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SOME PROPERTIES OF 1-DISTRIBUTIVE JOIN SEMILATTICES

By

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Abstract.

J. C. Varlet introduced the concept of 1-distributive lattices to generalize the notion of dual pseudo complemented lattices. A lattice L *with 1 is called a 1-distributive lattice if for all* $a,b,c \in L$, $a \vee b = 1 = a \vee c$ *imply* $a \vee (b \wedge c) = 1$. *Of course every distributive lattice with 1 is 1-distributive. Also every dual pseudo complemented lattice is 1-distributive.Recently, Shiuly and Noor extended this concept for directed below join semi lattices. A join semi lattice* S *is called directed below if for all* $a,b \in S$, *there exists* $c \in S$ *such that* $c \leq a,b$. *Again Y. Rav has extended the concept of 1-distributivity by introducing the notion of semi prime filters in a lattice. Recently, Noor and Ayub have studied the semi prime filters in a directed below join semi lattice. In this paper we have included several characterizations and properties of 1-distributive join semi lattices. We proved that for a join sub semi lattice* A *of* S, $A^1 = \{x \in S : x \vee a = 1 \text{ for some } a \in A\}$ *is a semi prime filter of* S *if and only if* S *is 1-distributive. We also showed that a directed below join semi lattice with 1 is 1-distributive if and only if for all* $a,b \in S$, $(a)^1 \cap (b)^1 = (d)^1$ *for some* $d \in S, d \leq a,b$. *Introducing the notion of* α*- filters and using different equivalent conditions of 1-distributive join semi lattices we have given a 'Separation theorem' for* α*- filters.*

Keywords and Phrases : *1-distributive join semi lattice, Semi prime filter, Prime ideal, Maximal ideal,* α*-filter.*

Introduction: J. C. Varlet [8] first introduced the concept of 1-distributive lattices. Then many authors including [1, 2, 7] studied them for lattices and semilattices. By [7], a join semilattice S with 1 is called 1-distributive if for all $a,b,c \in S$ with $a \vee b = 1 = a \vee c$ imply $a \vee d = 1$ for some $d \leq b,c$. We also know that a 1-distributive join semilattice is directed below. A join semi lattice S is called *directed below* if for all $a,b \in S$, there exists $c \in S$ such that $c \leq a,b$. A non-empty subset F of a directed below join semilattice S is called an *up set* if for

$x \in F$ and $y \geq x$ $(y \in S)$ imply $y \in F$. Up set F is called a *filter* if for $x, y \in F$, there exists $z \leq x, y$ such that $z \in F$.

A non-empty subset I of S is called a *down set* if $x \in I$ and $y \leq x$ $(y \in S)$ imply $y \in I$. A down set I of S is called an *ideal* if for all $x, y \in I$, $x \vee y \in I$. A filter (up set) Q is called a prime filter (down set) if $a \vee b \in Q$ implies either $a \in Q$ or $b \in Q$. An ideal P of S is called prime if $S - P$ is a prime filter.

An ideal I of S is called a maximal ideal if $I \neq S$ and it is not contained by any other proper ideal of S. A prime up set P is called a minimal prime up set if it does not contain any other prime up set of S.

Following Lemmas in lattices are due to [2] and [6], and also hold for join semi lattices by [1, 7].

Lemma 1: *A proper subset I of a join semilattice S is a maximal ideal if and only if $S - I$ is a minimal prime up set.* ☐

Lemma 2: *Let I be a proper ideal of a join semilattice S with 1. Then there exists a maximal ideal containing I.* ☐

Following result is due to [1, Lemma 5]

Lemma 3: *Let I be an ideal and F be a filter of a directed below join semilattice S, such that $F \cap I = ¨$. Then I is a maximal ideal disjoint from F if and only if for each $a \notin I$, there exists $b \in I$ such that $a \vee b \in F$.*

