ISSN (Online) : 2454-7190 **ISSN (Print) : 0973-8975**
J.Mech.Cont.& Math. Sci., Vol.-10, No.-1, October (2015) Pages 1469-1480

NUMERICAL SIMULATION OF LAMINAR CONVECTION FLOW AND HEAT TRANSFER AT THE LOWER STAGNATION POINT OF A SOLID SPHERE.

BY

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Abstract.

A numerical algorithm is presented for studying laminar convection flow and heat transfer at the lower stagnation point of a solid sphere. By means of similarity transformation, the original nonlinear partial differential equations of flow are transformed to a pair of nonlinear ordinary differential equations. Subsequently they are reduced to a first order system and integrated using Newton Raphson and adaptive Runge-Kutta methods. The computer codes are developed for this numerical analysis in Matlab environment. Velocity and temperature profiles for various values of Prandtl number and at a fixed conjugate parameter are illustrated graphically. The results of the present simulation are then compared with previous results available in literature with good agreement.

Keywords: Free Convection, Fluid Flow, Heat Transfer, Matlab, Numerical Simulation, Solid Sphere, Stagnant Point.

List of Symbols

a radius of the sphere, m,
a_1, a_2 initial values eq (21),
f function defined in eq (11),
f_1, f_2 functions defined in eq (23),
g acceleration due to gravity, m/s^2,
Gr Grashof number, dimensionless,
h heat transfer coefficient, W/m^2.K,
k thermal conductivity, W/m.K,
Pr Prandtl number, dimensionless,
$\bar{r}$ radial distance from the symmetrical axis to the surface of the sphere, m,
r dimensionless coordinate in $\bar{r}$,
T fluid temperature, K,
T_w surface temperature, K,

T_∞ free streams temperature, K,
$\bar{u}$ velocity component in $\bar{x}$, m/s,
u dimensionless velocity in x,
$\bar{v}$ velocity component in $\bar{y}$, m/s,
v dimensionless velocity in y,
$\bar{x}$ coordinate along the surface, m,
x dimensionless coordinate in $\bar{x}$,
$\bar{y}$ coordinate normal to the surface, m,
y dimensionless coordinate in $\bar{y}$,
z_1, z_2, z_3, z_4, z_5 variables, eq (18)

Greek Symbols
θ function defined in eq (5), dimensionless,
coefficient of thermal expansion, 1/K,
α thermal diffusivity, m^2/s,
¬ kinematic viscosity, m^2/s,
ψ stream function, m^2/s,
ρ density, kg/m^3,
ρ_∞ free stream density, kg/m^3,
¥ conjugate parameter, dimensionless,

1. Introduction
There have been a number of studies on natural convection over a sphere, vertical and horizontal plate due to its relevance to a variety of industrial applications and naturally occurring processes, such as solar collectors, pipes, ducts, electronic packages, airfoils, turbine blades etc. Convection about a sphere has been reported by a number of researchers in the last two decades [1-10]. The problem is also discussed in several text books [11-13].

In the present numerical investigation, a simple accurate numerical simulation of laminar free-convection flow and heat transfer at the lower stagnant point of a sphere is developed.

The paper is organized as follows: Mathematical model of the problem, its solution procedure, development of code in Matlab, interpretation of the results along with comparison with previous works available in literature.

2. Mathematical Model
We consider a heated solid sphere immersed in a quiescent fluid. We assume the natural convection flow to be steady, laminar, two-dimensional, no dissipation, and the fluid to be Newtonian with constant properties, including density, with one exception: the density difference $° - °_\infty$ is to be considered since it is this density difference between

the inside and the outside of the boundary layer that gives rise to buoyancy force and sustains flow. (This is known as the *Boussinesq approximation.*) We take the distance along the surface of the sphere from the lower stagnant point to be $\bar{x}$ and the direction normal to surface to be $\bar{y}$, as shown in Figure 1.

