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GLIVENKO CONGRUENCE ON A 0-DISTRIBUTIVE MEET SEMILATTICE

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Abstract:

In This paper the author studies the Glivenko congruence R in a 0-distributive meet semilattice. It is proved that a meet semilattice S with 0 is 0-distributive if and only if the quotient semilattice $\frac{S}{R}$ *is distributive. Hence S is 0-distributive if and only if (0] is the Kernel of some homomorphism of S onto a distributive meet semilattice with 0.*

Key words and phrases: *Glivenko congruence, 0-distributive semilattice, distributive meet semilattice.*

Introduction:

J.C.Varlet [7] first introduced the concept of 0-distributive lattices. Then many authors including [1,2,3,5] studied them for lattices and semilattices. By [2], a meet semilattice S with 0 is called 0-distributive if for all $a,b,c \in S$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge d = 0$ for some $d \geq b,c$. A meet semi lattice S is called *directed above* if for all $a,b \in S$, there exists $c \in S$ such that $c \geq a,b$. We know that all modular and distributive semilattices have the directed above property. Moreover, [3] have shown that every 0-distributive meet semilattice is directed above.

Let S be a meet semilattice with 0. For a non-empty subset A of S, we define $A^{\perp} = \{x \in S \mid x \wedge a = 0 \quad \textit{for all } a \in A\}$.This is clearly a down set, but we can not prove that this is an ideal even in a distributive meet semilattice, when A is infinite.

By [2,3] we know that, for any $a \in S$,$\{a\}^{\perp}$ is an ideal if and only if S is 0-distributive.

We define a relation R on a meet semilattice S by $a \equiv b(R)$ if and only if $(a]^{\perp} = (b]^{\perp}$. In other words, $a \equiv b(R)$ is equivalent to "for each $x \in S$, $a \wedge x = 0$ if and only if $b \wedge x = 0$".

We will show below that this is a congruence on the meet semilattice S. We call it Glivenko congruence. In this paper we establish some results on this congruence in a meet semilattice.

We start with the following result which is due to [3]. We include its proof for the convenience of the reader.

Lemma 1: Let S be a meet-semilattice with *0*. Again let $A, B \subseteq S$ and $a, b \in S$ then we have the followings:

(i) If $A \cap B = (0]$, then $B \subseteq A^{\perp}$

(ii) $A \cap A^{\perp} = (0]$,

(iii) $A \subseteq B$ imply that $B^{\perp} \subseteq A^{\perp}$

(iv) If $a \le b$ imply that $\{b\}^{\perp} \subseteq \{a\}^{\perp}$ and $\{a\}^{\perp\perp} \subseteq \{b\}^{\perp\perp}$

(v) $\{a\}^{\perp} \cap \{a\}^{\perp\perp} = (0]$

(vi) $\{a \wedge b\}^{\perp\perp} = \{a\}^{\perp\perp} \cap \{b\}^{\perp\perp}$

(vii) $A \subseteq A^{\perp\perp}$

(viii) $A^{\perp\perp\perp} = A^{\perp}$

Proof: (i) Let $b \in B$. Then $a \wedge b = 0$ for all $a \in A$, as $A \cap B = \{0\}$. Thus $b \in A^{\perp}$. Hence $B \subseteq A^{\perp}$.

(ii) Let $x \in A \cap A^{\perp}$.

$= x \in A$ and $x \wedge a = 0$ for all $a \in A$

$= x \wedge x = 0$

$= x = 0$

(iii) Let $A \subseteq B$

$\therefore A \cap B^{\perp} \subseteq B \cap B^{\perp} = (0]$

$\Rightarrow A \cap B^{\perp} = (0]$

So, by (i), $B^{\perp} \subseteq A^{\perp}$.

(iv) Let $x \in \{b\}^{\perp}$. Then $b \wedge x = 0$ for some $x \in S$. Since $a \le b$, then we have $a \wedge x = 0$ for some $x \in S$, which imply that $x \in \{a\}^{\perp}$. Hence,

$\{b\}^{\perp} \subseteq \{a\}^{\perp}$.

Now let $x \in \{a\}^{\perp\perp}$. Then $y \wedge x = 0$ for all $y \in \{a\}^{\perp}$, which implies that $y \wedge x = 0$ for all $y \in \{b\}^{\perp}$ as $\{b\}^{\perp} \subseteq \{a\}^{\perp}$ Thus $x \in \{b\}^{\perp\perp}$.

Hence,

$\{a\}^{\perp\perp} \subseteq \{b\}^{\perp\perp}$.

(v) Let $x \in \{a\}^{\perp} \cap \{a\}^{\perp\perp}$. Then $x \in \{a\}^{\perp}$ and $x \in \{a\}^{\perp\perp}$ which implies that $x \wedge a = 0$ and $x \wedge y = 0$ for all $y \in \{a\}^{\perp}$. Thus $x \wedge x = 0$. Hence

$\{a\}^{\perp} \cap \{a\}^{\perp\perp} = (0]$.

