

## **EXPANSION OF A SPHERICAL CAVITY AT THE CENTRE OF A NON-HOMOGENEOUS SPHERE OF DUCTILE METAL WITH EFFECT OF WORK-HARDENING UNDER INTERNAL AND EXTERNAL PRESSURES**

**BY**

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### **Abstract**

The aim of this paper is to investigate the distribution of stresses due to expansion of a spherical cavity at the centre of a non-homogeneous metallic sphere of finite radius for an elasto-plastic solid with effect of work-hardening under an increasing internal pressure, the external pressure remaining constant. The non-homogeneity of the elastic material is characterised by supposing that the Lamé constants vary exponentially as the function of radial distance. The case of ideal plastic solid has also been deduced from this general case.

### **1. INTRODUCTION**

The radially symmetric problems for small elasto-plastic deformations have been considered by a number of investigators. By applying finite deformation theory HILL (1) has presented the problem of expansion of a spherical shell under the action of internal pressure only. Radially symmetric deformation and corresponding stress in a spherical shell of isotropic solid under the influence of internal and external pressures are determined by LAMÉ and given in LOVE. (2) The problem for a spherically anisotropic material was solved by SAINT-VENANT (3) and presented in the standard book of LEKHNITSKII (4). HOPKINS (5) studied the problem of dynamic expansion of cavities in ductile metals of infinite dimensions. The problems of radial deformations of non-homogeneous spheres and spherical shells of elastic solids with concentric spherical inclusions were also investigated by SENGUPTA (6). ROY (7) calculated the stresses due to expansion of a spherical cavity at the centre of a homogeneous sphere under internal and external pressures including the effect of work-hardening of a material in which the thermal effect on the elastic parameters  $\lambda$  and  $\mu$  was ignored.

In this paper, the authors have investigated the distribution of stresses due to expansion of a spherical cavity at the centre of a non-homogeneous metallic sphere of finite radius under internal and external pressures including the effect of work-hardening of the material, considering the Lamé constants are variable quantities and vary exponentially as the function of radial distance of the cavity. This problem will be a

realistic one if it is supposed that an explosion takes place at the centre of the cavity of a sphere immersed in a liquid or placed in a high pressure atmosphere. The liquid or atmosphere will then exert constant equal pressure on the external surface of the sphere while due to explosion at the centre of the cavity, the internal surface will be under a continuous increasing pressure.

## 2. MATHEMATICAL FORMULATION OF THE PROBLEM AND BOUNDARY CONDITIONS :

We consider a sphere of radius "b" under constant external pressure  $p_e$ . A small cavity of radius "a" is considered at the centre of the sphere. The cavity is supposed to be formed under the action of high explosive charge placed at the centre of the sphere and this surface of the small sphere is under constant increasing pressure.

It should be noted that there are some similarities and at the same time important differences between the cognate problems of explosions in various media, such as soils, metals, water and air. The problem of radially symmetric cavity formation due to an explosion taken place inside a large block is of course one of interaction, involving an exchange of energy between the explosion gas products and the surrounding mass of metal. It is supposed that initiation occurs at the centre, the explosion products are confined initially to the volume originally occupied by the solid explosive charge.

Taking spherical polar co-ordinates  $(r, \theta, \phi)$  and the corresponding displacement components  $(u, v, w)$ , we suppose for radially symmetric deformation

$$u = u(r), v = 0, w = 0 \dots\dots\dots(2.1)$$

As the Lamé constants vary exponentially as the function of radial distance, we suppose

$$\lambda = \lambda_0 e^{\alpha r} \text{ and } \mu = \mu_0 e^{\alpha r} \text{ where } \lambda_0 \text{ and } \mu_0 \text{ are Lamé constants when the radial distance } r = 0 \text{ and } \alpha \begin{matrix} > \\ < \end{matrix} 0$$

