NUMERICAL STUDY AND CFD SIMULATIONS OF INCOMPRESSIBLE NEWTONIAN FLOW BY SOLVING STEADY NAVIER-STOKES EQUATIONS USING NEWTON'S METHOD.

By

K. M. Helal

Department of Mathematics, Comilla University, Comilla
Email : mailtohelal@yahoo.com

Abstract

In this paper, incompressible Newtonian flow is numerically studied by approximating the solution of the steady Navier-Stokes equations in two dimensional case. Computational Fluid Dynamics (CFD) simulations are carried out by using the finite element method. Newton's method is applied to solve the Navier-Stokes equations where the finite element solutions of Stokes equations is considered as the initial guess to obtain the convergence of Newton's sequence. The numerical simulations are presented in terms of the contours of velocity, pressure and streamline. All the meshes and simulations are implemented on the general finite element solver FreeFem++. A two-dimensional benchmark flow was computed with posteriori estimates. It has also been established that the free access solver FreeFem++ can provide a reasonable good numerical simulations of complicated flow behavior.

Keywords and Phrases: Navier-Stokes equations, CFD simulation, finite element method, Newton-Raphson method.

Introduction:

The aim of this work is to analyze numerically and simulate computationally the incompressible Newtonian flow which is modeled by the steady Navier-Stokes equations in two dimensional case. CFD describes the fluid flow in terms of mathematical models such as Navier-Stokes equations which consist of constitutive equations in the form of non-linear system of partial differential equations . In the present paper, Newton's method is applied to solve the Navier-Stokes system which is discretized using the Hood-Taylor finite elements. The numerical simulations are obtained computationally by the implementation of the finite element method where the finite element solution of

Stokes equations is chosen as the initial approximation of Newton's method. It is obtained that the Newton's sequence converges quadratically to the unique solution of Navier-Stokes equations for sufficiently small mesh size h and a moderate Reynolds number Re which is in good agreement with the results discussed by Kim et al. in [9], Ghia et al. [7] and many authors. In [9], Kim et al. discussed theoretically the Newton's method for the Navier-Stokes equations with finite element initial guess of Stokes equations. We briefly discuss the mathematical and numerical analysis, and analyze the approach problem in the context of finite element method ([10], [5], [11]). All the numerical simulations are implemented with our own script developed in FreeFem++ using the equivalent iterative variational formulation of Navier-Stokes problem. The solutions are obtained computationally and graphically in terms of velocity, pressure and streamline contours. A two-dimensional benchmark problem is computed and posteriori estimates for the rate of convergence is established. Finally, we draw some conclusions.

The Constitutive Equations and Problem Formulation: For a simple, isotropic, incompressible fluid, the Cauchy stress tensor $\mathbf{T}$ can be expressed as

$$\mathbf{T} = -p\mathbf{I} + \tau_s$$

where p is the hydrostatic pressure, τ_s is the extra stress tensor and $\mathbf{I}$ is the identity matrix or Kronecker tensor. For a Newtonian fluid, the dissipative effects of frictional forces can be described by a linear relation between extra stress tensor and rate of strain tensor, i.e.,

$$\tau_s = 2\eta\mathbf{D}(\mathbf{u}) \text{ (Stokes law)}$$

where $\eta > 0$ is the dynamic viscosity coefficient expressing the fluid's resistance which it offers to shear strain during the displacement ($[\eta] = Pa\,s$), $\mathbf{D}(\mathbf{u}) = \dfrac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^t]$ is the symmetric part of the velocity gradient. So, the Cauchy stress tensor can be written in the form

$$\mathbf{T} = -p\mathbf{I} + 2\eta\mathbf{D}(\mathbf{u}) = -P\mathbf{I} + \eta[\nabla\mathbf{u} + (\nabla\mathbf{u})^t]$$

(1)

The Navier-Stokes equations for incompressible fluid is a system of non-linear equations formed by the law of conservation of mass and the momentum equations. Considering $\mathbf{T}$, as in (1), the Navier-Stokes equations which model the incompressible Newtonian flow can be formed as

$$\begin{cases} \ldots \dfrac{\partial \mathbf{u}}{\partial t} + \ldots (\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \cdot \mathbf{T} + \ldots \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{T} = -P\mathbf{I} + \sim[\nabla\mathbf{u} + (\nabla\mathbf{u})^t] \end{cases} \Leftrightarrow \begin{cases} \ldots \dfrac{\partial \mathbf{u}}{\partial t} + \ldots (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \nabla \cdot \sim[\nabla\mathbf{u} + (\nabla\mathbf{u})^t] + \ldots \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \ldots \dfrac{\partial \mathbf{u}}{\partial t} + \ldots (\mathbf{u} \cdot \nabla)\mathbf{u} - \sim \Delta\mathbf{u} + \nabla p = \ldots \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \qquad (2)$$

