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NUMERICAL SIMULATION ON LAMINAR CONVECTION FLOW AND HEAT TRANSFER OVER AN ISOTHERMAL HORIZONTAL PLATE

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Abstract

A numerical algorithm is presented for studying laminar convection flow and heat transfer over an isothermal horizontal plate. By means of similarity transformation, the original nonlinear coupled partial differential equations of flow are transformed to a pair of simultaneous nonlinear ordinary differential equations. Subsequently they are reduced to a first order system and integrated using Newton Raphson and adaptive Runge-Kutta methods. The computer codes are developed for this numerical analysis in Matlab environment. Velocity, and temperature profiles for various Prandtl number are illustrated graphically. Flow, and heat transfer parameters are derived as functions of Prandtl number alone. The results of the present simulation are then compared with experimental data in literature with good agreement.

Keywords: Free Convection, Heat Transfer, Isothermal Horizontal Plate, Matlab, Numerical Simulation.

List of Symbols

a_1, a_2 initial values eq (25)
c_p specific heat capacity, J/Kg.K
$C_{f,x}$ friction coefficient, dimensionless
f function defined in eq (6)
g gravitational acceleration, 9.81 m/s^2
h heat transfer coefficient, W/m^2.K
k thermal conductivity, W/m.K
L length of the plate, m
Nu_x Nusselt number at x, dimensionless
Nu_L Nusselt number at L, dimensionless

Pr Prandtl number, dimensionless
Q_w Total heat flux of the plate, W/m^2
q_w heat flux of the plate, W/m^2
Re_x Reynolds number at x, dimensionless
Re_L Reynolds number at L, dimensionless
T   temperature, K
T_w surface temperature, K
T_1 free streams temperature, K
u velocity component in x, m/s
u_1 free stream velocity in x, m/s
v velocity component in y, m/s
x coordinate from the leading edge, m
y coordinate normal to plate, m
z_1, z_2, z_3, z_4, z_5 variables, eq (22)

Greek Symbols

θ function defined in eq (18), dimensionless
coefficient of thermal expansion, 1/K
δ boundary layer thickness, m
μ dynamic viscosity, $N.s/m^2$
α thermal diffusivity, m^2/s
¬ kinematic viscosity, m^2/s
η similarity variables, eq (7)
τ shear stress, N/m^2
ψ stream function, m^2/s
ρ density, kg/m^3

1. Introduction

There have been a number of studies on natural convection over an isothermal horizontal plate due to its relevance to a variety of industrial applications and naturally occurring processes, such as solar collectors, pipes, ducts, electronic packages, airfoils, turbine blades etc. The earliest analytical investigation was a similarity analysis of the boundary layer equations by Blasius [1]. Many other methods of attack are chronicled in the text books by Meksyn [2] and Rosenhead [3]. The problem is also discussed in several text books [4-10].

In the present numerical investigation, a simple accurate numerical simulation of laminar free-convection flow and heat transfer over an isothermal horizontal plate is developed.

The paper is organized as follows: Mathematical model of the problem, its solution procedure, development of code in Matlab, interpretation of the results, comparison with experimental data.

2. Mathematical Model

We consider the flow of a fluid of velocity u_1 over an isothermal horizontal plate having temperature T_w. We assume the natural convection flow to be steady, laminar, two-dimensional, having no dissipation, and the fluid to be Newtonian with constant properties, including density, with one exception: the density difference $\rho - \rho_{\infty}$ is to be considered since it is this density difference between the inside and the outside of the boundary layer that gives rise to buoyancy force and sustains flow, known in the literature as *Boussinesq approximation*. We take the direction along the plate to be *x*, and the direction normal to surface to be *y*, as shown in Fig 1.