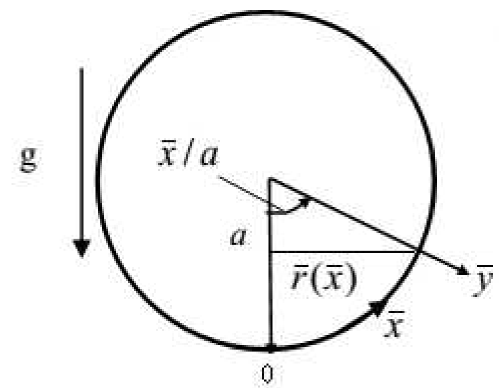

Fig. 1 Physical Model and its coordinate system

The equations governing the flow are

$$\frac{\partial}{\partial \bar{x}}(\bar{r}\bar{u}) + \frac{\partial}{\partial \bar{y}}(\bar{r}\bar{v}) = 0 \tag{1}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \nu\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + g\ \ (T - T_\infty)\sin(\frac{\bar{x}}{a}) \tag{2}$$

$$\bar{u}\frac{\partial T}{\partial \bar{x}} + \bar{v}\frac{\partial T}{\partial \bar{y}} = \alpha\frac{\partial^2 T}{\partial \bar{y}^2} \tag{3}$$

The boundary conditions on the solution are:

$$\begin{aligned} &\bar{y} = 0, \quad \bar{u} = \bar{v} = 0, \quad -k\frac{\partial T}{\partial \bar{y}} = h(T_w - T_\infty) \\ &\bar{y} \to \infty, \quad \bar{u} = 0, \quad T = T_\infty \end{aligned} \tag{4}$$

where $\bar{r}(\bar{x})$ is the radial distance from the symmetrical axis to the surface of the sphere. We now introduce the following non-dimensional variables:

$$x = \frac{\bar{x}}{a}, \quad y = Gr^{1/4}(\frac{\bar{y}}{a}), \quad r = \frac{\bar{r}}{a} \qquad u = \frac{a}{\nu}Gr^{-1/2}\bar{u}, \quad v = \frac{a}{\nu}Gr^{-1/4}\bar{v}, \quad \theta = \frac{T_w - T_\infty}{T - T_\infty} \tag{5}$$

where $Gr = g\ \ (T_w - T_\infty)\frac{a^3}{\nu^2}$, is Grashof number.

In terms of these new variables, eqs (1). (2) and (3) can be written as

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial y} = 0 \tag{6}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2}+\theta\sin x \tag{7}$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{\Pr}\frac{\partial^2 \theta}{\partial y^2} \tag{8}$$

where $\Pr=\frac{\nu}{\alpha}$, is Prandtl number. The boundary conditions (4) become

$$y=0,\quad u=v=0,\quad \frac{\partial\theta}{\partial y}=-\gamma(1-\theta)$$
$$y\to\infty,\quad u=0,\quad \theta=0 \tag{9}$$

where $\gamma=\frac{ahGr^{-1/4}}{k}$, is conjugate number. The continuity equation (1) is automatically satisfied through introduction of the stream function ψ where

$$u=\frac{1}{r}\frac{\partial\psi}{\partial y},\quad v=-\frac{1}{r}\frac{\partial\psi}{\partial x} \tag{10}$$

Again, we introduce the following variable

$$f(x,y)=\frac{\psi}{xr(x)} \tag{11}$$

Substituting the expressions (10) and (11) into eqs (7) and (8), we then obtain (with a prime denoting differentiation with respect to y)

$$x\left(f'\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial f}{\partial x}f''\right)=f'''+(1+x\cot x)ff''-(f')^2+\frac{\sin x}{x}\theta \tag{12}$$

$$x\left(f'\frac{\partial\theta}{\partial x}-\theta'\frac{\partial f}{\partial x}\right)=\frac{1}{\Pr}\theta''+(1+x\cot x)f\theta' \tag{13}$$

The appropriate boundary conditions are:

$$y=0,\quad f=f'=0,\quad \theta'=-\gamma(1-\theta)$$
$$y\to\infty,\quad f'=\theta=0 \tag{14}$$

At the lower stagnant point o, of the sphere, x=0 and eqs (12) and (13) reduces to the following equations:

$$f'''+2ff''-(f')^2+\theta=0 \tag{15}$$

$$\frac{1}{\Pr}\theta''+2f\theta'=0 \tag{16}$$

The boundary conditions (14) become

$$\begin{aligned} &at \quad y=0, \quad f = f' = 0, \quad \theta' = -\gamma(1-\theta(0)) \\ &at \quad y \to \infty, \quad f' = \theta = 0 \end{aligned} \tag{17}$$

