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NUMERICAL SIMULATION ON LAMINAR FREE-CONVECTION FLOW AND HEAT TRANSFER OVER A VERTICAL PLATE WITH CONSTANT HEAT FLUX

By

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Abstract

In the present numerical study, laminar free-convection flow and heat transfer over a vertical plate with constant heat flux is presented. By means of similarity transformation, the original nonlinear coupled partial differential equations of flow are transformed to a pair of simultaneous nonlinear ordinary differential equations. Then, they are reduced to first order system. Finally, Newton-Raphson method and adaptive Runge-Kutta method are used for their integration. The computer codes are developed for this numerical analysis in Matlab environment. Velocity and temperature profiles for various Prandtl number are illustrated graphically. Flow and heat transfer parameters are derived as functions of Prandtl number alone. The results of the present simulation are then compared with experimental data published in literature and find a good agreement.

Keywords: Constant Heat Flux, Free Convection, Heat Transfer, Matlab, Numerical Simulation, Vertical Plate.

List of Symbols

c_1, c_2 constants eq (7)

c_p specific heat capacity, J/Kg.K

F function defined in eq (6)

g gravitational acceleration, 9.81 m/s^2

Gr_x Grashof number based on x, dimensionless

Gr_x^* modified Grashof number, dimensionless

h heat transfer coefficient, W/m^2.K

k thermal conductivity, W/m.K

L height of the plate, m

Nu_x Nusselt number at x, dimensionless

Pr Prandtl number, dimensionless

q_w heat flux of the plate, W/m^2

T temperature, K

T_w surface temperature, K

T_∞ free streams temperature, K

u velocity component in x, m/s

v velocity component in y, m/s

x coordinate from the leading edge, m

y coordinate normal to plate, m

z_1, z_2, z_3, z_4, z_5 variables, eq (14)

Greek Symbols

θ function defined in eq (10), dimensionless

coefficient of thermal expansion, 1/K

μ dynamic viscosity, N.s/m^2

¬ kinematic viscosity, m^2/s

η similarity variables, eq (7)

τ shear stress, N/m^2

ψ stream function, m^2/s

ρ density, kg/m^3

Subscript

w plate surface

∞ free stream

1. Introduction

There have been a lot of studies on natural convection from a vertical plate with constant heat flux due to its relevance to a variety of industrial applications and naturally occurring processes such as solar collectors, pipes, ducts, electronic packages, walls and windows etc. The earliest analytical investigation was a similarity analysis of the boundary layer equations by Sparrow and Gregg [1]. The results were validated by Pohlhausen [2], Dotson [3], Goldstein and Eckert [4], Fujii et al. [5], Pitman et. al. [6], and Aydin et. al. [7]. The problem is also discussed in several text books [8-12].

In the present numerical investigation, a simple accurate numerical simulation of laminar free-convection flow and heat transfer over a vertical plate with constant heat flux is developed.

The paper is organized as follows: Mathematical model of the problem, its solution procedure, development of code in Matlab, interpretation of the results, comparison with experimental data.

2. Mathematical Model

We consider a vertical hot flat plate with constant heat flux immersed in a quiescent fluid body. We assume the natural convection flow to be steady, laminar, two-dimensional, no dissipation, and the fluid to be Newtonian with constant properties, including density, with one exception: the density difference $\rho - \rho_\infty$ is to be considered since it is this density difference between the inside and the outside of the boundary layer that gives rise to buoyancy force and sustains flow. (This is known as the *Boussinesq approximation.*). We take the upward direction along the plate to be *x,* and the direction normal to surface to be *y,* as shown in Figure 1.