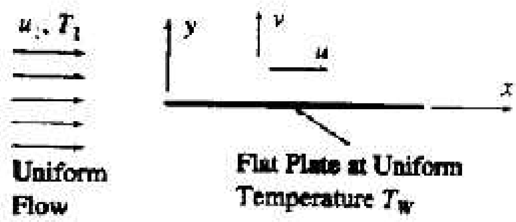

Fig 1. Physical Model and its coordinate system

The equations governing the flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

The boundary conditions on the solution are:

At y=0: u=v=0, $T=T_w$

For large y: $u \rightarrow u_1$, $T \rightarrow T_1$ (4)

The continuity equation (1) is automatically satisfied through introduction of the stream function:

$$u \equiv \frac{\partial \psi}{\partial y} \qquad v \equiv -\frac{\partial \psi}{\partial x} \tag{5}$$

A similarity solution is possible if

$$\psi = u_1 \sqrt{\frac{\nu x}{u_1}} f(\eta) \tag{6}$$

where, η is the similarity variable

$$\eta = \frac{y}{x}\sqrt{\mathrm{Re}_x} = y\sqrt{\frac{u_1}{\nu x}} \tag{7}$$

The basic form of the velocity profiles at different values of x, as shown in Fig 2, are all the same.

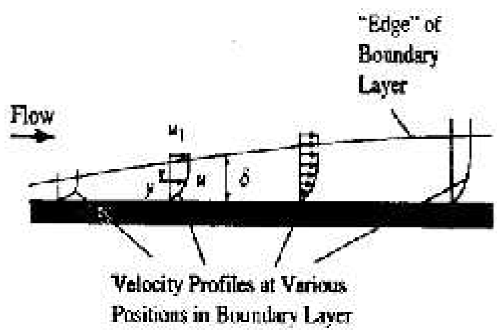

Fig 2. Velocity profiles at various positions in the boundary layer over a flat plate

The velocity profiles at all points in the boundary layer are assumed as follows:

$$\frac{u}{u_1} = function(\frac{y}{\delta}) \tag{8}$$

The boundary layer assumptions indicate that:

$$\frac{\delta}{x} = o(\frac{1}{\sqrt{\mathrm{Re}_x}}) \tag{9}$$

Therefore, the similar velocity profile assumptions given in Eq(8) can be rewritten as

$$\begin{aligned}\frac{u}{u_1} &= function(\frac{y}{x}\sqrt{\mathrm{Re}_x}) \\ &= function(\sqrt{\frac{u_1}{\nu x}}) = function(\eta)\end{aligned} \tag{10}$$

Hence, the velocity profile can be uniquely determined by the similarity variable η, which depends on both x and y.

From equations (5) through (7), we get

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = u_1\sqrt{\frac{\nu x}{u_1}}\frac{df}{d\eta}\sqrt{\frac{u_1}{\nu x}} = u_1\frac{df}{d\eta} \qquad (11)$$

and

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial x} = -(u_1\sqrt{\frac{\nu x}{u_1}}\frac{df}{dx} + \frac{u_1}{2}\sqrt{\frac{\nu}{u_1 x}}f)$$

$$= \frac{1}{2}\sqrt{\frac{\nu u_1}{x}}(\eta\frac{df}{d\eta} - f) \qquad (12)$$

By differentiating the velocity components, it may also be shown that

$$\frac{\partial u}{\partial x} = -\frac{u_1}{2x}\eta\frac{d^2 f}{d\eta^2} \qquad (13)$$

$$\frac{\partial u}{\partial y} = u_1\sqrt{\frac{u_1}{\nu x}}\frac{d^2 f}{d\eta^2} \qquad (14)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_1^2}{\nu x}\frac{d^3 f}{d\eta^3} \qquad (15)$$

Substituting these expressions into eq(2), we then obtain (with a prime denoting differentiation with respect to η)

$$2f''' + ff'' = 0 \qquad (16)$$

Hence the velocity boundary layer problem is reduced to an ordinary differential equation. This confirms the assumptions that velocity profiles are similar. The appropriate boundary conditions are:

at y = 0: u = 0 i.e., at η = 0: $f' = 0$

at y = 0: v = 0 i.e., at η = 0: f = 0

for large y: u → u_1 i.e., for large η: $f' \to 1$ (17)

To solve eq (3), we nondimensionlize the temperature according to the following

$$\theta = \frac{T_w - T}{T_w - T_1} \qquad (18)$$

The assumptions that the temperature profiles are similar is equivalent to assuming that θ depends only on the similarity variable, η, because the thermal boundary layer thickness is also of the order $x/\sqrt{\mathrm{Re}_x}$.