(vi) Let $x \in \{a\}^{\perp\perp} \cap \{b\}^{\perp\perp}$ and $y \in \{a \wedge b\}^{\perp}$. Then we get $(y \wedge a) \wedge b = 0$, which implies that $(y \wedge a) \in \{b\}^{\perp}$. Since $x \in \{b\}^{\perp\perp}$, we get $(x \wedge y) \wedge a = 0$.

Hence $x \wedge y \in \{a\}^{\perp}$. Since $x \in \{a\}^{\perp\perp}$, we get $x \wedge y \in \{a\}^{\perp\perp}$. Thus $x \wedge y \in \{a\}^{\perp} \cap \{a\}^{\perp\perp} = (0]$. Hence $x \wedge y = 0$ for all $y \in (a \wedge b)^{\perp}$. Therefore $x \in (a \wedge b)^{\perp\perp}$. Thus $\{a\}^{\perp\perp} \cap \{b\}^{\perp\perp} \subseteq \{a \wedge b)\}^{\perp\perp}$.

Conversely we can write that $a \wedge b \leq a$, which implies by (i) $(a \wedge b)^{\perp\perp} \subseteq \{a\}^{\perp\perp}$. Similarly $\{a \wedge b\}^{\perp\perp} \subseteq \{b\}^{\perp\perp}$. Therefore we have, $\{a \wedge b\}^{\perp\perp} \subseteq \{a\}^{\perp\perp} \cap \{b\}^{\perp\perp}$.

(vii) Let $x \in A$, consider any $r \in A^{\perp}$, then we get $x \wedge a = 0$ for all $a \in A$ which implies that $r \wedge x = 0$. Since $x \wedge r = 0$ for all $r \in A^{\perp}$. Thus $x \in A^{\perp\perp}$. Hence $A \subseteq A^{\perp\perp}$.

(viii) Since by (vii) $A \subseteq A^{\perp\perp}$. So by (iii) $(A^{\perp\perp})^{\perp} \subseteq A^{\perp}$.

Hence $A^{\perp\perp\perp} \subseteq A^{\perp}$. Since by (vii) $A^{\perp} \subseteq (A^{\perp})^{\perp\perp} = A^{\perp\perp\perp}$. Therefore we have $A^{\perp} = A^{\perp\perp\perp}$.

Hence the proof is completed. □

Theorem 2: R is a meet congruence on S.

Proof: It is clearly an equivalent relation.

Let $a \equiv b(R)$ and $t \in S$

Then $(a]^{\perp} = (b]^{\perp}$, so by using Lemma 1, we have $(a \wedge t]^{\perp} = (a \wedge t]^{\perp\perp\perp}$

$$=\{(a]^{\perp\perp} \wedge (t]^{\perp\perp}\}^{\perp}$$

$$=\{(b]^{\perp\perp}\wedge(t]^{\perp\perp}\}^{\perp}$$

$$=(b\wedge t]^{\perp\perp\perp}=(b\wedge t]^{\perp}$$

This implies $a\wedge t\equiv b\wedge t(R)$, and so R is a meet congruence on S.

A meet semilattice S with 0 is weakly complemented if for any pair of distinct elements a, b of S, there exists an element c disjoint from one of these elements but not from the other. In particular, if $a<b$, then there exists $c\in S$ such that $a\wedge c=0$ but $b\wedge c\neq 0$.

Theorem 3: If S is weakly complemented. Then R is an equality relation.

Proof: Suppose $a,b\in S$ with $a\neq b$. Since S is weakly complemented, so there exist $x\in S$, $a\wedge x=0$ but $b\wedge x\neq 0$. This implies $(a,b)\notin R$. Hence R is an equality relation.

Theorem 4: For any meet semilattice S. $\frac{S}{R}$ is also a meet semilattice. Moreover S is directed above if and only if $\frac{S}{R}$ is directed above.

Proof: For $[a],[b]\in\frac{S}{R}$, define $[a]R\wedge[b]R=[a\wedge b]R$. Thus $\frac{S}{R}$ is a meet semilattice.

Now let $a,b\in S$. If S is directed above, then there exists $d\geq a,b$.

Now, $[a]R\wedge[d]R=[a\wedge d]R=[a]R$ and $[b]R\wedge[d]R=[b\wedge d]R=[b]R$

Implies $[d]R\geq[a]R,[b]R$. Thus, $\frac{S}{R}$ is also directed above.

Conversely suppose $\frac{S}{R}$ is directed above. Let $a,b\in S$

Then $[a],[b]\in\frac{S}{R}$. Since $\frac{S}{R}$ is directed above, so there exists $C\in\frac{S}{R}$ such that $C\geq[a]R,[b]R$. Then there exists $d\in C$, such that $[d]=C$ and $d\geq a,b$. So S is directed above.

A meet semilattice S is called a *distributive semilattice* if $w\geq a\wedge b$ implies that there exist $x\geq a$, $y\geq b$ in S such that $w=x\wedge y$.