3. Solution Procedure

Eqs (15) and (16) are coupled nonlinear ordinary differential equations for the velocity and temperature functions, f' and θ. No analytic solution is known, so numerical integration is necessary. Values of f and f' at the surface of the sphere (y = 0), and that of f'' and θ far away from the surface (y → ∞) are known. The value of θ' at the surface of the sphere depends on $\theta(0)$ and γ. One must find the proper values of $f''(0)$ and $\theta(0)$ which cause the velocity and temperature to vanish for large y. The Prandtl number, Pr and the conjugate parameter, γ are parameters.

3.1 Reduction of Equations to First-order System

This is done easily by defining new variables:

$$\begin{aligned} &z_1 = f \\ &z_2 = z_1' = f' \\ &z_3 = z_2' = z_1'' = f'' \\ &z_3' = z_2'' = z_1''' = f''' = -2z_1z_3 + (z_2)^2 - z_4 \\ &z_4 = \theta \\ &z_5 = z_4' = \theta' \\ &z_5' = z_4'' = \theta'' = -2\,\mathrm{Pr}\, z_1 z_5 \end{aligned} \tag{18}$$

Therefore from eqs (15) and (16), we get the following set of differential equations

$$\begin{aligned} &z_1' = f' \\ &z_2' = z_1'' = f'' \\ &z_3' = z_2'' = z_1''' = f''' = -2z_1z_3 + (z_2)^2 - z_4 \\ &z_4' = \theta' \\ &z_5' = z_4'' = \theta'' = -2\,\mathrm{Pr}\, z_1 z_5 \end{aligned} \tag{19}$$

with the following boundary conditions:

$$\begin{aligned} &z_1(0) = f(0) = 0 \\ &z_2(0) = z_1'(0) = f'(0) = 0 \\ &z_2(\infty) = z_1'(\infty) = f'(\infty) = 0 \\ &z_4(\infty) = \theta(\infty) = 0 \end{aligned}$$

$$z_5(0) = z_4'(0) = \text{¯}'(0) = -\text{¥}(1-\text{¯}(0)) = -\text{¥}(1-z_4(0)) \tag{20}$$

Eq (15) is third-order and is replaced by three first-order equations, whereas eq (16) is second-order and is replaced with two first-order equations.

3.2 Solution to Initial Value Problems

To solve eqs (19), we denote the two unknown initial values of f'' and ¯ by a_1 and a_2 respectively, the set of initial conditions is then:

$$\begin{aligned}
&z_1(0) = f(0) = 0 \\
&z_2(0) = z_1'(0) = f'(0) = 0 \\
&z_3(0) = z_2'(0) = z_1''(0) = f''(0) = a_1 \\
&z_4(0) = \text{¯}(0) = a_2 \\
&z_5(0) = z_4'(0) = \text{¯}'(0) = -\text{¥}(1-a_2)
\end{aligned} \tag{21}$$

If eqs (21) are solved with adaptive Runge-Kutta method using the initial conditions in eq (21), the computed boundary values at y=∞ depend on the choice of a_1 and a_2 respectively. We express this dependence as

$$\begin{aligned}
&z_2(\infty) = z_1'(\infty) = f'(\infty) = f_1(a_1) \\
&z_4(\infty) = \text{¯}(\infty) = f_2(a_2)
\end{aligned} \tag{22}$$

The correct choice of a_1 and a_2 yields the given boundary conditions at y=∞; that is, it satisfies the equations

$$\begin{aligned}
&f_1(a_1) = 0 \\
&f_2(a_2) = 0
\end{aligned} \tag{23}$$

These nonlinear equations can be solved by the Newton-Raphson method. A value of 10 is fine for infinity, even if we integrate further nothing will change.

3.3 Program Details

This section describes a set of Matlab routines for the solution of eqs (19) along with the initial conditions (21). They are listed in Table 1.

Table 1. A set of Matlab routines used sequentially to solve Equations (19).