Considering $\ldots$ as a constant, we define the kinematic viscosity by $\in = \dfrac{\sim}{\ldots}$ (m^2/s)

and the scaled pressure $p = \dfrac{p}{\ldots}$ (m^2/s^2) still denoted by p and we obtain the Navier-Stokes equations for steady flow as

$$\begin{cases} \ldots(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \in \Delta\mathbf{u} = \ldots \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

$$(3)$$

The above equations are non-dimensionalized as follows:

$$\mathbf{x} = \dfrac{\mathbf{x}^{'}}{L}, \quad t = \dfrac{t^{'}}{T} = \dfrac{Ut^{'}}{L}, \quad \mathbf{u} = \dfrac{\mathbf{u}^{'}}{U}, \quad p = \dfrac{p^{'}L}{\sim U}, \quad \mathbf{f} = \dfrac{\mathbf{f}^{'}L^2}{\sim U}$$

$$\mathrm{Re} = \ldots \dfrac{UL}{\sim} = \dfrac{UL}{\in}$$

where L is the characteristic length, U is the characteristic velocity, T is the characteristic time and Re is the Reynold's number which is the ratio of inertial to viscous forces.

The non-dimensional governing equations is of the form

$$\begin{cases} \mathrm{Re}(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \Delta\mathbf{u} = \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

The mathematical analysis of Navier-Stokes equations can be found in ([11], [12]).
Nomenclature:
Before discussing the boundary conditions and variational formulation, we introduce some notations of different function spaces in the following table, details of which can be found in ([4],[1]).

$C^0(\Omega)$ or $C(\Omega)$	The vector space of all continuous functions on $\Omega \subset R^d$ $(d = 2, 3)$
$C^m(\Omega)$	Vector space of all functions, where all the partial derivatives D^r of order $0 \le |r \le m|$ are continuous on Ω
$C^\infty(\Omega)$	The vector space of all infinitely differentiable functions
$C_0^m(\Omega)$	The space of all functions in $C^m(\Omega)$ with compact support
$L^p(\Omega)$	The Lebesgue spaces
$W^{m,p}(\Omega)$, where $m \ge 0$ be an integer and $1 \le p \le \infty$	The standard Sobolev spaces
$\|\cdot\|_{m,p}$	Norms of $W^{m,p}(\Omega)$
$H^m(\Omega)$	$W^{m,p}(\Omega)$, for $p = 2$
$L^p(\Omega)$	$W^{0,p}(\Omega)$

Boundary Conditions:

To close mathematical formulation and obtain a well-posed problem, the equations (3) need to be supplemented by some boundary conditions. For simplicity, we consider the case in which the system of differential equations (3) is equipped with the Dirichlet boundary conditions

$$\mathbf{u} = \mathbf{g} \text{ on } \partial\Omega \text{ (adherence conditions)}$$

For the incompressible fluids, the Dirichlet boundary condition $\mathbf{g}$ satisfies the compatibility condition [3]

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. We take the homogeneous Dirichlet boundary conditions (no-slip boundary conditions) which describes a fluid confined into a domain with fixed boundary (the boundary is at rest). With the homogeneous Dirichlet boundary conditions defined over Ω, we can write the steady Navier-Stokes problem as follows:

Given $\mathbf{f} \in \mathbf{L}^2(\Omega)$, find $(\mathbf{u}, p)$ such that

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p-\in\Delta\mathbf{u}=\mathbf{f}, & \text{in}\quad \Omega \\ \nabla\cdot\mathbf{u}=0 & \text{in}\quad \Omega \\ \mathbf{u}=0 & \text{on}\quad \partial\Omega \end{cases}$$

(4)

If the velocity of the flow is small enough, then the non-linear convective term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ is negligible. So, for slow viscous flows, we obtain the following Stokes problem:

Given $\mathbf{f}\in\mathbf{L}^2(\Omega)$, find $(\mathbf{u},\,p)$ such that

$$\begin{cases} \nabla p-\in\Delta\mathbf{u}=\mathbf{f}, & \text{in}\quad \Omega \\ \nabla\cdot\mathbf{u}=0 & \text{in}\quad \Omega \\ \mathbf{u}=0 & \text{on}\quad \partial\Omega \end{cases}$$

(5)

