

RELATION BETWEEN LATTICE AND SEMIRING

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Abstract

In this paper, connection between lattice and semiring are investigated. This is done by introducing some examples of lattice and semirings. Examples and results are illustrated. In some cases we have used MATLAB.

Keywords and phrases : lattice, semiring, MATLAB.

বিবৃত সার (Bengali version of the Abstract)

এই পত্রে আমরা আংশিক α - ঢাকনিযুক্তের (ফনডস. সতক্ষিভূতর অ* ট্রি ড্রতথভশফ)

ধারণাকে, সংক্ষেপে $p\alpha$ - ঢাকনিযুক্তের এবং আংশিক α -সংহত (ফনডস. সতক্ষিভূতর অ* ট্রি দষলসতদচ), সংক্ষেপে $p\alpha$ - সংহত (ফনডস. $p\alpha^*$ ট্রি দষলসতদচ) ফাজি সেটকে (পয়াঁ
ডনচড) উপস্থাপন করেছি এবং ফাজি টেপোলজীয় দেশে (পয়াঁ চৰসমৰব্যভূতত ডস্তদনড) ইহাদের বহুবিধ বৈশিষ্ট্যকে অনুসন্ধান করেছি ।

1. Introduction

The notion of semiring was first introduced by Vandiver in 1934. Vandiver introduced an algebraic system, which consists of non empty set S with two binary operations addition (+) and multiplication (.). The system ($S; +, .$) satisfies both distributive laws but does not satisfy cancellation law of addition. The system he constructed was ring like but not exactly ring. Vandiver called this system a ‘Semiring’. Additively inverse semirings are studied by Kervelles [8]. Luce [9] ,

Rutherford [2] , Petrich [6], Goodearl [4], Rentenner [1], Fang [5] have studied semiring. Birkhoff [3] has studied lattice theory.

2. Preliminaries:

In this section, we present some definitions and examples of lattice.

Definition 2.1: Suppose R is any non empty set and \leq is any partial order relation on R. Then $(R; \leq)$ is called a poset iff

$$\forall x, y, z \in R ;$$

Reflexive : $x \leq x$

Anti Symmetric : $x \leq y$ and $y \leq x \Rightarrow x = y$

Transitive: $x \leq y$ and $y \leq z \Rightarrow x \leq z$.

Definition 2.2: Suppose $(P : \leq)$ is a poset . It is called a **chain** iff

$$\forall a, b \in P; \text{ either } a \leq b \text{ or } b \leq a.$$

Definition 2.3: Let $(L; \leq)$ be a poset. Then L is called a **lattice** iff

$$\forall a, b \in L;$$

$$a + b = \sup \{a, b\} \in L$$

$$\text{and } a.b = \inf \{a, b\} \in L.$$

Example 2.3(a) : $(L = \{1, 2, 3, 4, 6, 8, 12, 24\}; |, +, .)$ is a lattice, where

$$a + b = \text{lcm}\{a, b\}, a.b = \text{gcd}\{a, b\}.$$

The MATLAB function scripts are not shown. Outputs are presented below.

```
>> A=[1 2 3 4 6 8 12 24];
```

```
>> join(A)
```

ans =

| + | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 24 |
|----|----|----|----|----|----|----|----|----|
| 1 | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 24 |
| 2 | 2 | 2 | 6 | 4 | 6 | 8 | 12 | 24 |
| 3 | 3 | 6 | 3 | 12 | 6 | 24 | 12 | 24 |
| 4 | 4 | 4 | 12 | 4 | 12 | 8 | 12 | 24 |
| 6 | 6 | 6 | 6 | 12 | 6 | 24 | 12 | 24 |
| 8 | 8 | 8 | 24 | 8 | 24 | 8 | 24 | 24 |
| 12 | 12 | 12 | 12 | 12 | 12 | 24 | 12 | 24 |
| 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 |

