for all* $x, y \in S_1$, $x \wedge y = 0$ *implies* $U\{x^*, x^{**}\} = \{1\}$.

Proof: Let S_1 be an S-semilattice and $x, y \in S_1$ such that $x \wedge y = 0$. Then $y \leq x^*$ and hence $y^* \geq x^{**}$. This implies $U\{x^*, y^*\} \subseteq U\{x^*, x^{**}\} = \{1\}$, and so $U\{x^*, y^*\} = \{1\}$.

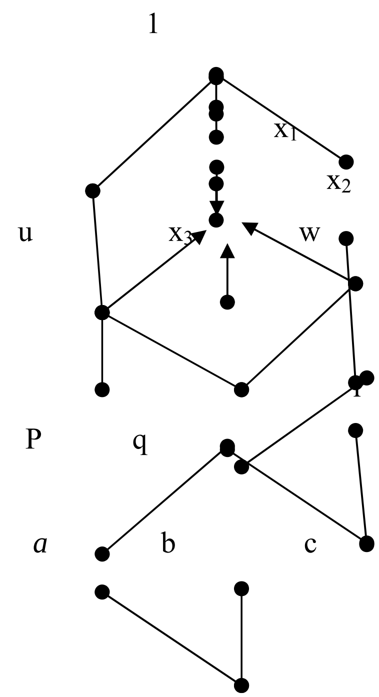
Conversely, suppose the condition holds and $x \in S_1$. Since $x \wedge x^* = 0$, we have $U\{x^*, x^{**}\} = \{1\}$. Thus S_1 in an S- semilattice.

Theorem 15: *Every S-semilattice is comaximal.*

Proof: Let S_1 be an S-semilattice and let P and Q be two distinct minimal prime ideals of S_1. Choose $a \in P - Q$. Since $a \wedge a^* = 0 \in Q$, we have $a^* \in Q$ as Q is prime. Since $a \in P$, by Theorem 13 $a^{**} \in P$. Since $U\{a^*, a^{**}\} = \{1\}$, so $1 \in P \vee Q$. Thus $P \vee Q = S_1$. In other word, S_1 is comaximal.

Remark 1: We have the following observation:

A comaximal pseudo complemented meet semilattice is not necessarily S-semilattice. For example consider the pseudo complemented meet semilattic S_3 given by diagram in Figure 3. Clearly, the ideals $(u], (q]$ and $(w]$ are the only prime ideals, and they are comaximal. Hence S_3 is comaximal. Now $U\{q^*, q^{**}\} = U\{b, q\} = \{1, x_1, x_2, ...\} \neq \{1\}$. Hence S_3 is not an S-semilattiice.



S_3

Figure- 3.

We conclude the paper with the following result.

Theorem 16: Every S-semilattice is normal.

Proof: Let A be an S-semilattice. If it is not normal, then there exists a prime ideal P containing two minimal prime ideals Q and R. Then there exists $q \in Q$ such that $q \notin R$. Now, $q \wedge q^* = 0 \in R$ implies $q^* \in R$ as R is prime. Again by Theorem-13, $q^{**} \in Q$ as Q is minimal. Hence $q^*, q^{**} \in P$.

Since P is an ideal, there exists $d \in P$ such that $d \geq q^*, q^{**}$, But $d \neq 1$ as P is a prime ideal. Hence $U\{q^*, q^{**}\} \neq \{1\}$, which implies that A is not an S-semilattice. This gives a contradiction. Therefore, A must be normal.

Remark: A pseudo complemented normal meet semilattice is not necessarily an S-semilattice. For example consider the pseudo complemented meet semilattic S_3

given by diagram in Figure 3. As mentioned in Remark 1, the ideals $(u], (q]$ and $(w]$ are the only prime ideals, so this is a normal meet semilattice, it is not S-semilattice.

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