Variational formulation of Steady Navier-Stokes Problem: The variational formulation of the Navier-Stokes equations consists of integral equations which is obtained by taking integral over the domain of the scalar product of the momentum equation and the continuity equation with appropriate test functions, and applying the Green integration formula. Following Ladyzhenskaya (1959), we assume that $\mathbf{u}\in C^2(\Omega)\bigcap C^0(\overline{\Omega})$ and $p\in C^1(\Omega)\bigcap C^0(\overline{\Omega})$ are the classical (or strong) solution of (4). Consider two Hilbert spaces $\mathbf{V}=\mathbf{H}_0^1(\Omega)$ and $Q=L_0^2(\Omega)$ and let $\mathbf{v}\in\mathbf{V}$ and $q\in Q$ be two arbitrary test functions. The variational formulation of the Navier-Stokes problem (4) reads :

Given $\mathbf{f}\in\mathbf{H}^{-1}(\Omega)$, find $(\mathbf{u},\,p)\in\mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)$ such that

$$\begin{cases} \displaystyle\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{u}.\mathbf{v}-\int_\Omega p\nabla.\mathbf{v}+\in\int_\Omega\nabla\mathbf{u}:\nabla\mathbf{v}=\int_\Omega\mathbf{f}.\mathbf{v}, & \forall\mathbf{v}\in\mathbf{V} \\ \displaystyle\int_\Omega\nabla\cdot\mathbf{u}\,q=0, & \forall q\in Q \end{cases}$$

(6)

Problems (4) and (6) are equivalent.

We reformulate the variational formulation of the previous problem in a general abstract formulation that is suitable for many elliptic problems. We introduce the continuous and coercive $(\mathbf{V} - elliptic)$ bilinear form:

$$a(\mathbf{u}, \mathbf{v}) = \in (\nabla\mathbf{u}, \nabla\mathbf{v}) = \in \int_{\Omega} \nabla\mathbf{u} : \nabla\mathbf{v}$$

and continuous bilinear form

$$b(\mathbf{v}, p) = -(p, \nabla \cdot \mathbf{v}) = -\int_{\Omega} p\nabla \cdot \mathbf{v},$$

And we also introduce continuous trilinear form

$$c(\mathbf{w}; \mathbf{u}, \mathbf{v}) = ((\mathbf{w} \cdot \nabla)\mathbf{u}, \mathbf{v}) = \int_{\Omega} (\mathbf{w} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}$$

Taking into account the above forms, we can reformulate the variational formulation of the Navier-stokes problem as abstract formulation which can be written as follows:

Given $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, find $\mathbf{u} \in \mathbf{V}, p \in Q$ such that

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}; \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u}, q) = 0, & \forall q \in Q \\ \mathbf{u} = 0 & on\ \partial\Omega \end{cases}$$

(7)

Existance and uniqueness of the solution: It can be proved [12] that the problem (7) is well-posed and equivalent to (4). The existence and uniqueness of theorem for the solutions of Navier-Stokes system can be found in (Galdi (1994) [6], Girault and Raviart (1986) [11], Temam (1984) [12]).

Numerical Analysis for the Navier-Stokes Problem: We use finite element method (FEM) to approximate the numerical solutions of Navier-stokes problem (7).

Dividing the domain of solution into a finite numberof subdomains, the finite elements, the approach variational problem is defined over a finite-dimensional subspace Vh of V (infinite-dimensional function space where theexact solution exists) where h is a discretization parameter. The solution $(\mathbf{u}, p)$ of the problem (7) lives in a space of infinite dimension. In this case, it is normally impossible to

calculate the exact solution. Rather we determine the approximate value $\mathbf{u}_h$ and p_h, of $\mathbf{u}$ and p, each one defined in finite dimensional subspaces V_h. These spaces are formed by polynomials and for all function v_h in V_h (in particular $\mathbf{u}_h$ and p_h for the appropriate spaces) we have

$$v_h = \sum_{i=1}^{n} \ulcorner_i \{_i, \ulcorner \in IR, i = 1, \cdots, n \text{ where } \{\{_1, \{_2, \cdots, \{_n\} \text{ is a basis of } V_h.$$

Let T_h be a non-degenerated triangulations of Ω, with $h > 0$ the discretization parameter and let $\mathbf{V}_h$ and Q_h be two finite-dimensional spaces for the velocity and the pressure, respectively, such that $\mathbf{V}_h \in \mathbf{H}^1(\Omega)$ and $Q_h \in L_0^2(\Omega)$. We define pair o discrete space $\mathbf{V}_h^0 = \mathbf{V}_h \cap \mathbf{H}_0^1(\Omega)$ and $M_h = Q_h \cap L_0^2(\Omega)$.

In these spaces, the discrete problem can be written as:

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h^0 \times M_h$ such that

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + c(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{V}_h^0, \\ b(\mathbf{u}_h, q_h) = 0, & \forall q_h \in M_h, \end{cases}$$

(8)

The existence and uniqueness of the problem (8) is generated by the fact that the discrete space $\mathbf{V}_h^0$ and M_h verify a compatibility condition known as 'consistency condition', 'inf-sup condition' or LBB-condition [11]. The next theorem deals with the error estimate for the Navier-Stokes approximation of (8) using the Hood-Taylor finite element method. Proof can be found in [11].