The energy eq (3) can be written in terms of θ as (T_w, T_1 are constants)

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{\nu}{\Pr}\frac{\partial^2 \theta}{\partial y^2} \qquad (19)$$

The boundary conditions are then

y = 0: θ = 0

y large: θ → 0 (20)

Using the relations for the velocity components previously derived, eq (19) gives

$$f'\frac{d\theta}{d\eta}\frac{d\eta}{dx}+\left[\frac{1}{2}\sqrt{\frac{\nu}{xu_1}}(\eta f'-f)\right]\frac{d\theta}{d\eta}\frac{d\eta}{dy}=\frac{\nu}{\Pr}\frac{1}{u_1}\frac{d^2\theta}{d\eta^2}\left[\frac{d\eta}{dy}\right]^2$$

After rearrangement, it becomes

$$\theta''+\frac{1}{2}\Pr\theta' f=0 \tag{21}$$

with the following boundary conditions

η = 0: θ = 0

η large: θ → 1 (22)

Thus the energy eq (3) has been reduced to an ordinary differential equation. This confirms the assumptions that temperature velocity profiles are similar.

3. Solution Procedure

Eqs (16) and (21) are nonlinear ordinary differential equations for the velocity and temperature functions, f' and θ, and are independent of each other. Eq (16) is solved first and then eq (21). No analytic solution is known, so numerical integration is necessary [11]. There are two unknown initial values at the wall. One must find the proper values of $f''(0)$ and $\theta'(0)$ which cause the velocity and temperature to their respective free stream values for large η. The Prandtl number is a parameter in the second case.

3.1 Reduction of Equations to First-order System

This is done easily by defining new variables:

$$z_1 = f$$

$$z_2 = z_1' = f'$$

$$z_3 = z_2' = z_1'' = f''$$

$$z_3' = z_2'' = z_1''' = f''' = -\frac{1}{2}ff'' = -\frac{1}{2}z_1 z_3$$

$$z_4 = \theta$$

$$z_5 = z_4' = \theta'$$

$$z_5' = z_4'' = \theta'' = -\frac{1}{2}\Pr f\theta' \tag{23}$$

Therefore from eqs (16) and (21), we get the following set of differential equations

$$z_1' = f'$$
$$z_2' = z_1'' = f''$$
$$z_3' = z_2'' = z_1''' = f''' = -\frac{1}{2} ff'' = -\frac{1}{2} z_1 z_2$$
$$z_4' = {}^{-}{}'$$
$$z_5' = {}^{-}{}'' = -\frac{1}{2} \Pr f {}^{-}{}' \tag{24}$$

with the following boundary conditions:

$$z_1(0) = f(0) = 0$$
$$z_2(0) = z_1'(0) = f'(0) = 0$$
$$z_2(\infty) = z_1'(\infty) = f'(\infty) = 1$$
$$z_4(0) = {}^{-}(0) = 0$$
$$z_4(\infty) = {}^{-}(\infty) = 1 \tag{25}$$

Eq (16) is third-order and is replaced by three first-order equations, whereas eq (21) is second-order and is replaced with two first-order equations.

3.2 Solution to Initial Value Problems

To solve eqs (24), we denote the two unknown initial values by a_1 and a_2, the set of initial conditions is then:

$$z_1(0) = f(0) = 0$$
$$z_2(0) = z_1'(0) = f'(0) = 0$$
$$z_3(0) = z_2'(0) = z_1''(0) = f''(0) = a_1$$
$$z_4(0) = {}^{-}(0) = 0$$
$$z_5(0) = z_4'(0) = {}^{-}{}'(0) = a_2 \tag{26}$$

If eqs (24) are solved with adaptive Runge-Kutta method using the initial conditions in eq (26), the computed boundary values at ¦ $= \infty$ depend on the choice of a_1 and a_2 respectively. We express this dependence as

$$z_2(\infty) = z_1'(\infty) = f'(\infty) = f_1(a_1)$$
$$z_4(\infty) = {}^{-}(\infty) = f_2(a_2) \tag{27}$$

The correct choice of a_1 and a_2 yields the given boundary conditions at ¦ $= \infty$; that is, it satisfies the equations

$$f_1(a_1) = 1$$
$$f_2(a_2) = 1 \tag{28}$$

These nonlinear equations can be solved by the Newton-Raphson method. A value of 6 is fine for infinity, even if we integrate further nothing will change.

3.3 Program Details

This section describes a set of Matlab routines for the solution of eqs (24) along with the boundary conditions (26). They are listed in Table 1.

Table 1. A set of Matlab routines used sequentially to solve Equations (24).