With these value of  $\lambda$  and  $\mu$ , the non-vanishing components of stress are

$$\left. \begin{aligned} \sigma_r &= (\lambda_0 + 2\mu_0)e^{\alpha r} \frac{\partial u}{\partial r} + 2\lambda_0 e^{\alpha r} \frac{u}{r} \\ \sigma_\theta &= \sigma_\phi = \lambda_0 e^{\alpha r} \frac{\partial u}{\partial r} + 2(\lambda_0 + \mu_0)e^{\alpha r} \frac{u}{r} \end{aligned} \right\} \dots\dots\dots(2.2)$$

The only equation of equilibrium in absence of the body forces is

$$\frac{\partial \sigma_r}{\partial r} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0 \dots\dots\dots(2.3)$$

Now eliminating  $\sigma_r$  and  $\sigma_\theta$  from equations (2.2) and (2.3), we obtain a differential equation satisfied by the displacement component  $u$  as

$$r^3 \frac{d^2 u}{dr^2} + (\alpha r^2 + 2r) \frac{du}{dr} + (s \alpha r - 2) u = 0 \dots\dots\dots(2.4)$$

$$\text{where } s = \frac{2\lambda_0}{\lambda_0 + 2\mu_0}$$

The boundary concitions for the problem are as follows :

$$[\sigma_r]_{r=a} = p_i, \quad [\sigma_r]_{r=b} = p_e \dots\dots\dots (2.5)$$

$p^i \gg p^e$  and  $p^i$  being the yielding pressure

### 3. SOLUTION OF THE PROBLEM :

The solution of the differential equation (2.4) is

$$u = Ar p(r, s) + \frac{B}{r^2} q(r, s) \dots\dots\dots (3.1)$$

Where

$$\begin{aligned} p(r, s) = & 1 - \frac{\alpha(s+1)}{1.4} r + \frac{\alpha^2(s+1)(s+2)}{1.2.4.5} r^2 - \frac{\alpha^3(s+1)(s+2)(s+3)}{1.2.3.4.5.6} r^3 + \dots\dots\dots \text{to } \alpha \\ q(r, s) = & 1 - \frac{\alpha^3(s-2)(s-1)s}{1^3.2^2.3} r^3 \left( \frac{1}{s-2} + \frac{1}{s-1} + \frac{1}{s} - \frac{1}{3} \right) + \\ & + \frac{\alpha^4(s-2)(s-1)s(s+1)}{1^3.2^2.3.4} r^4 \left( \frac{1}{s-2} + \frac{1}{s-1} + \frac{1}{s} - \frac{1}{s+1} - 1 - \frac{1}{3} - \frac{1}{4} \right) - \\ & - \frac{\alpha^5(s-2)(s-1)s(s+1)(s+2)}{1^3.2^3.3.4.5} r^5 \left( \frac{1}{s-2} + \frac{1}{s-1} + \frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2} - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} \right) + \\ & + \frac{\alpha^6(s-2)(s-1)s(s+1)(s+2)(s+3)}{1^3.2^3.3^2.4.5.6} r^6 \left( \frac{1}{s-2} + \frac{1}{s-1} + \frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+3} - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} \right) - \\ & - \frac{\alpha^7(s-2)(s-1)s(s+1)(s+2)(s+3)(s+4)}{1^3.2^3.3^2.4^2.5.6.7} r^7 \times \\ & \times \left( \frac{1}{s-2} + \frac{1}{s-1} + \frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+3} + \frac{1}{s+4} - 2 - \frac{2}{3} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} \right) + \\ & + \frac{\alpha^8(s-2)(s-1)s(s+1)(s+2)(s+3)(s+4)(s+5)}{1^3.2^3.3^2.4^2.5^2.6.7.8} r^8 \times \\ & \times \left( \frac{1}{s-2} + \frac{1}{s-1} + \frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+3} + \frac{1}{s+4} + \frac{1}{s+5} - 2 - \frac{2}{3} - \frac{2}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} \right) - \dots\dots\dots \text{to } \alpha \end{aligned}$$