Theorem 1:

Let the solution $(\mathbf{u}, p)$ of the Navier-Stokes system (4) satisfy

$$\mathbf{u} \in \mathbf{H}^{k+1}(\Omega) \cap \mathbf{H}_0^1(\Omega), \ p \in H^k(\Omega) \cap L_0^2(\Omega), \quad k = 1, 2.$$

If the triangulation T_h is regular and it has no triangle with two edges on $\partial\Omega$, then

the solution $(\mathbf{u}_h, p_h)$ of the problem (12) with $\mathbf{V}_h^0$ and M_h satisfies the following error estimates:

$$\left|\mathbf{u}-\mathbf{u}_h\right|_{\mathbf{H}^1(\Omega)} + \left|p - p_h\right|_{L^2(\Omega)} \leq C_1 h^k \left(\left|\mathbf{u}\right|_{\mathbf{H}^{k+1}(\Omega)} + \left|p\right|_{H^k(\Omega)}\right), \quad k = 1, 2.$$

To make the problem (8) numerically stable, we add the additional term

$$\frac{1}{2}((\nabla \cdot \mathbf{u}_h)\mathbf{u}_h, \mathbf{v}_h) = \frac{1}{2}\int_\Omega (\nabla \cdot \mathbf{u}_h)\mathbf{u}_h \cdot \mathbf{v}_h$$

to the equation $(8)_1$ to make it consistent, since for the incompressibility condition the above additional term reduces to zero. In that case the modification is consistent and the modified approximate problem can be written as follows:

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h^0 \times M_h$ such that

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + c(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + \dfrac{1}{2}((\nabla \cdot \mathbf{u}_h)\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h^0, \\ b(\mathbf{u}_h, q_h) = 0, \qquad \forall q_h \in M_h, \\ \mathbf{u}_h = 0 \end{cases}$$

(9)

Algebraic System and Algorithm: We choose the Hood-Taylor finite elements to discretize the Navier-Stokes problem. Let $\dim(V_h^0) = N$, $\dim(M_h) = M$ and $\mathbf{V}_h^0 = V_h^0 \times V_h^0$ and suppose $\{\{_i\}_{i=1,\cdots,N}$ and $\{Œ_i\}_{j=1,\cdots,M}$ be the Lagrange bases of the spaces $\mathbf{V}_h^0$ and M_h respectively. Let us write the approximate solutions $\mathbf{u}_h = (u_{1,h}, u_{2,h})$ and p_h in the basis of $\mathbf{V}_h^0$ and M_h as

$$\mathbf{u}_h = (u_{1,h}, u_{2,h}) = \left(\sum_{j=1}^N (u_1)_j, \{_j, \sum_{j=1}^N (u_2)_j, \{_j\right), \quad p_h = \sum_{l=1}^M p_l \{_l$$

Let $(\mathbf{v}_h, q_h) = (v_{1,h}, v_{2,h}, q_h) = (\{_i, \{_i, Œ_k)$ be the test functions. Setting $\mathbf{v}_h = (v_{1,h}, v_{2,h}) = (\{\{_i\}, 0)$ and $(0, \{\{_i\})$, for $i = 1, \cdots, N$, and $q_h = \{Œ_l\}_{l=1,\cdots,M}$, we obtain the following system:

$$
\begin{cases}
\int_\Omega \left[\nu \nabla u_{1,h} \cdot \nabla \varphi_i + (\mathbf{u}_h \cdot \nabla) u_{1,h} \varphi_i + \frac{1}{2} (\nabla \cdot \mathbf{u}_h) u_{1,h} \varphi_i - p_h \frac{\partial \varphi_i}{\partial x} - f_1 \varphi_i \right] = 0, \ i = 1, \cdots, N \\[2mm]
\int_\Omega \left[\nu \nabla u_{2,h} \cdot \nabla \varphi_i + (\mathbf{u}_h \cdot \nabla) u_{2,h} \varphi_i - \frac{1}{2} (\nabla \cdot \mathbf{u}_h) u_{2,h} \varphi_i - p_h \frac{\partial \varphi_i}{\partial y} - f_2 \varphi_i \right] = 0, \ i = 1, \cdots, N \\[2mm]
\int_\Omega (\nabla \cdot \mathbf{u}_h) \psi_l = 0, \ l = 1, \cdots, M
\end{cases}
$$