Matlab code	Brief Description
deqs.m	Defines the differential equations (19).
incond.m	Describes initial values for integration, a_1 and a_2 are guessed values, γ is given, eq (21)
runKut5.m	Integrates as initial value problem using adaptive Runge-Kutta method.
residual.m	Provides boundary residuals and approximate solutions.
newtonraphson.m	Provides correct values a_1 and a_2 using approximate solutions from residual.m
runKut5.m	Again integrates eqs (19) using correct values of a_1 and a_2.

The final output of the code runKut5.m gives the tabulated values of f, f', f''(velocity profile), and ¯ and ¯′ (temperature profile) as function of y for various values of Prandtl number, Pr and a fixed value of conjugate number, γ=0.1. Three different values of Prandtl number, Pr = 0.7, 7, 100 corresponding to air, water and engine oil respectively are considered.

4. Interpretation of the Results

4.1 Dimensionless Velocity and Temperature Profiles

Physical quantities are related to the dimensionless functions f and ¯ through eqs (5), (10) and (11). The complete numerical solution of eqs (15) and (16) for three different values of Prandtl number, Pr=0.7, 7, 100 and a fixed value of conjugate number, γ=0.1 is given in Table 2. From this we can find all the flow parameters of interest to the lower stagnant point of a sphere.

Table 2a. Computed values of f, f', f'', ¯ *and* ¯′ for Pr = 0.7.

Pr = 0.7					
y	f	f'	f''	¯	¯′
0.00000	0.00000	0.00000	0.26182	0.23829	-0.07617
0.10000	0.00127	0.02500	0.23837	0.23067	-0.07617
0.57159	0.03570	0.11315	0.13849	0.19486	-0.07541
1.03191	0.09950	0.15807	0.06008	0.16076	-0.07228
1.50793	0.17912	0.17178	0.00142	0.12775	-0.06589
1.98893	0.26030	0.16286	-0.03486	0.09815	-0.05682
2.47110	0.33395	0.14130	-0.05176	0.07323	-0.04647
2.95764	0.39634	0.11491	-0.05493	0.05315	-0.03621
3.45740	0.44706	0.08849	-0.04988	0.03744	-0.02694
3.98232	0.48703	0.06461	-0.04081	0.02545	-0.01910
4.54690	0.51755	0.04448	-0.03060	0.01654	-0.01283
5.16980	0.53998	0.02852	-0.02105	0.01013	-0.00808
5.87730	0.55562	0.01661	-0.01311	0.00571	-0.00470
6.70837	0.56567	0.00839	-0.00719	0.00284	-0.00244
7.70040	0.57120	0.00339	-0.00335	0.00116	-0.00111
8.79856	0.57339	0.00096	-0.00134	0.00035	-0.00046
9.98351	0.57386	0.00001	-0.00042	0.00000	-0.00018
10.00000	0.57386	0.00000	-0.00041	0.00000	-0.00018

Table 2b. Computed values of f, f', f'', ‾ *and* ‾′ for Pr = 7.

Pr = 7					
y	f	f'	f''	‾	‾′
0.00000	0.00000	0.00000	0.11797	0.14480	-0.08552
0.10000	0.00057	0.01109	0.10392	0.13625	-0.08550
0.56453	0.01482	0.04609	0.04989	0.09703	-0.08204
1.00706	0.03884	0.05994	0.01531	0.06311	-0.06970
1.49124	0.06867	0.06155	-0.00615	0.03434	-0.04843
2.01257	0.09936	0.05539	-0.01567	0.01513	-0.02618
2.43846	0.12145	0.04821	-0.01743	0.00688	-0.01353
2.86829	0.14058	0.04083	-0.01661	0.00282	-0.00614
3.28107	0.15606	0.03432	-0.01486	0.00111	-0.00260
3.70619	0.16937	0.02844	-0.01282	0.00039	-0.00099
4.16309	0.18110	0.02306	-0.01074	0.00012	-0.00032
4.67280	0.19155	0.01812	-0.00872	0.00003	-0.00009
5.26405	0.20086	0.01356	-0.00679	0.00001	-0.00002
5.98205	0.20901	0.00937	-0.00498	0.00000	0.00000
6.90579	0.21580	0.00559	-0.00332	0.00000	0.00000
8.16754	0.22064	0.00238	-0.00190	0.00000	0.00000
9.74000	0.22249	0.00023	-0.00094	0.00000	0.00000
10.00000	0.22252	0.00000	-0.00084	0.00000	0.00000

Table 2c. Computed values of f, f', f'', $\bar{\ }$ and $\bar{\ }'$ for Pr = 100.