A and B being constants to be determined. Therefore the non-vanishing components of stress are

$$\left. \begin{aligned} \sigma_r &= L(r,s)e^{ar} A - M(r,s)e^{ar} B \\ \sigma_\theta &= \sigma_\phi = L_1(r,s)e^{ar} A - M_1(r,s)e^{ar} B \end{aligned} \right\} \dots\dots\dots(3.2)$$

where

$$L(r,s) = \{3p(r,s) + rp'(r,s)\}\lambda_0 + 2\{p(r,s) + rp'(r,s)\}\mu_0$$

$$M(r,s) = \left\{ \frac{4q(r,s)}{r^3} - \frac{q'(r,s)}{r^2} \right\} \lambda_0 - 2 \left\{ \frac{2q(r,s)}{r^3} - \frac{q'(r,s)}{r^2} \right\} \mu_0$$

$$L_1(r,s) = \{3p(r,s) + rp'(r,s)\}\lambda_0 + 2\mu_0 p(r,s)$$

$$M_1(r,s) = \left\{ \frac{q'(r,s)}{r^2} \lambda_0 + 2\mu_0 \frac{q(r,s)}{r^3} \right\}$$

p(r,s) and q(r,s) being the derivatives of p(r,s) and q(r,s) respectively w.r.t.r.

The constants A and B can be determined with the help of the Boundary conditions (2.5). Determining these constants, the non-vanishing components of stress and displacement become

$$\left. \begin{aligned} \sigma_r &= \frac{L(b,s)M(r,s) - L(r,s)M(b,s)}{L(a,s)M(b,s) - L(b,s)M(a,s)} p_i e^{a(r-s)} + \\ &+ \frac{L(r,s)M(a,s) - M(r,s)L(a,s)}{L(a,s)M(b,s) - L(b,s)M(a,s)} p_e e^{u(r-b)} \\ \sigma_\theta &= \sigma_\phi = -p_i \frac{L_1(r,s)M(b,s) + M_1(r,s)L(b,s)}{L(a,s)M(b,s) - L(b,s)M(a,s)} e^{a(r-s)} + \\ &+ p_e \frac{L_1(r,s)M(a,s) + M_1(r,s)L(a,s)}{L(a,s)M(b,s) - L(b,s)M(a,s)} e^{a(r-b)} \\ u &= -p_i \frac{M(b,s)p(r,s)r^3 + L(b,s)q(r,s)}{r^2\{L(a,s)M(b,s) - L(b,s)M(a,s)\}} e^{-as} + \\ &+ p_e \frac{M(a,s)p(r,s)r^3 + L(a,s)q(r,s)}{r^2\{L(a,s)M(b,s) - L(b,s)M(a,s)\}} e^{-ab} \end{aligned} \right\} \dots\dots\dots(3.3)$$

According to Hencky and Von-Mises, the yielding commences when the maximum of  $|\sigma_\theta - \sigma_r|$  reaches a critical value Y where Y is material constant.

Now

$$|\sigma_\theta - \sigma_r| = \left\{ \frac{4q(r,s)}{r^3} \lambda_0 - \frac{2q(r,s)}{r^3} \mu_0 + \frac{2q'(r,s)}{r^2} \right\} e^{ar} B - 2\mu_0 rp'(r,s)e^{ar} A$$

is maximum at  $r = a$

From physical standpoint, due to explosion at the centre of the cavity, pressure is exerted on the cavity surface and this pressure increases contially. This consideration also shows that yielding beging at  $r = a$ . Hence, yielding begins at  $r = a$  and the corresponding pressure  $p_i$  is given by

$$p_i = Y \frac{a^3 [L(a,s)M(b,s) - M(a,s)L(b,s)]}{2\mu_0 M(b,s)a^4 p'(a,s) - (4\lambda_0 - 2\mu_0)L(b,s)q(a,s) - 2\mu_0 L(b,s)aq'(a,s)} +$$

$$+ p_e \frac{[2\mu_0 M(a,s)a^4 p'(a,s) - (4\lambda_0 - 2\mu_0)L(a,s)q(a,s) - 2\mu_0 L(a,s)aq'(a,s)]}{2\mu_0 M(b,s)a^4 p'(a,s) - (4\lambda_0 - 2\mu_0)L(b,s)q(a,s) - 2\mu_0 L(b,s)aq'(a,s)} e^{a(a-b)} \quad (3.4)$$