The above system of equations can be written as a non-symmetric matricial equation:

$$
\begin{bmatrix}
\epsilon\mathbf{A} & 0 & \mathbf{B}_x \\
0 & \epsilon\mathbf{A} & \mathbf{B}_y \\
\mathbf{B}_x^t & \mathbf{B}_y^t & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_1 \\
\mathbf{u}_2 \\
\mathbf{p}
\end{bmatrix}
+
\begin{bmatrix}
c(\mathbf{u}_1) \\
c(\mathbf{u}_2) \\
0
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{F}_x \\
\mathbf{F}_y \\
0
\end{bmatrix}
$$

$$(10)$$

where $\mathbf{u}_i^t = \left[u_i^1, \cdots, u_i^N \right]$, $i = 1, 2$, for N nodal velocities, $\mathbf{P}^t = \left[p_1, \cdots, p_M \right]$, for M nodal pressure and

$$
\mathbf{A} = \left[\mathbf{A}_{ij} \right]_{N \times N} = \int_\Omega \frac{\partial \{_i}{\partial x} \frac{\partial \{_j}{\partial x} + \frac{\partial \{_i}{\partial y} \frac{\partial \{_j}{\partial y}, \ i, j = 1, \cdots, N
$$

$$
\mathbf{B}_x = \left[\mathbf{B}_{xil} \right]_{N \times M} = \left[\int_\Omega \frac{\partial \{_i}{\partial x} \mathbb{E}_l \right]_{N \times M}
$$

$$
\mathbf{B}_y = \left[\mathbf{B}_{yil} \right]_{N \times M} = \left[\int_\Omega \frac{\partial \{_i}{\partial y} \mathbb{E}_l \right]_{N \times M}, \ i = 1, \cdots, N, l = 1, \cdots, \dots M
$$

$$
\mathbf{F}_x = \left[\int_\Omega f_1 \{_i \right]_{N \times 1}
$$

$$
\mathbf{F}_y = \left[\int_\Omega f_2 \{_i \right]_{N \times 1}
$$

The nonlinear $N \times 1$ vector is given as

$$
c_i(\mathbf{u}_l) = \int_\Omega \left[\left(\sum_{j=1}^N u_{1,j} \varphi_j \right) \left(\sum_{k=1}^N (u_l)_k \frac{\partial \varphi_k}{\partial x} \right) + \left(\sum_{j=1}^N u_{2,j} \varphi_j \right) \left(\sum_{k=1}^N (u_l)_k \frac{\partial \varphi_k}{\partial y} \right) \right]
$$
$$
+ \frac{1}{2} \int_\Omega \varphi_i \left[u_{1,j} \frac{\partial \varphi_j}{\partial x} + u_{2,j} \frac{\partial \varphi_j}{\partial y} \right] \left(\sum_{k=1}^N (u_l)_k \varphi_k \right), \ i = 1, \cdots, N
$$

To solve the nonlinear system (10) we use the Newton-Raphson algorithm. For this Navier-Stokes system, in fact, we want to solve the nonlinear vector field function $\mathbf{H}(\mathbf{u}, p) = 0$, where

$$\mathbf{H}(\mathbf{u}, p) = \begin{bmatrix} \epsilon\mathbf{A} & 0 & \mathbf{B}_x \\ 0 & \epsilon\mathbf{A} & \mathbf{B}_y \\ \mathbf{B}_x^t & \mathbf{B}_y^t & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} c(\mathbf{u}_1) \\ c(\mathbf{u}_2) \\ 0 \end{bmatrix} - \begin{bmatrix} \mathbf{F}_x \\ \mathbf{F}_y \\ 0 \end{bmatrix}$$

$$(11)$$

Considering the initial data $\mathbf{u}^0$, p^0 are known, we obtain

$$\begin{bmatrix} \mathbf{u}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{u}^n \\ p^n \end{bmatrix} - \mathbf{J}^{-1}(\mathbf{u}^n, p^n)\mathbf{H}(\mathbf{u}^n, p^n), n \geq 0$$

Taking $\mathbf{J}^{-1}(\mathbf{u}^n, p^n)\mathbf{H}(\mathbf{u}^n, p^n) = \begin{bmatrix} \mathsf{u}\mathbf{u}^n \\ \mathsf{u}p^n \end{bmatrix}$,

we have

$$\begin{bmatrix} \mathbf{u}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{u}^n \\ p^n \end{bmatrix} - \begin{bmatrix} \mathsf{u}\mathbf{u}^n \\ \mathsf{u}p^n \end{bmatrix}$$

So, we can define the algorithm

1. Given $(\mathbf{u}^0, p^0) \in \mathbf{V} \times Q$.

2. Repeat

$$\text{Solve } \mathbf{J}\begin{bmatrix} \mathsf{u}\mathbf{u}^n \\ \mathsf{u}p^n \end{bmatrix} = \mathbf{H}(\mathbf{u}^n, p^n)$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n - \mathsf{u}\mathbf{u}^n$$

$$p^{n+1} = p^n - \mathsf{u}p^n$$

until $\left\| (\mathsf{u}\mathbf{u}^n, \mathsf{u}p^n) \right\| < TOL$.