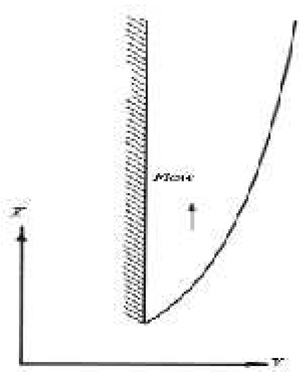


Fig 1. Physical Model and its coordinate system

The equations governing the flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + \quad g(T - T_\infty) \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial u}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

The boundary conditions on the solution are:

At y=0: u=v=0, $\quad -k\frac{\partial T(y=0)}{\partial y} = q_w$

For large y: $u \to 0$, $T \to T_\infty$ (4)

The continuity equation (1) is automatically satisfied through introduction of the stream function:

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x} \tag{5}$$

A similarity solution is possible if

$$\psi = c_2 x^{4/5} F(\eta) \tag{6}$$

where

$$\eta = \frac{c_1 y}{x^{1/5}}$$

$$c_1^5 = \frac{g\; q_w}{5k\nu^2}$$

$$c_2^5 = \frac{5^4 g\; q_w \nu^3}{k} \tag{7}$$

Then the velocity components can be written as

$$u = c_1 c_2 x^{3/5} F'(\eta) \tag{8}$$

$$v = c_2 \frac{\eta F'(\eta) - 4F(\eta)}{5x^{1/5}} \tag{9}$$

Now, with the dimensionless temperature

$$\theta = c_1 \frac{T_\infty - T}{x^{1/5} q_w / k} \tag{10}$$

the partial differential equations (2) and (3) are transformed to ordinary differential equations (with a prime denoting differentiation with respect to η)

$$F''' - 3(F')^2 + 4FF'' - \theta = 0 \tag{11}$$

$$\theta'' + \Pr(4\theta' F - \theta F') = 0 \tag{12}$$

where $\Pr = \frac{c_p \mu}{k}$, Prandtl number of the fluid.

Eqs (11) and (12) constitute a pair of simultaneous nonlinear ordinary differential equations for the velocity and temperature functions, F’ and θ. They must be solved subject to the following boundary conditions:

At y = 0: u = 0 i.e., at η = 0: F’ = 0

At y = 0: v = 0 i.e., at η = 0: F = 0

At y = 0: $-k\dfrac{\hat{o}T(y=0)}{\hat{o}y} = q_w$ i.e., at η = 0: θ = 1

For large y: u→ 0 i.e., for large η: F' = 0

For large y: T→ T_∞ i.e., for large η: θ = 0 (13)

The fact that the original partial differentials have been reduced to a pair ordinary differential equations confirms the assumptions that similarity solutions do in fact exist.

3. Solution Procedure

Eqs (11) and (12) are coupled and must be solved simultaneously, which is always the case in free-convection problems. No analytic solution is known, so numerical integration is necessary. There are two unknown initial values at the wall. One must find the proper values of F''(0) and θ(0) which cause the velocity and temperature to vanish for large η. The Prandtl number is a parameter.

3.1 Reduction of Equations to First-order System

This is done easily by defining new variables:

$$z_1 = F$$

$$z_2 = \dot{z}_1 = \dot{F}$$

$$z_3 = \dot{z}_2 = \ddot{z}_1 = \ddot{F}$$

$$\dot{z}_3 = \ddot{z}_2 = \dddot{z}_1 = \dddot{F} = 3z_2^2 - 4z_1z_3 + z_4$$

$$z_4 = \overline{}$$

$$z_5 = \dot{z}_4 = \dot{\overline{}}$$

$$\dot{z}_5 = \ddot{z}_4 = \ddot{\overline{}} = -\Pr[4z_1z_5 - z_2z_4] \quad (14)$$

with the following boundary conditions:

$$z_1(0) = F(0) = 0$$

$$z_2(0) = \dot{z}_1(0) = \dot{F}(0) = 0$$

$$z_2(\infty) = \dot{z}_1(\infty) = \dot{F}(\infty) = 0$$

$$z_4(\infty) = \overline{}(\infty) = 0$$

$$z_5(0) = \dot{z}_4(0) = \dot{\overline{}}(0) = 1 \quad (15)$$

Eq (11) is third-order and is replaced by three first-order equations, whereas eq (12) is second-order and is replaced with two first-order equations.