Pr = 100					
y	f	f'	f''	$\bar{\ }$	$\bar{\ }'$
0.00000	0.00000	0.00000	0.04398	0.08552	-0.09145
0.10000	0.00021	0.00399	0.03589	0.07637	-0.09132
0.38983	0.00259	0.01155	0.01754	0.05052	-0.08512
0.66102	0.00621	0.01472	0.00681	0.02960	-0.06731
0.95555	0.01074	0.01571	0.00069	0.01364	-0.04091
1.19786	0.01454	0.01557	-0.00154	0.00611	-0.02217
1.41168	0.01782	0.01514	-0.00237	0.00265	-0.01110
1.62831	0.02104	0.01458	-0.00268	0.00101	-0.00478
1.83392	0.02399	0.01403	-0.00274	0.00036	-0.00189
2.04238	0.02685	0.01346	-0.00272	0.00011	-0.00066
2.26252	0.02975	0.01286	-0.00266	0.00003	-0.00019
2.50382	0.03277	0.01223	-0.00258	0.00001	-0.00004
2.77915	0.03605	0.01153	-0.00250	0.00000	-0.00001
3.10894	0.03971	0.01073	-0.00240	0.00000	0.00000
3.52795	0.04400	0.00975	-0.00227	0.00000	0.00000
4.06410	0.04891	0.00857	-0.00212	0.00000	0.00000
4.36999	0.05144	0.00794	-0.00203	0.00000	0.00000
4.70391	0.05398	0.00728	-0.00194	0.00000	0.00000
5.10152	0.05672	0.00653	-0.00184	0.00000	0.00000
5.40214	0.05860	0.00598	-0.00176	0.00000	0.00000
5.71699	0.06039	0.00544	-0.00169	0.00000	0.00000
6.04942	0.06211	0.00489	-0.00161	0.00000	0.00000
6.32538	0.06340	0.00445	-0.00155	0.00000	0.00000
6.60750	0.06460	0.00402	-0.00149	0.00000	0.00000
6.91089	0.06575	0.00358	-0.00143	0.00000	0.00000
7.21810	0.06678	0.00315	-0.00137	0.00000	0.00000
7.46555	0.06752	0.00282	-0.00132	0.00000	0.00000
7.71745	0.06819	0.00249	-0.00128	0.00000	0.00000
8.00613	0.06886	0.00213	-0.00123	0.00000	0.00000
8.32014	0.06946	0.00175	-0.00117	0.00000	0.00000
8.55305	0.06984	0.00148	-0.00113	0.00000	0.00000
8.79290	0.07016	0.00122	-0.00110	0.00000	0.00000
9.07728	0.07047	0.00091	-0.00105	0.00000	0.00000
9.36018	0.07068	0.00062	-0.00101	0.00000	0.00000
9.64803	0.07082	0.00033	-0.00097	0.00000	0.00000
9.90963	0.07087	0.00008	-0.00094	0.00000	0.00000
10.00000	0.07088	0.00000	-0.00092	0.00000	0.00000

Some typical velocity and temperature profiles for various values of Pr obtained from this code are shown in Figs (2) and (3).