Let S be a join semilattice with 1. For a non-empty subset A of S, we define $A^{\perp^d} = \{x \in S \mid x \vee a = 1 \quad \textit{for all } a \in A\}$ .This is clearly an up set, but we can not prove that this is a filter even in a distributive join semilattice. If L is a lattice with 1, then it is well known that L is 1-distributive if and only if $D(L)$*, the lattice of all filters of* L is 0-distributive.Unfortunately, we can not prove or disprove that when S is a 1-distributive join semi lattice, then $D(S)$ is 0-distributive. But if $D(S)$ is 0-distributive, then it is easy to prove that S is also 1-distributive. We define $A^1 = \{x \in S \mid x \vee a = 1 \textit{ for some } a \in A\}$.This is obviously an up set. Moreover, $A \subseteq B$ implies $A^1 \subseteq B^1$. For any $a \in S$, it easy to check that $(a)^{\perp^d} = (a)^1 = (a]^1$.

Following result is due to [7].

Theorem 4: *Let S be a directed below join semilattice with 1. Then the following conditions are equivalent.*

(i) S *is 1-distributive*

(ii) *For each* $a \in S$, $(a] \vee (d] \subseteq (a] \vee (b]$ *is a filter.*

(iii) *Every maximal ideal of* S *is prime.* □

Since in a 1-distributive join semilattice S, for each $a \in S$, $(a)^{\perp^d}$ is a filter, so we prefer to denote it by $[a)^{*d}$. Y Rav [6] have generalized the concept of 1-distributive lattices and introduced the notion of semi prime ideals and filters in lattices. Recently, [3] have studied many properties of semi prime ideals in lattices. In a directed below join semilattice S, a filter F is called a *semi prime filter* if for all $x, y, z \in S$, $x \vee y \in F$, $x \vee z \in F$ imply $x \vee d \in F$ for some $d \leq y, z$. In a distributive semilattice, every filter is semi prime. Moreover, the semilattice itself is obviously a semi prime filter. Also, every prime filter of S is semi prime.

Theorem 5: *For any join sub semilattice* A *of a directed below join semi lattice* S *with 1,* A^1 *is a semi prime filter of* S *if and only if* S *is 1-distributive.*

Proof: Suppose S is 1-distributive. We already know that A^1 is an up set, Now let $x, y \in A^1$. Then $x \vee a = 1 = y \vee b$ for some $a, b \in A$. Then $x \vee a \vee b = 1 = y \vee a \vee b$. Since S is 1-distributive, so $(a \vee b) \vee d = 1$ for some $d \leq x, y$. Now $a \vee b \in A$ implies $d \in A^1$, and so A^1 is an ideal. Finally let $x \vee y \in A^1$, and $x \vee z \in A^1$. Then $x \vee y \vee a_1 = 1 = x \vee z \vee b_1$ for some $a_1, b_1 \in A$. Thus $x \vee a_1 \vee b_1 \vee y = 1 = x \vee a_1 \vee b_1 \vee z$. Then by the 1-distributive property, $x \vee a_1 \vee b_1 \vee d_1 = 1$ for some $d_1 \leq y, z$. Thus $x \vee d_1 \in A^1$ as $a_1 \vee b_1 \in A$. Therefore A^1 is semi prime. Conversely, if A^1 is a semi prime filter for every join sub semilattice A of S, then in particular $(a)^1$ is a filter. Thus S is 1-distributive by Theorem 4. □

Following characterization of semi prime filters is due to [1].

Theorem 6: *Let* S *be a directed below join Semilattice with 1 and* F *be a filter of* S.
Then the following conditions are equivalent.

(i) F *is semi prime*

(ii) *Every maximal ideal disjoint to* F *is prime.* □

Thus we have the following separation theorem.

Theorem 7: *Let S be a 1-distributive join semi lattice and A be a join subsemilattice of S. Then for an ideal I disjoint from A^1, there exists a prime filter containing A^1 and disjoint from I.*

Lemma 8: *Let A and B be ideals of a directed below join semilattice S with 1, such that $A \cap B^1 = \varphi$. Then there exists a minimal prime up set containing B^1 and disjoint from A.*

Proof: Observe that $1 \notin A \vee B$. For if $1 \in A \vee B$, Then $1 \le a \vee b$ for some $a \in A$, $b \in B$. That is, $a \vee b = 1$, which implies $a \in B^1$ gives a contradiction. It follows that $A \vee B$ is a proper ideal of S. Then by Lemma 2, $A \vee B \subseteq M$ for some maximal ideal M. If $x \in M \cap B^1$, Then $x \in M$ and $x \vee b_1 = 1$ for some $b_1 \in B \subseteq M$. This implies $1 \in M$ which is a contradiction as M is maximal. Thus, $M \cap B^1 = \varphi$. Then by Lemma 1, $S - M$ is a minimal prime up set containing B^1. Moreover, $(S - M) \cap A =$ ¨ .