Three of the boundary conditions are specified at $\eta = 0$, while the remaining two at $\eta = \infty$. One way to overcome the lack of starting conditions is to guess the missing boundary values. Since F'' and θ' are proportional to the velocity gradient and temperature gradient, F'' should be positive and θ' negative at $\eta = 0$. The initial guessed values are altered to obtained to obtained solutions at higher values of Prandtl number.

3.2 Solution to Initial Value Problems

To solve Eq(14), we denote the two unknown initial values by u_1 and u_2, the set of initial conditions is then:

$$z_1(0) = F(0) = 0$$

$$z_2(0) = \dot{z}_1(0) = \dot{F}(0) = 0$$

$$z_3(0) = \dot{z}_2(0) = \ddot{z}_1(0) = \ddot{F}(0) = u_1$$

$$z_4(0) = \theta(0) = u_2$$

$$z_5(0) = \dot{z}_4(0) = \dot{\theta}(0) = 1 \tag{16}$$

If Eqs (11) and (12) are solved with adaptive Runge-Kutta method using the initial conditions in Eq(16), the computed boundary values at $\eta = \infty$ depend on the choice of u_1 and u_2. We express this dependence as

$$z_2(\infty) = \dot{z}_1(\infty) = \dot{F}(\infty) = f_1(u_1, u_2)$$

$$z_4(\infty) = \theta(\infty) = f_2(u_1, u_2)$$

The correct choice of u_1 and u_2 yields the given boundary conditions at $\eta = \infty$; that is, it satisfies the equations

$$f_1(u_1, u_2) = 0$$

$$f_2(u_1, u_2) = 0 \tag{17}$$

These are simultaneous nonlinear equations that can be solved by the Newton-Raphson method.

A value of 10 is fine for infinity, even if we integrate further nothing will change.

3.3 Program Details

This section describes a set of Matlab routines for the solution of eqns (11) and (12) along with the boundary conditions (13). They are listed in Table 1.

Table 1. A set of Matlab routines used sequentially to solve Equations (11) & (12).

Matlab code	Brief Description
deqs.m	Defines the differential equations (11) and (12).
incond.m	Describes initial values for integration, u_1 and u_2 are guessed values, eq (16)
runKut5.m	Integrates the initial value problem (14) using adaptive Runge-Kutta method.
residual.m	Provides boundary residuals and approximate solutions.
newtonraphson.m	Provides correct values u_1 and u_2 using approximate solutions from residual.m
runKut5.m	Again integrates the initial value problem (14) using correct values of u_1 and u_2.

The output of the code runKut5.m gives the tabulated values of F, F', F'', θ, θ' as function of η for various values of Prandtl number.

4 Interpretation of the Results

Physical quantities are related to the dimensionless functions F and θ through Eqs (7), (8) and (9). F and θ are now known.

4.1 Computed Values of the Parameters

Some accurate initial values from this computation are listed in Table 2. These theoretical computations are in good agreement with results published in literatures [4].

Table 2. Computed parameters from Eqs (11) & (12)

Pr	F''(0)	-θ(0)
0.001	4.7526	9.1196
0.005	3.8956	7.1520
0.01	3.3435	5.9604
0.05	2.0750	3.4921
0.1	1.6430	2.7514
0.5	0.9298	1.6522
0.72	0.8142	1.4873
1	0.7220	1.3574
10	0.3064	0.7676
20	0.2353	0.6573
30	0.2014	0.6017
50	0.1653	0.5392
60	0.1540	0.5187
70	0.1450	0.5021
80	0.1376	0.4881
90	0.1320	0.4773
100	0.1290	0.4587
1000	0.1103	0.2890

4.2 Dimensionless Velocity and Temperature Profiles

Some typical velocity and temperature profiles obtained using this procedure are shown in Figs (2) and (3).