>> meet(A)

ans =

| . | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 24 |
|----|---|---|---|---|---|---|----|----|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 1 | 2 | 2 | 2 | 2 | 2 |
| 3 | 1 | 1 | 3 | 1 | 3 | 1 | 3 | 3 |
| 4 | 1 | 2 | 1 | 4 | 2 | 4 | 4 | 4 |
| 6 | 1 | 2 | 3 | 2 | 6 | 2 | 6 | 6 |
| 8 | 1 | 2 | 1 | 4 | 2 | 8 | 4 | 8 |
| 12 | 1 | 2 | 3 | 4 | 6 | 4 | 12 | 12 |
| 24 | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 24 |

Proposition 2.4: A poset $(L; \leq)$ is a lattice and $\forall a, b \in L; a + b = a \vee b$, $a.b = a \wedge b$.

Then L satisfies

- (1) $a + a = a$ and $a.a = a$ [idempotent law]
- (2) $a + b = b + a$ and $a.b = b.a$ [commutative law]
- (3) $a + (b + c) = (a + b) + c$ and $a.(b.c) = (a.b).c$ [associative law]
- (4) $a.(a + b) = a$ and $a + (a.b) = a$ [absorption law]

Proof: Trivial.

Proposition 2.5: $\forall n \in \mathbb{N}$;

Let $D(n) = \{k \in \mathbb{N} : k \mid n\}$. Define $\forall p, q \in D(n)$;

$$+(p, q) = \text{lcm}(p, q)$$

$$\cdot(p, q) = \gcd(p, q).$$

Then $(D(n); +, \cdot)$ is a lattice under $p \mid q \Leftrightarrow p \cdot q = p$.

Proof: Trivial.

Definition 2.6: Suppose $(L; +, \cdot)$ is a lattice. Then L is called **distributive lattice** iff $\forall a, b, c \in L; a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

3. Semiring

In this section we prove some basic properties. We also present how lattices are used to construct semirings.

Definition 3.1: Let S be non empty set with two binary operations $+$ and \cdot . Then the algebraic structure $(S; +, \cdot)$ is called a **semiring** iff

- (i) $(S; +)$ is a semigroup
- (ii) $(S; \cdot)$ is a semigroup
- (iii) $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(b+c) \cdot a = b \cdot a + c \cdot a; \forall a, b, c \in S$

Example 3.1(a): Consider $(S = \{0, 1, 2, 3, 4, 5\}; +_6, \times_6)$. Then $(S; +_6, \times_6)$ is a semiring.

Example 3.1(b) : $(A = \{1, 2, 4, 8, 16\}; |, +, \cdot)$ is a semiring.

Definition 3.2: Let $(S, +, \cdot)$ be a semiring. Then S is called

- (i) **additively commutative** iff $\forall x, y \in S, x + y = y + x$.
- (ii) **multiplicatively commutative** iff $\forall x, y \in S, x \cdot y = y \cdot x$.

$(S; +, \cdot)$ is called a **commutative semiring** iff both (i) and (ii) hold.

Definition 3.3 : Let $(S; +, .)$ be a commutative semiring with zero (0) and identity (1). Then $(S; +, .)$ is called **idempotent semiring** iff $\forall x \in S$,

$$x + x = x = xx.$$

Proposition 3.4 : $\forall n \in \mathbb{N}$; let $D(n) = \{k \in \mathbb{N} : k \mid n\}$. Define

$$+ (p, q) = \text{lcm}(p, q)$$

$$\cdot (p, q) = \text{gcd}(p, q).$$

Then $(D(n); +, \cdot)$ is a semiring.

Proof:

$\forall n \in \mathbb{N}$; let $D(n) = \{k \in \mathbb{N} : k \mid n\}$. Define

$$+ (p, q) = \text{lcm}(p, q)$$

$$\cdot (p, q) = \text{gcd}(p, q).$$

By Proposition 2.5, $(D(n); +, \cdot)$ is a lattice.