Lemma 9: *Let A be an ideal of a directed below join semilattice S with 1. Then A^1 is the intersection of all the minimal prime up sets disjoint from A.*

Proof: Let N be any minimal prime up set disjoint from A. If $x \in A^1$, then $x \vee a = 1$ for some $a \in A$ and so $x \in N$ as N is prime.
Conversely, let $y \in A^1$. Then $y \vee a \neq 1$ for all $a \in A$. Hence $A \vee (y]$ is a proper ideal of S. Then by Lemma 2, $A \vee (y] \subseteq M$ for some maximal ideal M. Thus by Lemma 1, $S - M$ is a minimal prime up set. Clearly $(S - M) \cap A =$ ¨ and $y \notin S - M$.

Now we include some characterization of 1-distributive join semilattices.

Theorem 10: *Let S be a directed below join semilattice with 1. Then the following statements are equivalent.*

(i) *S is 1-distributive.*

(ii) *For each $a \in S$, $(a)^1$ is a semi prime ideal.*

(iii) *For any three ideals A, B, C of S,*
$(A \vee (B \cap C))^1 = (A \vee B)^1 \cap (A \vee C)^1$

(iv) *For all $a, b, c \in S$, $((a] \vee ((b] \cap (c]))^1 = ((a] \vee (b])^1 \cap ((a] \vee (c])^1$*

(v) *For all $a, b, c \in S$, $(a \vee d)^1 = (a \vee b)^1 \cap (a \vee c)^1$ for some $d \le b, c$.*

Proof: (i) $\Leftrightarrow$ (ii). Follows by theorem 4.

(i) $\Rightarrow$ (iii). Let $x \in (A \vee B)^1 \cap (A \vee C)^1$. Then $x \in (A \vee B)^1$ and $x \in (A \vee C)^1$. Thus $x \vee f = 1 = x \vee g$ for some $f \in A \vee B$ and $g \in A \vee C$. Then $f \leq a_1 \vee b$, and $g \leq a_2 \vee c$ for some $a_1, a_2 \in A$, $b \in B$, $c \in C$. This implies $x \vee a_1 \vee b = 1 = x \vee a_2 \vee c$ and so $x \vee a_1 \vee a_2 \vee b = 1 = x \vee a_1 \vee a_2 \vee c$. Since S is 1-distributive, so $x \vee a_1 \vee a_2 \vee d = 1$ for some $d \leq b, c$. Now $a_1 \vee a_2 \in A$ and $d \in B \cap C$. Therefore, $((a_1 \vee a_2) \vee d \in A \vee (B \cap C)$ and so $x \in (A \vee (B \cap C))^1$. The reverse inclusion is trivial as $A \vee (B \cap C) \subseteq A \vee B, A \vee C$. Hence (iii) holds.

(iii) $\Rightarrow$ (iv) is trivial by considering $A = (a]$, $B = (b]$ and $C = (c]$ in (iii).

(iv)$\Rightarrow$(v). Let (iv) holds. Suppose $x \in (a \vee b)^1 \cap (a \vee c)^1$. Then by (iv) $x \in ((a] \vee (b])^1 \cap ((a] \vee (c])^1 = ((a] \vee ((b] \cap (c]))^1$. This implies $x \vee f = 1$ for some $f \in (a] \vee ((b] \cap (c])$. Then $f \leq a \vee d$ for some $d \in (b] \cap (c]$. That is, $f \leq a \vee d$ for some $d \leq b, c$. It follows that $x \vee a \vee d = 1$ and so $x \in (a \vee d)^1$. On the other hand, $(a] \vee (d] \subseteq (a] \vee (b]$ and $(a] \vee (d] \subseteq (a] \vee (c]$ implies that $(a \vee d)^1 \subseteq (a \vee b)^1 \cap (a \vee c)^1$. Therefore (v) holds as the reverse inclusion is trivial.