Numerical Results: We developed our own script in FreeFem++ to implement the Newton's method applied to the non-dimensional Navier-Stokes problem

$$\begin{cases} \text{Re}(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \in \Delta\mathbf{u} = \mathbf{f}, & \text{in} \quad \Omega \\ \mathbf{u} = \mathbf{u}_0, & \text{on} \quad \partial\Omega \end{cases}$$

To validate the solution, we fix the velocity and pressure

$$\left. \begin{aligned} \mathbf{u}(\mathbf{x}) &= \left((x^2 - x)^2(y^2 - y)(2y - 1), -(x^2 - x)(y^2 - y)^2(2x - 1) \right) \\ p(\mathbf{x}) &= x + y \end{aligned} \right\} \tag{12}$$

and we evaluate external force $\mathbf{f} = (f_1, f_2)$ to verify the Navier-Stokes equations with $\text{Re} = 1$.

We consider that the fluid is confined into a squared domain $\Omega = [0,1]^2$ and the prescribed Dirichlet boundary conditions agree with the exact solution according to (12).

To guarantee the quadratic convergence of Newton's method applied to Navier-Stokes equations, we should choose an initial approximation nearby the exact solution[11]. If we choose the initial approximation as the finite-element solution of Stokes equations, then the Newton's sequence converges quadratically to the unique solution to Navier-Stokes equations for sufficiently small mesh size h and a moderate Reynolds number Re [9]. The problem has been solved using four grids obtained by successive refinements dividing each triangle into four new triangles starting with a coarse mesh with 32-elements.

We use the following four meshes
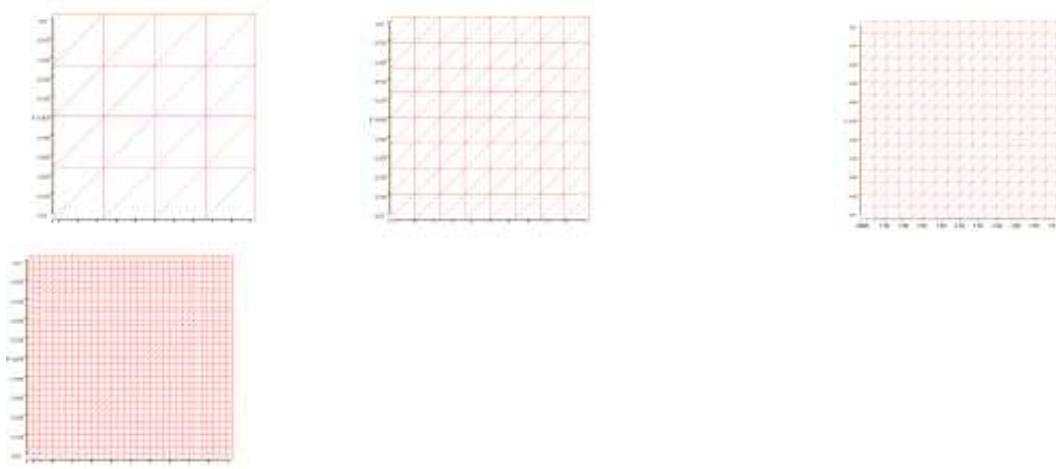
(a) 32-elements   (b) 128-elements   (c) 512-elements   (d) 2048-elements

Figure 1: Meshes over the square $[0,1]^2$.

The characterization of the meshes through the diameter, number of elements and degree of freedom:

Grid	h	No. of elements	$\mathbb{P}_2$ nodes	$\mathbb{P}_1$ nodes
Grid1	0.353553	32	81	25
Grid2	0.176777	128	289	81
Grid3	0.0883883	512	1089	289
Grid4	0.0441942	2048	4225	1089

Table 1: Characterization of the grids

In each case, we evaluate the error of the fluid velocity in $\mathbf{H}^1$-norm and pressure in L^2-norm which are respectively defined by

$$err_u = \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}^1(\Omega)} = \sum_{i=1}^{2}\left(\|u_i - u_{h,i}\|_{L^2(\Omega)} + \|\nabla(u_i - u_{h,i})\|_{L^2(\Omega)}\right)$$

and
$$err_p = \|p - p_h\|_{L^2(\Omega)} = \left[\int_\Omega (p - p_h)^2\right]^{1/2}.$$