Following result gives some characterizations of distributive meet semilattices which are due to [4, Theorem 1.1.6], also see [6].

Lemma 5: *For a meet semilattice S, the following conditions are equivalent.*

i) *S is distributive.*

ii) $w \geq a \wedge b$ *implies that there exists* $y \in S$ *such that* $y \geq b$, $y \geq w$ *and* $y \wedge a = a \wedge w$.

iii) $a \wedge b = b \wedge c$ *implies that there exists* $y \in S$ *such that* $y \geq b$, $y \geq c$ *and* $y \wedge a = a \wedge c$.

Theorem 6: For any meet semilattice S, the quotient meet semilattice $\frac{S}{R}$ is weakly complemented. Furthermore, S is 0-distributive if and only if $\frac{S}{R}$ is distributive.

Proof: First part: For any meet semilattice S, when A< B in $\frac{S}{R}$, there exists a in A and b in B such that a< b, and by the definition of R, there is an element c such that $c \wedge a = 0$ and $c \wedge b \neq 0$. Since the minimum class of $\frac{S}{R}$ has the only element 0, the class C of c satisfies $A \wedge C = [0]$ and $C \wedge B \neq [0]$. Therefore, $\frac{S}{R}$ is weakly complemented.

For second part: Let S be 0-distributive. Suppose $B \geq A \wedge C$ in $\frac{S}{R}$. So there exists $b \in B$, $a \in A$, $c \in C$ such that $b \geq a \wedge c$ and $B = [b]R$, $A = [a]R$, $C = [c]R$. Suppose $a \wedge b \wedge x = 0$. Then $a \wedge c \wedge x = 0$. Since S is 0-distributive, so there exists $d \geq b, c$ such that $a \wedge d \wedge x = 0$. On the other hand, for any $d \geq b, c$, $a \wedge d \wedge x = 0$ implies $a \wedge d \wedge x \wedge b = a \wedge b \wedge x = 0$. Therefore, $a \wedge b \equiv a \wedge d(R)$ for some $d \geq b, c$. In other words, $A \wedge B = A \wedge D$ where $D = [d] \geq B, C$. Therefore by [4, Theorem 1.1.6 (ii)], $\frac{S}{R}$ is distributive.

Conversely, suppose $\frac{S}{R}$ is distributive. Let $a, b, c \in S$ with $a \wedge b = a \wedge c = 0$. Then $[a] \wedge [b] = [a] \wedge [c] = [0]R$. Since [0] contains only the element 0, so $A \wedge B = A \wedge C = 0$, where $A = [a]$, $B = [b]$, $C = [c]$. Then $B \geq A \wedge C$. Since $\frac{S}{R}$ is distributive, so $B = A_1 \wedge C_1$ for some $A_1 \geq A$, $C_1 \geq C$. Moreover, $B = A_1 \wedge C_1$ implies $C_1 \geq B$. Thus $0 = A \wedge B = A \wedge A_1 \wedge C = A \wedge C_1$.

Now $C_1 \geq B, C$ implies $C_1 = [d]R$ for some $d \geq b, c$. Therefore, $a \wedge d = 0$ for some $d \geq b, c$ and so S is 0-distributive.

We conclude the paper with the following result.

Theorem 7: Let S be a meet semilattice. Then the following conditions are equivalent

(i) S is 0-distributive.

(ii) (0] is the kernel of some homomorphism of S onto a distributive semilattice with 0.

(iii) (0] is the kernel of a homomorphism of S onto a 0-distributive semilattice.

Proof: (i) $\Rightarrow$(ii). Suppose S is 0-distributive. Then by Theorem 1, the binary relation R defined by $x \equiv y(R)$ iff $(x]^{\perp} = (y]^{\perp}$ is a congruence on S. Moreover by Theorem 5, $\frac{S}{R}$ is a distributive meet semilattice. Clearly the map $a \rightarrow [a]R$ is a homomorphism. Now let $a \equiv 0(R)$. Then $0 \wedge a = 0$ implies $a = a \wedge a = 0$. Here [0]R contains only 0 of S. That is, (0] is a complete congruence class modulo R.

(ii) $\Rightarrow$(iii) is obvious as every distributive semilattice with 0 is 0-distributive.

(iii) $\Rightarrow$(i). Let $^{-}$ be a congruence on S for which (0] is the zero element of the 0-distributive semilattice $S/^{-}$. Then $x \wedge y = 0 = x \wedge z$ imply $[x]^{-} \wedge [y]^{-} = [x \wedge y]^{-} = [x \wedge z]^{-} = [x]^{-} \wedge [z]^{-}$. Thus, $[x]^{-} \wedge [y]^{-} = (0] = [x]^{-} \wedge [z]^{-}$. Hence by the 0-distributivity of $\frac{S}{=}$, $[x]^{-} \wedge [d]^{-} = (0]$, for some $[d]^{-} \geq [y]^{-}, [z]^{-}$. This implies $x \wedge d \in (0]$ and so $x \wedge d = 0$, where $d \geq y, z$. Therefore, S is 0-distributive. □

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