So by Proposition 2.4, $D(n)$ satisfies

1. Idempotent Law
2. Commutative law
3. Absorption Law
4. Associative Law

By the Definition of lattice

$$+ (p, q) \in D(n)$$

$$\cdot (p, q) \in D(n).$$

Again by associative law and by Definition of lattice, we get

$(D(n); +)$ and $(D(n); \cdot)$ are semigroups.

Distributive law :

Let $p, q, r \in D(n)$

Now

$$p|n, q|n, r|n$$

We know

$$\Rightarrow (p, q) \mid p$$

Again

$$\cdot(p,q) \mid q \mid +(q,r)$$

S₀

Again

$$\begin{aligned} \gcd(p, r) &\mid p \\ \Rightarrow (p, r) &\mid p \end{aligned}$$

Again

$$\cdot(p, r) \mid r \mid +(q, r)$$

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From (ii) and (iv) we get

From (i) and (iii) we get

Hence from (v) and (vi) we get

$$\langle p, \pm(q, r) \rangle \equiv \pm(\langle p, q \rangle, \langle p, r \rangle).$$

Similarly

$$\cdot(\pm(q, r), p) \equiv \pm(\cdot(q, p), \cdot(r, p)).$$

Hence $(P(\eta), \pm, \cdot)$ is a semiring.

Proposition 3.5 : Let L be any distributive lattice and $R = L \times \mathbb{N}$. We define

$\forall x, y \in L$ and $i, j \in \mathbb{N}$,

$$(x, i) + (y, j) = (x \vee y, \max\{i, j\})$$

$$(x, i) \cdot (y, j) = (x \wedge y, \max\{i, j\}).$$

Then $(R; +, \cdot)$ is an idempotent commutative semiring. But R is not a lattice.

Proof: We have L is a distributive lattice.

So $\forall x, y \in L$;

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

(R; +) is a semigroup:

$$\forall (x, i), (y, j), (z, k) \in L \times \mathbb{N};$$

$$(x, i) + (y, j) = (x \vee y, \max\{i, j\}) \in L \times \mathbb{N}$$

Again

$$\begin{aligned} (x, i) + ((y, j) + (z, k)) &= (x, i) + (y \vee z, \max\{j, k\}) \\ &= (x \vee (y \vee z), \max\{i, \max\{j, k\}\}) \\ &= ((x \vee y) \vee z, \max\{\max\{i, j\}, k\}) \\ &= (x \vee y, \max\{i, j\}) + (z, k) \\ &= ((x, i) + (y, j)) + (z, k) \end{aligned}$$

Therefore $(R, +)$ is a semigroup.

(R; .) is a semigroup:

$$\forall (x, i), (y, j), (z, k) \in L \times \mathbb{N};$$

$$(x, i) \cdot (y, j) = (x \wedge y, \max\{i, j\}) \in L \times \mathbb{N}$$

Again

$$\begin{aligned} (x, i) \cdot ((y, j) \cdot (z, k)) &= (x, i) \cdot (y \wedge z, \max\{j, k\}) \\ &= (x \wedge (y \wedge z), \max\{i, \max\{j, k\}\}) \\ &= ((x \wedge y) \wedge z, \max\{\max\{i, j\}, k\}) \end{aligned}$$

$$\begin{aligned}
 &= (x \wedge y, \max\{i, j\}) \cdot (z, k) \\
 &= ((x, i) \cdot (y, j)) \cdot (z, k)
 \end{aligned}$$

Therefore (R, \cdot) is a semi group

Distributive law :

$$\begin{aligned}
 (x, i) \cdot ((y, j) + (z, k)) &= (x, i) \cdot (y \vee z, \max\{j, k\}) \\
 &= (x \wedge (y \vee z), \max\{i, \max\{j, k\}\}) \\
 &= ((x \wedge y) \vee (x \wedge z), \max\{i, j, k\}) \quad [\because L \text{ is a distributive} \\
 &\quad \text{lattice}] \\
 &= ((x \wedge y) \vee (x \wedge z), \max\{\max\{i, j\}, \{i, k\}\}) \\
 &= (x \wedge y, \max\{i, j\}) + (x \wedge z, \max\{i, k\}) \\
 &= (x, i) \cdot (y, j) + (x, i) \cdot (z, k)
 \end{aligned}$$

Similarly

$$((x, i) + (y, j)) \cdot (z, k) = (x, i) \cdot (z, k) + (y, j) \cdot (z, k)$$

Hence $(R, +, \cdot)$ is a semiring.