Matlab code	Brief Description
deqs.m	Defines the differential equations (24).
incond.m	Describes initial values for integration, a_1 and a_2 are guessed values, eq (26)
runKut5.m	Integrates as initial value problem using adaptive Runge-Kutta method.
residual.m	Provides boundary residuals and approximate solutions.
newtonraphson.m	Provides correct values a_1 and a_2 using approximate solutions from residual.m
runKut5.m	Again integrates eqs (24) using correct values of a_1 and a_2.

The final output of the code runKut5.m gives the tabulated values of f, f', f'' as function of η for velocity profile, and ¯ and ¯′ as function of η for various values of Prandtl number.

4. Interpretation of the Results

4.1 Dimensionless Velocity and Temperature Profiles

Physical quantities are related to the dimensionless functions f and ¯ through eqs (7), (11) and (12). f and ¯ are now known. The complete numerical solution of eq (24) is given in Table 2 and Table 3. From this we can find all the flow parameters of interest to flat plate.

Table 2 Computed parameters from eqs (24)

η	f	f'	f''
0	0	0	0.3326
0.1000	0.0017	0.0333	0.3326
0.7064	0.0829	0.2344	0.3293
1.3108	0.2839	0.4293	0.3125
1.9050	0.5923	0.6049	0.2751
2.4914	0.9913	0.7504	0.2185
3.0892	1.4750	0.8611	0.1514
3.5750	1.9091	0.9220	0.1004
4.0103	2.3187	0.9573	0.0634
4.4487	2.7436	0.9788	0.0364
4.8572	3.1459	0.9900	0.0200
5.2529	3.5390	0.9959	0.0103
5.6481	3.9332	0.9988	0.0049
6.0000	4.2849	1.0000	0.0024

Table 3 Values of f(Pr) for various values of Pr

Pr	$f(\Pr) = -'\big\rvert_{=0}$	$0.3321\Pr^{0.34}$
0.6	0.277	0.279
0.7	0.293	0.294
0.8	0.307	0.308
0.9	0.320	0.320
1.0	0.332	0.332
1.1	0.343	0.343
1.4	0.373	0.372
2.0	0.442	0.420
2.5	0.456	0.454
3.0	0.485	0.483
5.0	0.577	0.574
7.0	0.646	0.644
10.0	0.728	0.727
15.0	0.834	0.834
20.0	0.918	0.920
50.0	1.247	1.257

Fig 3 shows a plot of f', f', *and* f'' and Fig 4 compares of the profile f' with the experiments of Liepmann [12]. The agreement is excellent.

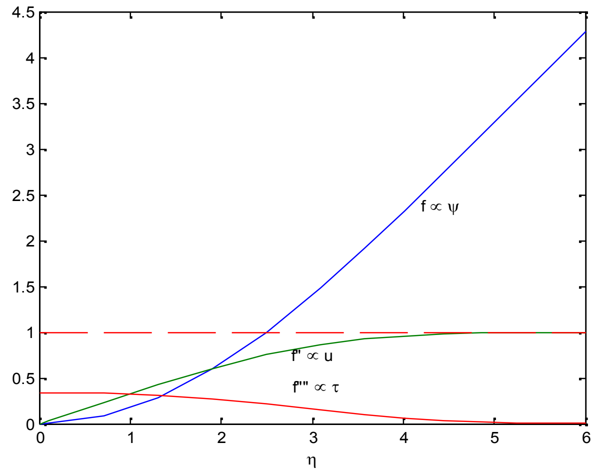

Fig. 3 Solutions f, f' *and* f''

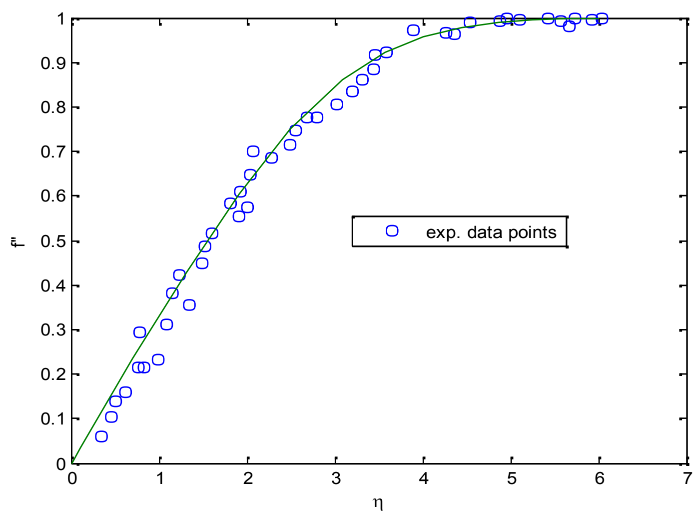

Fig. 4 Comparison of f' with experiments of Liepman [12]

Some typical variations of θ with η for various values of Pr obtained from the code are shown in Fig 5.