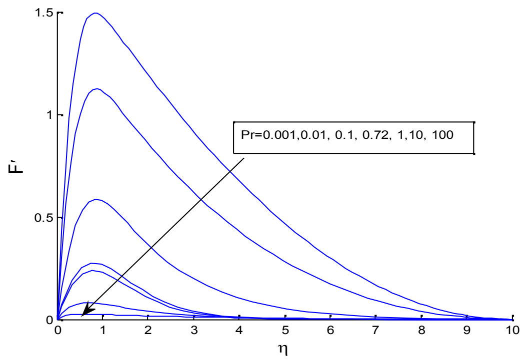


Fig 2. Dimensionless velocity distributions for various Prandtl numbers

The temperature distribution in the boundary layer can be written from eq(10)

$$\frac{T-T_\infty}{T_w-T_\infty}=\frac{\overline{\ }(\text{¦})}{\overline{\ }(0)} \tag{18}$$

The temperature profiles are plotted in Fig 3 for various Prandtl numbers.

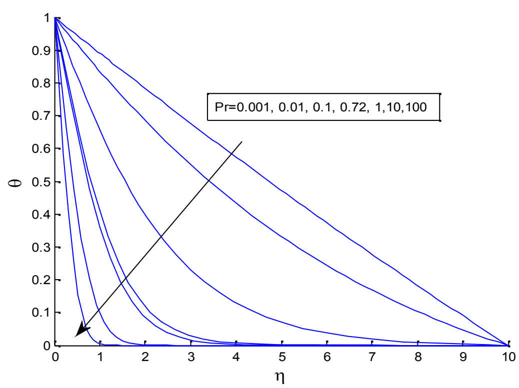


Fig 3. Dimensionless temperature distributions for various Prandtl numbers

4.3 Variation of the wall temperature

The temperature excess at the wall from eq(10) is

$$T_w(x)-T_\infty=-c_1^{-1}\frac{q_w}{k}-(0)x^{1/5}$$

$$=-5^{1/5}-(0)\frac{q_w x}{k}\left(\frac{g\ \ q_w x^4}{\neg^2 k}\right)^{-1/5} \tag{19}$$

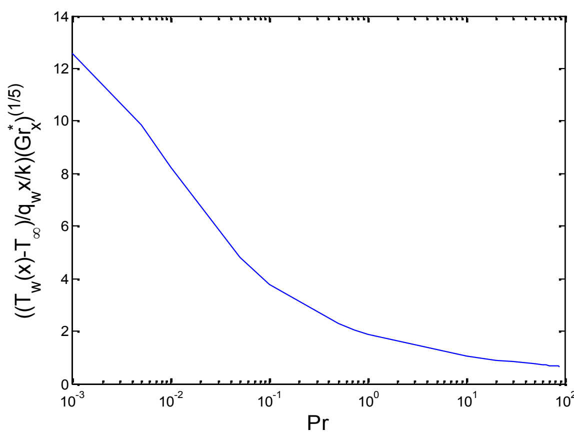
The dimensionless group in the bracket plays the role in this problem as does the Grashof number in the problem of free convection from a vertical plate having uniform wall temperature. The group is referred as modified Grashof number. It is related to conventional Grashof number as

$$Gr_x^* = Gr_x \times \Pr$$
$$= \frac{g}{\neg^2}(T_w - T_\infty)x^3 \times \frac{hx}{k} = \frac{g}{\neg^2 k}x^4 q_w \tag{20}$$

Eq (19) can be written as

$$\frac{T_w(x) - T_\infty}{\frac{q_w x}{k}}(Gr_x^*)^{1/5} = -5^{1/5} - (0) \tag{21}$$

Eq (21) is a function of Prandtl number only and plotted in Fig 4 using the computed values from Table 1.



$$\frac{T_w - T_\infty}{q_w x/k}(Gr_x^*)^{1/5}$$

Fig 4 Variation of with Prandtl number

From eq (19), we see that the wall temperature excess increases as $x^{1/5}$, distance from the leading edge and is proportional to $q_w^{4/5}$. This relationship can be rephrased as

$$T_w(x) - T_\infty = [T_w(L) - T_\infty](\frac{x}{L})^{1/5} \tag{22}$$

The variation of the wall temperature from the leading edge of the plate is shown graphically in Fig 5.