Idempotency:

$$\begin{aligned}
 (x, i) + (x, i) &= (x \vee x, \max\{i, i\}) \\
 &= (x, i)
 \end{aligned}$$

Again

$$\begin{aligned}
 (x, i) \cdot (x, i) &= (x \wedge x, \max\{i, i\}) \\
 &= (x, i)
 \end{aligned}$$

Therfore $(R, +, \cdot)$ is an idempotent semiring.

Commutativity:

$$\begin{aligned}
 (x, i) + (y, j) &= (x \vee y, \max\{i, j\}) \\
 &= (y \vee x, \max\{j, i\}) \\
 &= (y, j) + (x, i)
 \end{aligned}$$

Again

$$\begin{aligned}
 (x, i) \cdot (y, j) &= (x \wedge y, \max\{i, j\}) \\
 &= (y \wedge x, \max\{j, i\}) \\
 &= (y, j) \cdot (x, i)
 \end{aligned}$$

Therefore $(R, +, \cdot)$ is an idempotent commutative semiring.

2nd Part:

Let us show by an example R is not a lattice.

Let

$$(x, 2), (x, 3) \in R$$

Now

$$\begin{aligned}
 (x, 2) + ((x, 2) \cdot (x, 3)) &= (x, 2) + (x \wedge x, \max\{2, 3\}) \\
 &= (x, 2) + (x, 3) \\
 &= (x \vee x, \max\{2, 3\}) \\
 &= (x, 3) \\
 &\neq (x, 2); \forall x \in L
 \end{aligned}$$

Therefore absorption law does not hold.

Therefore R is not a lattice.

Proposition 3.6: $(\mathbb{N}; +, \cdot)$ is an idempotent commutative semiring.

Define $\forall a, b \in \mathbb{N}$;

$$a + b = \max\{a, b\}$$

$$a \cdot b = \max\{a, b\}.$$

But R is not a lattice.

Proof: $\forall a, b \in \mathbb{N}$;

There are six cases:

Case (i) : $a < b < c$

Case (ii) : $a < c < b$

Case (iii) : $c < b < a$

Case (iv) : $c < a < b$

Case (v) : $b < a < c$

Case (vi) : $b < c < a$

For case (i):

$$a+b = \max\{a,b\} = b \in \mathbb{N}$$

Again

$$\begin{aligned} a + (b + c) &= a + \max\{b, c\} \\ &= a + c \\ &= \max\{a, c\} \\ &= c \end{aligned}$$

$$\begin{aligned} (a + b) + c &= \max\{a, b\} + c \\ &= b + c \\ &= \max\{b, c\} \\ &= c \end{aligned}$$

Therefore $(\mathbb{N}; +)$ is a semigroup.

Again

$$a.b = \max\{a, b\} = b \in \mathbb{N}$$

$$\begin{aligned} a.(b.c) &= a.\max\{b, c\} \\ &= a.c \\ &= \max\{a, c\} \\ &= c \\ (a.b).c &= \max\{a, b\}.c \\ &= b.c \\ &= \max\{b, c\} \\ &= c \end{aligned}$$

Therefore $(\mathbb{N}; .)$ is a semigroup.

Distributive Law:

$$\begin{aligned} a.(b+c) &= a.\max\{b, c\} \\ &= a.c \\ &= \max\{a, c\} \\ &= c \end{aligned}$$

$$\begin{aligned}
 a.b + a.c &= \max\{a,b\} + \max\{a,c\} \\
 &= b + c \\
 &= \max\{b, c\} \\
 &= c
 \end{aligned}$$

Therefore $a.(b+c) = a.b + a.c$.