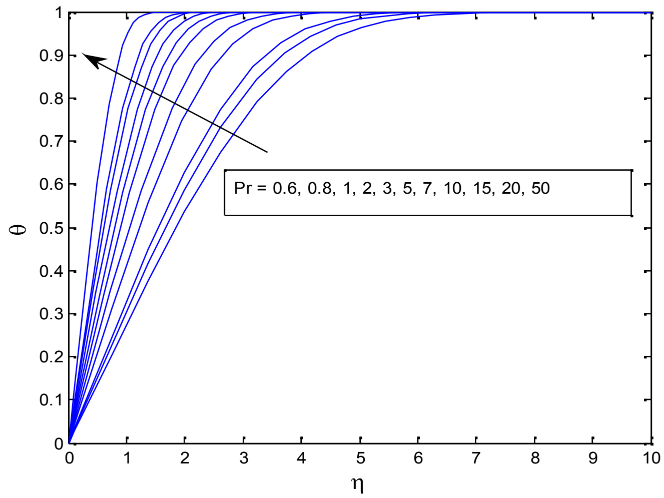

Fig 5. Solution of θ

4.2 Flow and Heat Transfer Parameters

Besides the velocity and temperature distributions, it is often desirable to compute other physically important quantities (for example, shear stress, drag, heat-transfer-rate) associated with the convection flow.

The boundary layer thickness, δ, is defined as the value of y at which $u=0.99u_1$. Fig 3 shows that $u=0.99u_1$, i.e., $f'=0.99$ when $\eta = 5$. From (7)

$$\delta\sqrt{\frac{u_1}{\nu x}} = 5$$

i.e.,

$$\frac{\delta}{x} = \frac{5}{\sqrt{\mathrm{Re}_x}} \tag{29}$$

From eq (14), wall shear stress may be expressed as

$$\tau_s = \mu\left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu u_1\sqrt{\frac{u_1}{\nu x}}\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0} \tag{30}$$

Hence from Table

$$\tau_s = 0.3326 u_1\sqrt{\frac{\rho\mu u_1}{x}} \tag{31}$$

The local friction coefficient is then

$$C_{f,x} = \frac{\tau_s}{\rho u_1^2 / 2} = 0.665\sqrt{\mathrm{Re}_x} \tag{32}$$

The heat transfer rate at the wall is given by

$$q_w = -k \left.\frac{\partial T}{\partial y}\right|_{y=0} \tag{33}$$

Hence, using eq (18)

$$\frac{q_w}{k(T_w - T_1)} = \left.\frac{\partial \theta}{\partial y}\right|_{y=0} = \left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0} \frac{\partial \eta}{\partial y}$$

i.e.,

$$\frac{q_w x}{k(T_w - T_1)} = \theta'\big|_{\eta=0} \sqrt{\mathrm{Re}_x} \tag{34}$$

i.e.,

$$Nu_x = \theta'\big|_{\eta=0} \sqrt{\mathrm{Re}_x} \tag{35}$$

Nu_x and Re_x being, of course, the local Nusselt and Reynolds numbers based on x.