The results obtained for $\mathbf{u}$ and p over different meshes are presented in the table 2. The good convergence of results of all kinematics can be confirmed by the slope value. We used the least square approximation to find the slope of the log-log plot
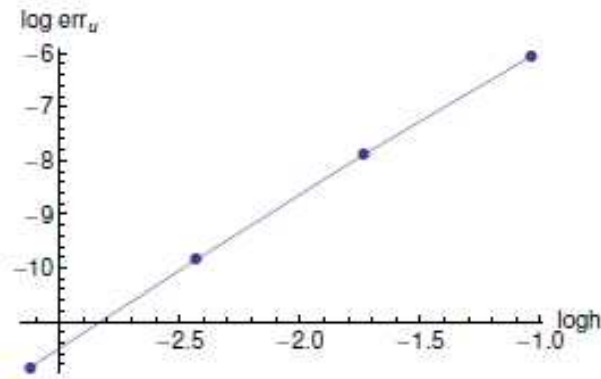
of the error of the fluid velocity and pressure from which the good convergence of results for all kinematics can be confirmed.

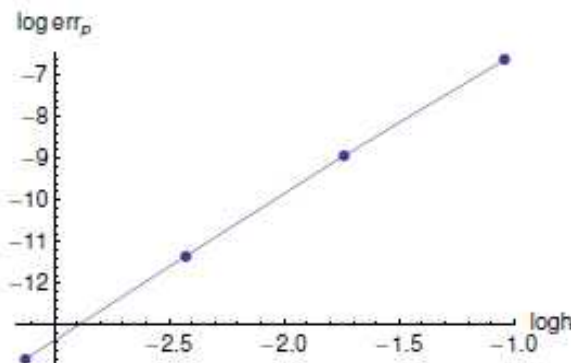The error of the fluid velocity and the pressure and the slope of the log-log plot of the errors:

Error	Grid1	Grid2	Grid3	Grid4	Slope of the log log plot
err_u	0.00234436	0.000377856	5.3719×10^{-5}	7.18033×10^{-6}	2.78906
err_p	0.00132182	0.000134722	1.19493×10^{-5}	1.02202×10^{-6}	3.45345

Table 2: Error of the velocity field and pressure.

The log-log plot of the errors of the velocity and the pressure:

(a) Log-log plot of the error err_u

(b) Log-log plot of the error err_p

Figure 2: Log-log plot of the error of the velocity and pressure.

Like our expectation, the rate of convergence (the slope) is positive (quadratic for the velocity) for both of the errors, and since the errors approaches zero as h tends to zero, so, our approximation converges to the exact solution with respect to the corresponding norms.

The exact and numerical solutions are illustrated graphically in the next figure.

$$u_1(x,y) = (x^2 - x)^2(y^2 - y)(2y - 1) \qquad u_2(x,y) = -(x^2 - x)(y^2 - y)^2(2x - 1)$$