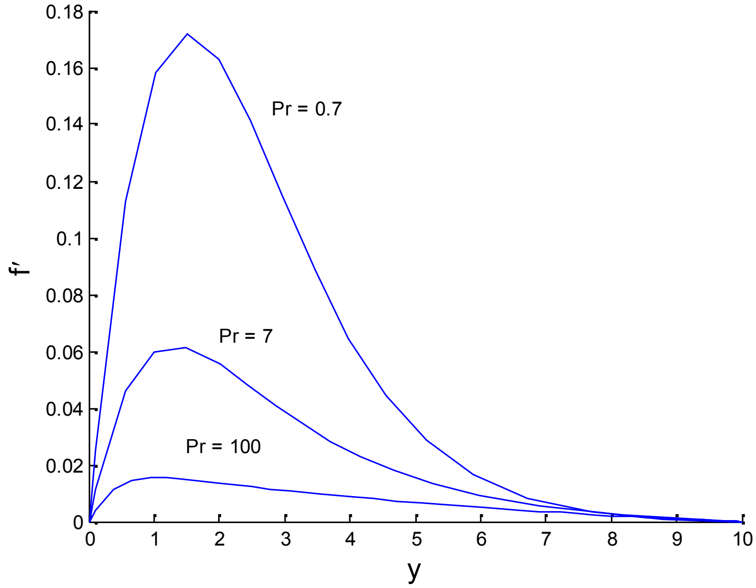

Fig 2. Dimensionless velocity distributions for various Prandtl numbers

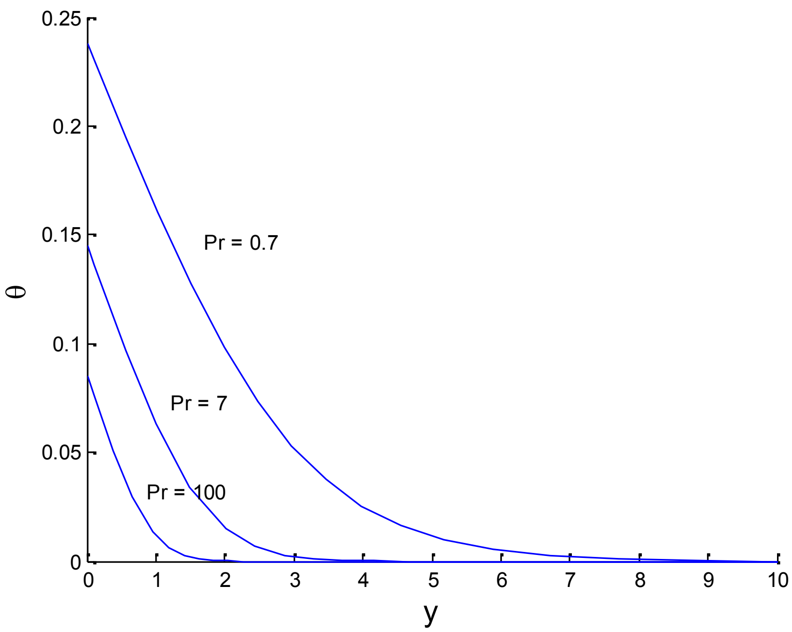

Fig 3. Dimensionless temperature distributions for various Prandtl numbers

4.2 Comparison of present results with previous work

Table 2 illustrates the computed values of the skin friction coefficient $f''(0)$ and the surface temperature $\overline{\ }(0)$ at the lower stagnation point of the sphere, x=0, when Pr=

0.7, 7, 100 and $\gamma=0.1$ for the present study as well as the numerical work by Alkasasbeh et al. [10]. The agreement is excellent.

Table 2 Comparison of the present computed values with previous work [10]

Pr	$f''(0)$		¯(0)	
	Present	Alkasasbeh et al.[10]	Present	Alkasasbeh et al.[10]
0.7	0.261816	0.261164	0.238288	0.236634
7	0.117968	0.117331	0.144798	0.148764
100	0.043980	0.042963	0.085515	0.090019

5. Conclusion

In the present numerical simulation, laminar convection flow and heat transfer at the lower stagnation point of a solid sphere is presented. Details of the solution procedure of the nonlinear partial differential equations of flow and heat transfer are discussed. The computer codes are developed for this numerical analysis in Matlab environment. Complete numerical solutions for fluid flow and heat transfer at the lower stagnant point of the sphere are presented for Prandtl numbers of 0.7, 7 and 100 (corresponding to air, water and engine oil) at a fixed conjugate number 0.1. Typical velocity and temperature profiles for these parameters are also illustrated graphically. A good agreement between the present results and the previous works indicates that the present numerical simulation may be an efficient and stable numerical scheme in natural convection.

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