Because θ depends only on η for a given Pr, $\theta'\big|_{\eta=0}$ depends only on Pr and its value can be obtained from the solution for the variation θ with η for any value of Pr. So, we can write eq (35) as

$$Nu_x = f(\mathrm{Pr})\sqrt{\mathrm{Re}_x} \tag{36}$$

Over the ranges of Prandtl numbers covered in the Table, it has been found that f(Pr) varies very nearly as $\mathrm{Pr}^{0.340}$ and from the table it can be represented by the following relation

$$f(\mathrm{Pr}) = 0.3321\,\mathrm{Pr}^{0.3402} \qquad 37$$

Hence eq (36) can be written as

$$Nu_x = 0.332\,\mathrm{Pr}^{0.332}\,\mathrm{Re}_x^{0.5} \qquad 38$$

The total heat transfer rate Q_w is related to the local heat transfer rate q_w by

$$Q_w = \int_0^L q_w\,dx \qquad 39$$

But eq (34) gives the local heat transfer rate as

$$q_w = f(\mathrm{Pr})k(T_w - T_1)\sqrt{\frac{u_1}{\nu x}}$$

Substituting this result into eq (38) then gives on carrying out integration

$$Q_w = 2f(\mathrm{Pr})k(T_w - T_1)\sqrt{\frac{u_1 L}{\neg}}$$

Therefore the mean heat transfer rate for the whole plate, $\bar{h}$, is

$$\bar{h} = \frac{Q_w}{L(T_w - T_1)} = \frac{2f(\mathrm{Pr})k}{k}\sqrt{\frac{u_1 L}{\neg}} \tag{40}$$

The mean Nusselt number for the whole plate, $\overline{Nu_L}$ is therefore, given by

$$\overline{Nu_L} = 2f(\mathrm{Pr})\sqrt{\mathrm{Re}_L}$$

$$\text{i.e., } \overline{Nu_L} = 0.664\,\mathrm{Pr}^{0.332}\sqrt{\mathrm{Re}_L} \tag{41}$$

where Re_L is the Reynolds number based on the plate length, L.

The results (38) and (41) are in reasonably good agreement with experimental results available in the literature [5,7,8,12].

5. Conclusion

In the present numerical simulation, laminar convection flow and heat transfer over an isothermal horizontal plate is presented. Details of the solution procedure of the nonlinear partial differential equations of flow are discussed. The computer codes are developed for this numerical analysis in Matlab environment. Velocity profile, and temperature profiles for Prandtl numbers of 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.4, 2.0, 2.5, 3.0, 5.0, 7.0, 10.0, 15.0, 20.0 and 50.0 are computed using these codes. The computed and experimental velocity and temperature distributions are in very good agreement with results published in literatures. Flow and heat transfer parameters (giving physically important quantities such as shear stress, drag, heat-transfer-rate) are derived. The computed results of the present simulation are compared with frequently used semi empirical heat transfer correlation and find a good agreement. A good agreement between the present results and the past indicates that the developed numerical simulation as an efficient and stable numerical scheme in natural convection.

References

1) Blasius, H., “Grenzschichten in Flussikeiten mitkleiner Reibung”, Z. Angew,Math. Phys., vol. 56, pp. 1-37, 1908 [English translation in NACA Technical Memo. 1256].

2) Meksyn, D., “New Methods in Laminar Boundary Layer Theory”, Pergamon, London, 1961.

3) Rosenhead, L. (ED), “Laminar Boundary Layers”, Oxford Univ. Press, London, 1963.

4) Hansen, A. G., “Similarity Analysis of Boundary Value Problems in Engineering”, Prentice-Hall, Englewood Cliffs, N. J., 1964.

5) Bejan, A., *Heat Transfer*, John Wiley, New York, 1993.

6) Sachdev, P. L., “Self-Similarity and Beyond: Exact Solutions of Nonlinear Problems”, CRC Press, Boca Ratton, Fla, 2000

7) Incropera, F.P. and DeWitt D.P., *Introduction to Heat Transfer*, Fourth edition, John Wiley, New York, 2002.

8) Çengel, Y.A., *Heat Transfer*, Second edition, McGraw–Hill, New York, 2003.

9) Lienhard IV, J.H. and Lienhard V, J.H., *A Heat Transfer Textbook*, Phlogiston Press, Cambridge, MA, 2003.

10) Nellis, G. and Klein, S., *Heat Transfer*, Cambridge University Press, London, UK, 2008.

11) Mitra A., “Numerical Simulation on Laminar Free-Convection Flow and Heat Transfer Over an Isothermal Vertical Plate,” International Journal of Research in Engineering & Technology, 04, 2015, pp 488-494.

12) Leipmann, H. W., “Investigation on Laminar Boundary-Layer Stabilty and Transition on Carved Boundaries,“ NACA Wartime Report W107 (ACR3H30), 1943 [see also NACA Technical Memo. 1196 (1947) and NACA Report 890 (1947)].