With increasing pressure a plastic region spreads into the shell. For reasons of symmetry, the plastic boundary must be a spherical surface. Let its radius at any moment be donated by  $c$ . Hence the region  $a \leq r \leq c$  must be plastic and the region  $c \leq r \leq b$  be elastic. In the elastic region the stresses and displacement are still of the form :

$$\left. \begin{aligned} \sigma_r &= L(r,s) e^{ar} A_1 - M(r,s) e^{ar} B_1 \\ \sigma_\theta &= \sigma_\phi = L_1(r,s) e^{ar} A_1 + M_1(r,s) e^{ar} B_1 \\ u &= A_1 r p(r,s) + \frac{B_1}{r^2} q(r,s) \end{aligned} \right\} \quad (3.5)$$

Where  $A_1$  and  $B_1$  are constants to be determined.

Now the material just on the elastic side of the plastic boundary must be on the point of yielding and satisfy the condition  $|\sigma_\theta - \sigma_r| = Y$  on  $r = c$  and the boundary condition  $\sigma_r = -p_e$  on  $r = b$ .

Using the conditions the constants  $A_1$  and  $B_1$  are calculated and then we obtain the stresses and displacement in the region  $c \leq r \leq b$  as follows :

$$\sigma_r = Y \frac{M(b,s)L(r,s) - L(b,s)M(r,s)}{N} e^{u(r-c)} -$$

$$- p_e \frac{[M_1(c,s) + M(c,s)]L(r,s) + [L_1(c,s) - L(c,s)]M(r,s)}{N} e^{a(r-b)}$$

$$\sigma_\theta = \sigma_\phi = Y \frac{M(b,s)L(r,s) + L(b,s)M_1(r,s)}{N} e^{a(r-c)} -$$

$$- p_e \frac{[M_1(c,s) + M(c,s)]L_1(r,s) - [L_1(c,s) - L(c,s)]M_1(r,s)}{N} e^{a(r-b)} \quad (3.6)$$

$$u = \left[ \frac{YM(b,s)rp(r,s)}{N} + \frac{YL(b,s)q(r,s)}{Nr^2} \right] e^{ar} -$$

$$- p_e \left[ \frac{\{M_1(c,s) + M(c,s)\}rp(r,s)}{N} - \frac{\{L_1(c,s) + L(c,s)\}q(r,s)}{Nr^2} \right] e^{-ab}$$

Where

$$N = L_1(c,s)M(b,s) - L(c,s)M(b,s) + M_1(c,s)L(b,s) + M(c,s)L(b,s)$$

#### 4. EFFECT OF WORK-HARDENING OF THE MATERIAL :

Let us now consider the plastic solid of work hardening material in the region  $a \leq r \leq c$ .

The equilibrium equation is

$$\frac{\partial \sigma_r}{\partial r} + \frac{2}{r} (\sigma_r - \sigma_\theta) = 0 \quad \dots\dots\dots (4.1)$$

The stresses must satisfy the compressibility equation

$$\sigma_r + 2\sigma_\theta = 3k_0 \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right) e^{\alpha r} \quad \dots\dots\dots (4.2)$$

where

$$k_0 = \lambda_0 + \frac{2\mu_0}{3} \text{ is the bulk modulus at } r = 0 \text{ and } k = k_0 e^{\alpha r} \text{ is the same at any radial distance } r$$

Following Hill [1] the stress-strain curve for a work-hardening material in uniaxial compression is of the form

$$\sigma = Y +$$

which are compressive stress and strain (both taken as +ve),  $Y$  is the initial yield stress,  $H$  is the degree of hardening expressed as a function of total strain. Evidently in a radial symmetric deformation, any element of the material is subject to a uniaxial, radial compression stress state together with hydrostatic

tensile stress  $\sigma_\theta$ . The latter stress by itself produces a positive elastic strain of amount  $\frac{1-2\nu}{E} \sigma_\theta$ .