(v) $\Rightarrow$ (i). Suppose (v) holds. Let $a, b, c \in S$ with $a \vee b = 1 = a \vee c$. Then $a \vee (a \vee b) = 1 = a \vee (a \vee c)$ implies $a \in (a \vee b)^1 \cap (a \vee c)^1 = (a \vee d)^1$ for some $d \leq b, c$. Thus, $a \vee (a \vee d) = 1$ for some $d \leq b, c$. That is $a \vee d = 1$ for some $d \leq b, c$ and so S is 1 –distributive.

Now we include few more characterizations of 1-distributive semilattices.

Theorem 11: *Let S be a directed below join semi lattice with 1. Then the following are equivalent.*

- *(i)* *S is 1-distributive.*
- *(ii)* *For any three ideals A, B, C of L.*
  $((A \cap B) \vee (A \cap C))^1 = A^1 \cap (B \vee C)^1$
- *(iii)* *For any two ideals A, B of S, $(A \cap B)^1 = A^1 \cap B^1$*
- *(iv)* *For all $a, b \in S$, $(a)^1 \cap (b)^1 = (d)^1$ for some $d \leq b, c$.*
- *(v)* *For all $a, b \in S$, $[a)^{*d} \cap [b)^{*d} = [d)^{*d}$ for some $d \leq b, c$.*

Proof: (i) $\Rightarrow$ (ii). Suppose S is 1-distributive, Since $(A \cap B) \vee (A \cap C) \subseteq A$ and $B \vee C$, so $((A \cap B) \vee (A \cap C))^1 \subseteq A^1 \cap (B \vee C)^1$. Now suppose $x \in A^1 \cap (B \vee C)^1$ Then $x \in A^1$ and $x \in (B \vee C)^1$. Thus $x \vee a = 1$ for some $x \in A$ and $x \vee d = 1$ for some $d \in B \vee C$. Now $d \in B \vee C$ implies $d \leq b \vee c$ for some $b \in B, c \in C$. Hence

$x \vee a = 1 = x \vee b \vee c$. Then $x \vee c \vee a = 1 = x \vee c \vee b$. Since S is 1-distributive, so $x \vee c \vee d_1 = 1$ for some $d_1 \leq a,b$. Then $d_1 \in A \vee B$. Now $x \vee a = 1$ implies $x \vee d_1 \vee a = 1 = x \vee d_1 \vee c$. Again by the 1-distributivity, $x \vee d_1 \vee d_2 = 1$ for some $d_2 \leq a,c$ that is $d_2 \in A \vee C$. Therefore, $x \in ((A \cap B) \vee (A \cap C))^1$ and so (ii) holds.
(ii) $\Rightarrow$ (iii) is trivial by considering $B = C$ in (iii).
(iii) $\Rightarrow$ (iv). Choose $A = (a]$ and $B = (b]$ in (iii).

Now for all $d \leq a,b$, $(a] \supseteq (d]$ and $(b] \supseteq [d)$ and so $(d]^1 \subseteq (a)^1 \cap (b)^1$. Also by (iii), $(a)^1 \cap (b)^1 = ((a] \cap (b])^1$. Thus, $x \in (a)^1 \cap (b)^1$ implies $x \vee d_1 = 1$ for some $d_1 \leq a,b$ as S is directed below. That is, $x \in (d_1)^1$ for some $d_1 \leq a,b$. Thus (iv) holds.
(iv) $\Leftrightarrow$ (v) is obvious.
(v)$\Rightarrow$(i). Suppose (v) holds and for $a,b,c \in S$, $a \vee b = 1 = a \vee c$. Then $a \in [b)^{*d} \cap [c)^{*d} = [d)^{*d}$ for some $d \leq b,c$. Therefore, $a \vee d = 1$ and so S is 1-distributive.

A filter F in a directed below join semilattice S with 1 is called an α-filter if for each $x \in F$,$\{x\}^{\perp^d \perp^d} \subseteq F$.