$$p(x, y) = x + y$$

$$u_{h,1}(x, y) \approx u_1(x, y) \qquad\qquad\qquad u_{h,2}(x, y) \approx u_2(x, y)$$
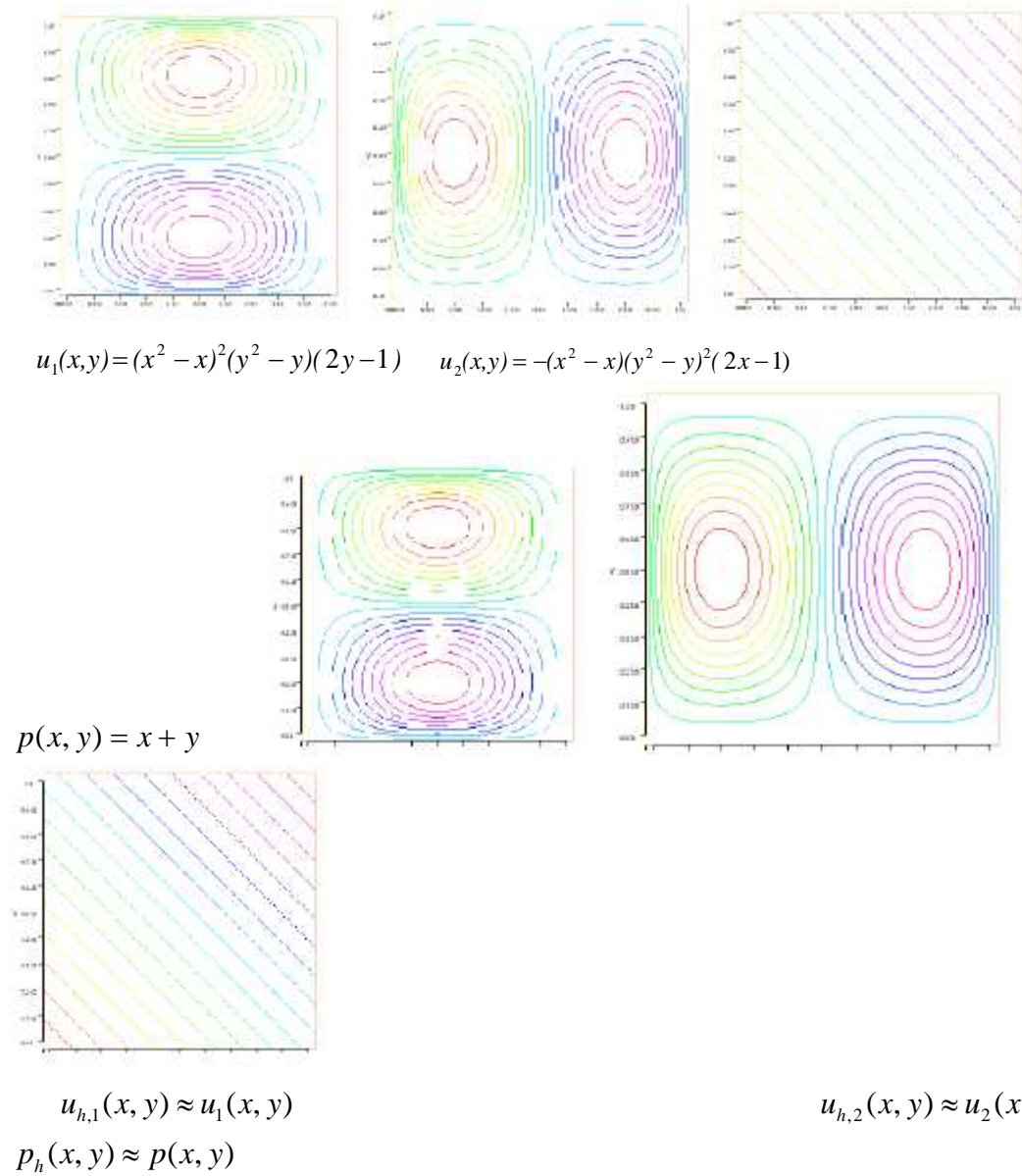
$$p_h(x, y) \approx p(x, y)$$

Figure 3: Exact and numerical solution for grid with 512 elements.

The above solutions (the velocity and the pressure) are obtained from the mesh with 512 elements. Here the contour of the first component of velocity is on the left, second component of velocity is on the centre and pressure is on the right. Here we observe that the behavior of the exact and numerical solutions is approximately same.

Exact and numerical streamlines:
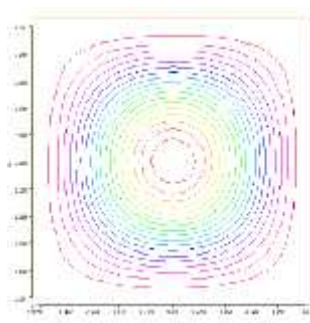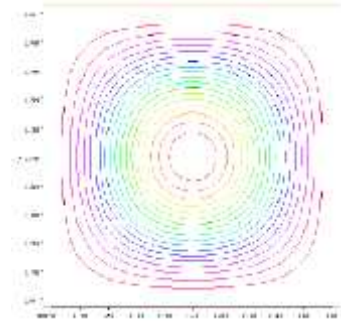
(a) Exact streamline   (b) Numerical streamline

Figure 4: Streamlines

We can see by the plot of the stream function, the fluid is rotating inside the domain with the same speed and also the qualitative behavior of the kinematics is almost same.

We can conclude the posteriori-estimates as

    i)   The rate of convergence is quadratic for the velocity.

    ii)   The rate of convergence is cubic for the pressure.

  ii) The errors approaches zero as h tends to zero (confirmed the theorem 1).

So, from all the above numerical and graphical results, we observe that the approximate solution (Newton's sequence) converges to the exact solution with respect to the corresponding norms.

Conclusion:

      In this paper, we have simulated incompressible Newtonian flow which is governed by the Navier-Stokes equations in two dimensional case. We obtained the approximate solution of steady Navier-Stokes equations using finite element

method with Newton's algorithm implemented in FreeFem++. We observed that, if we choose the finite element solutions of Stokes equations as initial guess, then the Newton's method converges to the exact solutions of Navier-Stokes problem. The numerical results are obtained by considering the benchmark problem over four different meshes. We have represented the solutions computationally and graphically , and also have established the posteriori-estimates. From the posteriori-estimates, we found that the approach solution converges to the exact solution and we have a very good rate of convergence. From the simulations results, it has also been established that FreeFem++ is capable to provide the better approximation for the incompressible Newtonian flow.

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