Recollecting the sign convention for  $\sigma$  and  $\epsilon$  the appropriate general yield criterion is

$$\sigma_\theta - \sigma_r = Y + H \left\{ -\epsilon_r + \frac{1-2\nu}{E_0} \sigma_\theta \right\}$$

If there is no Bauschinger effect, then  $H(\epsilon) = -H(-\epsilon)$  i.e.  $H(\epsilon)$  is an odd function of strain. Thus the general yield criteria for a work hardening material is

$$\sigma_\theta - \sigma_r = \alpha' Y + H \left\{ -\epsilon_r + \frac{1-2\nu}{E_0} \sigma_\theta \right\} \text{ where } \alpha' = \pm 1 \quad \dots\dots\dots (4.3)$$

In case of linear work-hardening,  $H$  is the function of total strain and an analytic discussion is possible. In such case the rate of work-hardening is constant and we suppose the yield criteria as

$$\sigma_\theta - \sigma_r = Y \left( 1 - \frac{E_0^0 e^{\alpha r}}{E_0 e^{\alpha r}} \right) + E_0^0 e^{\alpha r} \left\{ -\frac{\partial u}{\partial r} + \frac{1-2\nu}{E_0 e^{\alpha r}} \sigma_\theta \right\} \quad \dots\dots\dots (4.4)$$

Where  $H(\epsilon) = E_0^0 e^{\alpha r}$  = Gradient of the stress-strain curve in the positive range.

Solving (4.2) and (4.4) we obtain the stress-strain relation in the region  $a \leq r \leq c$  as follows :

$$\sigma_r = \frac{K_0 e^{\alpha r} \left( 1 - \frac{E_0^0}{3k_0} \right)}{1 - \frac{E_0^0}{9k_0}} \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right) - \frac{2Y}{3} \frac{1 - \frac{E_0^0}{E_0}}{1 - \frac{E_0^0}{9k_0}} + \frac{2E_0^0 e^{\alpha r}}{3 \left( 1 - \frac{E_0^0}{9k_0} \right)} \frac{\partial u}{\partial r}$$

$$\sigma_u = \sigma_r = \frac{K_0 e^{\alpha r}}{1 - \frac{E_1^0}{9k_0}} \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right) + \frac{Y}{3} \frac{1 - \frac{E_1^0}{E_0}}{1 - \frac{E_1^0}{9k_0}} - \frac{E_1^0 e^{\alpha r}}{3 \left( 1 - \frac{E_1^0}{9k_0} \right)} \frac{\partial u}{\partial r} \dots\dots\dots(4.5)$$

Substituting (4.5) in 4. ) we obtain a differential equation satisfied by u as

$$r^2 \frac{d^2 u}{dr^2} + (\alpha r^2 + 2r) \frac{du}{dr} + (2\alpha r - 2)u = -\frac{6Y \left( 1 - \frac{E_1^0}{E_0} \right)}{3k_0 + E_1^0} r e^{-\alpha r} \dots\dots\dots(4.6)$$

$$\text{where } t = \frac{3K_0 - E_1^0}{3K_0 + E_1^0}$$

The solution of the differential equation – (4.6)

$$u = C r g_1(r, t) + \frac{D}{r^2} g_2(r, t) + P(r, t) \dots\dots\dots(4.7)$$

where

$$g_1(r, t) = 1 - \frac{\alpha(2t+1)}{1.4} r + \frac{\alpha^2(2t+1)(2t+2)}{1.2.3.4} r^2 - \frac{\alpha^3(2t+1)(2t+2)(2t+3)}{1.2.3.4.5.6} r^3 +$$

$$+ \frac{\alpha^4(2t+1)(2t+2)(2t+3)(2t+4)}{1.2.3.4^2.5.6} r^4 - \frac{\alpha^5(2t+1)(2t+2)(2t+3)(2t+4)(2t+5)}{1.2.3.4^2.5^2.6.7} r^5$$

$$+ \dots\dots\dots \text{to } \alpha.$$

$$g_2(r, t) = 1 - \frac{\alpha^3(2t-2)(2t-1)2t}{1^2.2^2.3^2} r \left( \frac{1}{2t-2} + \frac{1}{2t-1} + \frac{1}{2t} - \frac{1}{3} \right) +$$