Similarly

$$(b+c).a = b.a + c.a.$$

For case (i)

$(\mathbb{N}; +, .)$ is a semiring.

Idempotency :

$$a+a = \max\{a,a\} = a$$

$$a.a = \max\{a,a\} = a$$

Therefore $(\mathbb{N}; +, .)$ is an idempotent semiring.

Commutativity:

$$a.b = \max\{a,b\} = \max\{b,a\} = b.a$$

$$a+b = \max\{a,b\} = \max\{b,a\} = b+a.$$

Therefore $(\mathbb{N}; +, .)$ is an idempotent commutative semiring.

Let us show by an example R is not a lattice:

$$2+2.3 = 2+\max\{2, 3\} = 2+3 = \max\{2, 3\} = 3 \neq 2$$

Therefore absorption law does not hold.

Hence R is not a lattice.

Similarly we can show for all cases

$(\mathbb{N}; +, .)$ is an idempotent commutative semiring.

Proposition 3.7 : $(\mathbb{R}^+, \oplus, \otimes)$ is a collection of non negative real numbers, where \oplus and \otimes are defined by $\forall x, y \in \mathbb{R}^+$;

$$x \oplus y = \max\{x, y\} \text{ and } x \otimes y = \min\{x, y\}.$$

Then $(\mathbb{R}^+, \oplus, \otimes)$ is a chain semiring.

Proof: Suppose $x < y$.

Then

$$x \oplus y = \max\{x, y\} = y.$$

$$\therefore x \oplus y \in \mathbb{R}^+$$

Again

$$x \otimes y = \min\{x, y\} = x$$

$$\therefore x \otimes y \in \mathbb{R}^+$$

Therefore closure law holds in \mathbb{R}^+ .

If $r \in \mathbb{R}^+$, then

$$x < y \Rightarrow r \otimes x < r \otimes y.$$

Now

$$(r \otimes x) \oplus (r \otimes y) = r \otimes y = r \otimes (x \oplus y).$$

If $r \in \mathbb{R}^+$, then

$$x < y \Rightarrow x \otimes r < y \otimes r.$$

Again

$$(x \otimes r) \oplus (y \otimes r) = y \otimes r = (x \oplus y) \otimes r.$$

Therefore distributive law holds in \mathbb{R}^+ .

There are six cases :

Case (i) : $x < y$ and $y < z$

Case (ii) : $x < y$ and $z < y$

Case (iii) : $y < x$ and $x < z$

Case (iv) : $y < x$ and $z < x$

Case (v) : $x < z$ and $z < y$

Case (vi) : $x < z$ and $y < z$

Case (i) :