Proposition 12: *If F is an ~-filter of a 1-distributive join semilattice S, then $F = \{y \in S \,|\, [y) \subseteq \{x\}^{\perp^d \perp^d}$ for some $x \in F\}$.*

Proof: Let $y \in R.H.S$. Then $[y) \subseteq \{x\}^{\perp^d \perp^d} \subseteq F$. This implies $y \in F$. Conversely, let $y \in F$. Since S is 1-distributive, so by theorem 4, $[y)^{\perp^d}$ is a filter and $[y) \cap [y)^{\perp^d} = [1)$. Thus, $[y) \subseteq [y)^{\perp^d \perp^d}$, which implies $y \in R.H.S.$
Prime separation theorem for α-ideals in 0-distributive lattices was given [4]. Now we include a prime separation theorem on α-filters for 1-distributive join semilattices.
Theorem 13: *Let I be an ideal and F be an ~-filter of a directed below join semilattice S with I, such that $I \cap F =$ ¨ . If D(S), the lattice of filters of S is 0-distributive, then there exists a prime ~-filter Q containing F such that $Q \cap I = \varphi$.*

Proof: By lemma 2, there exists a maximal ideal *M* containing I and disjoint to F. Thus $Q = S - M$ is a minimal prime up set containing F and disjoint to *M*. Now let $p,q \in S - M$. Then by lemma 3, there exist $a,b \in M$ such that

$a \vee p \in F$ and $b \vee q \in F$. Then by proposition 12, $[a \vee p) \subseteq (x]^{\perp^d \perp^d}$ and $[b \vee q) \subseteq [y)^{\perp^d \perp^d}$ for some $x, y \in F$. Thus $[a \vee p) \wedge [x)^{\perp^d} = [1) = [b \vee q) \wedge [y)^{\perp^d}$. This implies $[a \vee b) \wedge [x)^{\perp^d} \wedge [y)^{\perp^d} \wedge [p) = [1) = [a \vee b) \wedge [x)^{\perp^d} \wedge [y)^{\perp^d} \wedge [q)$, Now as F is a filter, so there exists $d_1 \leq x, y$ such that $d_1 \in F$. Again by Theorem 11(v), $[x)^{\perp^d} \wedge [y)^{\perp^d} = [d_2)^{\perp^d}$ for some $d_2 \leq x, y$. Then $d = d_1 \vee d_2 \in F$, and so $[d)^{\perp^d} \subseteq [x)^{\perp^d} \wedge [y)^{\perp^d} = [d_2)^{\perp^d} \subseteq [d)^{\perp^d}$ .Thus $[x)^{\perp^d} \wedge [y)^{\perp^d} = [d)^{\perp^d}$ for some $d \in F$, $d \leq x, y$. Then we have $[a \vee b) \wedge [d)^{\perp^d} \wedge [p) = [1) = [a \vee b) \wedge [d)^{\perp^d} \wedge [q)$. Since $D(S)$ is 0-distributive, so $[a \vee b) \wedge [d)^{\perp^d} \wedge ([p) \vee [q)) = [1)$. Then $[a \vee b) \wedge [d)^{\perp^d} \wedge [t) = [1)$ for some $t \leq p, q$. Thus $[a \vee b \vee t) \subseteq [d)^{\perp^d \perp^d} \subseteq F \subseteq S - M$ .But $a \vee b \in M$ implies $t \in S - M$ as $S - M$ is prime. Therefore $Q = S - M$ is a filter. Now let $x \in Q$. If $x \in F$, then $[x)^{\perp^d \perp^d} \subseteq F \subseteq Q$ as F is an α-filter. Finally if $x \in Q - F$. Then again by Lemma 3, there exists $a \in M$ such that $a \vee x \in F$. Thus $[a)^{\perp^d \perp^d} \vee [x)^{\perp^d \perp^d} \subseteq F \subseteq Q$. Since $a \notin Q$, so $[a)^{\perp^d \perp^d} \nsubseteq Q$. Therefore, $[x)^{\perp^d \perp^d} \subseteq Q$ as Q is prime, and hence Q is also an α-filter.

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