$$+ \frac{\alpha^4(2t-2)(2t-1)2t(2t+1)}{1^2.2^2.3.4} r^2 \left( \frac{1}{2t-2} + \frac{1}{2t-1} + \frac{1}{2t} + \frac{1}{2t+1} - 1 - \frac{1}{3} - \frac{1}{4} \right) -$$

$$- \frac{\alpha^5(2t-2)(2t-1)2t(2t+1)(2t+2)}{1^3.2^3.3.4.5} r^3 \times$$

$$\times \left( \frac{1}{2t-2} + \frac{1}{2t-1} + \frac{1}{2t} + \frac{1}{2t+1} + \frac{1}{2t+2} - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} \right) + \dots\dots\dots \text{to } \alpha.$$

and

$$P(r,t) = G_1(r,t) \int \frac{6Y \left(1 - \frac{E_t^0}{E_0}\right) G_2(r,t) e^{-\alpha r}}{G_1(r,t) G_2'(r,t) - G_1'(r,t) G_2(r,t)} dr +$$

$$+ G_2(r,t) \int \frac{6Y \left(1 - \frac{E_t^0}{E_0}\right) G_1(r,t) e^{-\alpha r}}{G_1'(r,t) G_2(r,t) - G_1(r,t) G_2'(r,t)} dr$$

in which

$$G_1(r,t) = r g_1(r,t), \quad G_2(r,t) = \frac{g_2(r,t)}{r^2}$$

C and D being constants to be determined. Now substituting (4.7) in (4.5) we obtain the stress components in the region  $a \leq r \leq c$ .

$$\left. \begin{aligned} \sigma_r &= V(r,t) e^{\alpha r} C + H(r,t) e^{\alpha r} D + F(r,t) \\ \sigma_\theta = \sigma_\phi &= V_1(r,t) e^{\alpha r} C + H_1(r,t) e^{\alpha r} D + F_1(r,t) \end{aligned} \right\} \dots\dots\dots(4.8)$$

Where

$$V(r,t) = \frac{(9K_0 - E_t^0) g_1(r,t) + (3K_0 + E_t^0) r g_1'(r,t)}{3 \left(1 - \frac{E_t^0}{9K_0}\right)}$$

$$H(r,t) = \frac{(3K_0 + E_t^0) r g_2'(r,t) - 4E_t^0 g_2(r,t)}{3 \left(1 - \frac{E_t^0}{9K_0}\right) r^3}$$

$$V_1(r,t) = \frac{(9K_0 - E_t^0) g_1(r,t) + (3K_0 - E_t^0) r g_1'(r,t)}{3 \left(1 - \frac{E_t^0}{9K_0}\right)}$$

$$H_1(r,t) = \frac{(3K_0 + E_t^0) r g_2'(r,t) + 2E_t^0 g_2(r,t)}{3 \left(1 - \frac{E_t^0}{9K_0}\right) r^3}$$

$$F(r,t) = \frac{\left[ (3K_0 + E_t^0) r P'(r,t) + 3K_0 P(r,t) \right] e^{\alpha r} - 2Y \left(1 - \frac{E_t^0}{E_0}\right) r}{3 \left(1 - \frac{E_t^0}{9K_0}\right) r}$$



and

$$F_1(r, t) = \frac{\left[ \left( 3K_0 + E^0_t \right) r P'(r, t) + 3K_0 P(r, t) \right] e^{ar} - Y \left( 1 - \frac{E^0_t}{E_0} \right) r}{3 \left( 1 - \frac{E^0_t}{9K_0} \right) r}$$

The constants C and D are calculated with the help of the following continuity condition which the normal component of the stress and the displacement must satisfy at the interface  $r = c$ .