$$x < y \text{ and } y < z$$

$$\therefore x < z$$

Now

$$x \oplus (y \oplus z) = x \oplus z = z$$

$$(x \oplus y) \oplus z = y \oplus z = z$$

$$\therefore (x \oplus y) \oplus z = x \oplus (y \oplus z)$$

Again

$$x \otimes (y \otimes z) = x \otimes y = x$$

$$(x \otimes y) \otimes z = x \otimes z = x$$

$$\therefore (x \otimes y) \otimes z = x \otimes (y \otimes z)$$

Case (ii) : (a) $x < y$ and $z < y$, $x < z$

(b) $x < y$ and $z < y$, $z < x$

$$(a) \quad x \oplus (y \oplus z) = x \oplus y = y$$

$$(x \oplus y) \oplus z = y \oplus z = y$$

$$\therefore (x \oplus y) \oplus z = x \oplus (y \oplus z)$$

Again

$$x \otimes (y \otimes z) = x \otimes z = x$$

$$(x \otimes y) \otimes z = x \otimes z = x$$

$$\therefore (x \otimes y) \otimes z = x \otimes (y \otimes z)$$

(b) $x \oplus (y \oplus z) = x \oplus y = y$

$$(x \oplus y) \oplus z = y \oplus z = y$$

$$\therefore (x \oplus y) \oplus z = x \oplus (y \oplus z)$$

Again

$$x \otimes (y \otimes z) = x \otimes z = z$$

$$(x \otimes y) \otimes z = x \otimes z = z$$

$$\therefore (x \otimes y) \otimes z = x \otimes (y \otimes z)$$

Case (iii) : $y < x$ and $x < z$, $y < z$

$$x \oplus (y \oplus z) = x \oplus z = z$$

$$(x \oplus y) \oplus z = x \oplus z = z$$

$$\therefore (x \oplus y) \oplus z = x \oplus (y \oplus z)$$

Again

$$x \otimes (y \otimes z) = x \otimes y = y$$

$$(x \otimes y) \otimes z = y \otimes z = y$$

$$\therefore (x \otimes y) \otimes z = x \otimes (y \otimes z)$$

Case (iv) : (a) $y < x$ and $z < x$, $y < z$

(b) $y < x$ and $z < x$, $z < y$

For (a):

$$x \oplus (y \oplus z) = x \oplus z = x$$

$$(x \oplus y) \oplus z = x \oplus z = x$$

$$\therefore (x \oplus y) \oplus z = x \oplus (y \oplus z)$$

Again

$$x \otimes (y \otimes z) = x \otimes y = y$$

$$(x \otimes y) \otimes z = y \otimes z = y$$

$$\therefore (x \otimes y) \otimes z = x \otimes (y \otimes z)$$

For (b):

$$x \oplus (y \oplus z) = x \oplus y = x$$

$$(x \oplus y) \oplus z = x \oplus z = x$$

$$\therefore (x \oplus y) \oplus z = x \oplus (y \oplus z)$$

Again

$$x \otimes (y \otimes z) = x \otimes z = z$$

$$(x \otimes y) \otimes z = y \otimes z = z$$

$$\therefore (x \otimes y) \otimes z = x \otimes (y \otimes z)$$

Case (v) : $x < z$ and $z < y$

$$\therefore x < y$$

$$x \oplus (y \oplus z) = x \oplus y = y$$

$$(x \oplus y) \oplus z = y \oplus z = y$$

$$\therefore (x \oplus y) \oplus z = x \oplus (y \oplus z)$$

Again

$$\begin{aligned}x \otimes (y \otimes z) &= x \otimes z = x \\(x \otimes y) \otimes z &= x \otimes z = x \\\therefore (x \otimes y) \otimes z &= x \otimes (y \otimes z)\end{aligned}$$

Case (vi) : (a) $x < z$ and $y < z$, $x < y$
 (b) $x < z$ and $y < z$, $y < x$

For (a):

$$\begin{aligned}x \oplus (y \oplus z) &= x \oplus z = z \\(x \oplus y) \oplus z &= x \oplus z = z \\\therefore (x \oplus y) \oplus z &= x \oplus (y \oplus z)\end{aligned}$$

Again

$$\begin{aligned}x \otimes (y \otimes z) &= x \otimes y = x \\(x \otimes y) \otimes z &= x \otimes z = x\end{aligned}$$

$$\therefore (x \otimes y) \otimes z = x \otimes (y \otimes z)$$

For (b):

$$\begin{aligned}x \oplus (y \oplus z) &= x \oplus z = z \\(x \oplus y) \oplus z &= x \oplus z = z \\\therefore (x \oplus y) \oplus z &= x \oplus (y \oplus z)\end{aligned}$$

Again

$$\begin{aligned}x \otimes (y \otimes z) &= x \otimes y = y \\(x \otimes y) \otimes z &= y \otimes z = y \\\therefore (x \otimes y) \otimes z &= x \otimes (y \otimes z)\end{aligned}$$

Therefore for all cases associative law holds.

Therefore $(\mathbb{R}^+, \oplus, \otimes)$ is a semiring.

It is clear that \mathbb{R}^+ is chain.

Hence $(\mathbb{R}^+, \oplus, \otimes)$ is a chain semiring. Δ

Acknowledgement:

The authors are grateful to the learned referees for their valuable suggestions which have led to improvement in the presentation.

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