$$[\sigma_r]_{c-0} = [\sigma_r]_{c+0}$$

$$[u]_{c-0} = [u]_{c+0} \dots \dots \dots (4.9)$$

Hence, we obtain the stress components and the displacement in the region  $a \leq r \leq c$  as follows :

$$\begin{aligned} \sigma_r = & \frac{V(r, t) e^{ar}}{H(c, t) c g_1(c, t) e^{ac} - \frac{V(c, t) g_2(c, t)}{c^2} e^{ac}} \left[ \frac{F(c, t) g_2(c, t)}{c^2} - \right. \\ & \left. - \frac{W(c, s) g_2(c, t)}{c^2} + Z(c, s) H(c, t) e^{ac} - P(c, t) H(c, t) e^{ac} \right] + \\ & + \frac{H(r, t) e^{ar}}{H(c, t) c g_1(c, t) e^{ac} - \frac{V(c, t) g_2(c, t)}{c^2} e^{ac}} \left[ W(c, s) c g_1(c, t) - \right. \\ & \left. - F(c, t) c g_1(c, t) - Z(c, s) V(c, t) e^{ac} + P(c, t) V(c, t) e^{ac} \right] + \\ & + F(r, t) \end{aligned}$$

$$\begin{aligned} \sigma_\theta = \sigma_\phi = & \frac{V_1(r, t) e^{ar}}{H(c, t) c g_1(c, t) e^{ac} - \frac{V(c, t) g_2(c, t)}{c^2} e^{ac}} \left[ \frac{F(c, t) g_2(c, t)}{c^2} - \right. \\ & \left. - \frac{W(c, s) g_2(c, t)}{c^2} + Z(c, s) H(c, t) e^{ac} - P(c, t) H(c, t) e^{ac} \right] + \\ & + \frac{H_1(r, t) e^{ar}}{H(c, t) c g_1(c, t) e^{ac} - \frac{V(c, t) g_2(c, t)}{c^2} e^{ac}} \left[ W(c, s) c g_1(c, t) - \right. \\ & \left. - F(c, t) c g_1(c, t) - Z(c, s) V(c, t) e^{ac} + P(c, t) V(c, t) e^{ac} \right] + \\ & + F_1(r, t) \end{aligned}$$

$$u = \frac{r g_1(r, t)}{H(c, t) c g_1(c, t) e^{ac} - \frac{V(c, t) g_2(c, t)}{c^2} e^{ac}} \left[ \frac{F(c, t) g_2(c, t)}{c^2} - \right.$$

$$\begin{aligned}
 & - \frac{W(c,s) g_2(c,t)}{c^2} + Z(c,s) H(c,t) e^{\alpha c} - P(c,t) H(c,t) e^{\alpha c} \Big] + \\
 & + \frac{g_2(r,t)}{r^2 \left[ H(c,t) c g_1(c,t) e^{\alpha c} - \frac{V(c,t) g_2(c,t) e^{\alpha c}}{c^2} \right]} \left[ W(c,s) c g_1(c,t) - \right. \\
 & \left. - F(c,t) c g_1(c,t) - Z(c,s) V(c,t) e^{\alpha c} + P(c,t) V(c,t) e^{\alpha c} \right] + \\
 & + P(r,t)
 \end{aligned}$$

where

$$\begin{aligned}
 W(c,s) &= \frac{Y}{N} \{ M(b,s) L(c,s) - L(b,s) M(c,s) \} - \frac{P_c}{N} [ \{ M_1(c,s) + \\
 & + M(c,b) \} L(c,s) + \{ L_1(c,s) - L(c,s) \} M(c,s) ] e^{\alpha(c-b)}
 \end{aligned}$$

and

$$\begin{aligned}
 Z(c,s) &= \left[ \frac{YM(b,s) cp(c,s)}{N} + \frac{YL(b,s) q(c,s)}{Nc^2} \right] e^{-\alpha c} - \\
 & - p_e \left[ \frac{\{ M_1(c,s) + M(c,s) \} cp(c,s)}{N} - \frac{\{ L_1(c,s) - L(c,s) \} q(c,s)}{Nc^2} \right] e^{-\alpha b}
 \end{aligned}$$

The internal pressure in this needed to produce a plastic flow to a radius  $c$  is given by

$$p_i^c = -[\sigma_r]_{r=a} \dots \dots \dots (4.11)$$

Now taking  $\alpha = 0$ ,  $p_e = 0$ ,  $E_1 = 0$  and tending  $b$  to infinity, the results obtained are in agreement with the results obtained by HOPKINS (5)

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