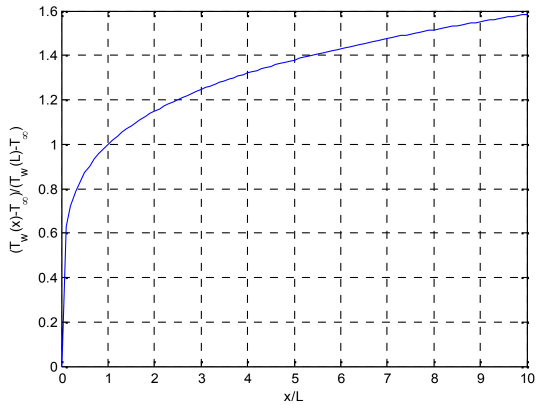


Fig 5 Variation $(T_w - T_\infty)$ along the plate (x=0 at the leading edge, x=L, length of the plate)

4.4 Flow and Heat Transfer Parameters

Besides the velocity and temperature distributions, it is often desirable to compute other physically important quantities (for example, shear stress, drag, heat-transfer-rate) associated with the free-convection flow.

Shear stress is defines as:

$$\tau = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{23}$$

From Eqs (7) and (8), we can write

$$\frac{\partial u}{\partial y} = \frac{\nu}{x^2} F''(\eta)\left(\frac{Gr_x^*}{5}\right)^{2/5}$$

Substituting this expression into (23), we get the flow parameter

$$\frac{\tau}{\frac{\mu\nu}{x^2}\left(\frac{Gr_x^*}{5}\right)^{2/5}} = F''(0) \tag{24}$$

The flow parameter is a function of Prandtl number only and is presented in Fig (6). From this figure, various flow quantities for a given set of conditions can be computed.

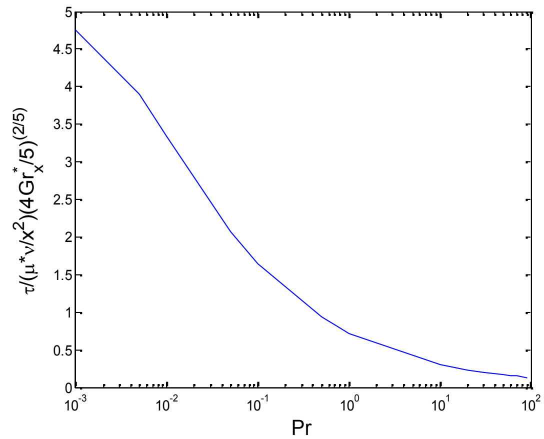


Fig 6. Dimensionless flow parameter as function of Prandtl number

4.5 Comparison with existing correlation

Local Nusselt number is defined as

$$Nu_x = \frac{hx}{k} = \frac{q_w}{T_w - T_\infty}\frac{x}{k} \tag{25}$$

From eq(21), we can write

$$\frac{Nu_x}{(Gr_x^*)^{1/5}} = -\frac{1}{5^{1/5}\,\overline{}(0)} \tag{26}$$

It is a function of only Prandtl number. Aydin [7] has recently examined the problem thoroughly and suggested the eq (27) as the fundamental correlation for laminar free convection from uniformly heated vertical plate

$$\frac{Nu_x}{(Gr_x^*)^{1/5}} = 0.630\left(\frac{\Pr^2}{0.670 + \Pr}\right)^{1/5} \tag{27}$$

Figure 7 illustrates the comparison of the present computation (26) with the latest semi-empirical formula (27).

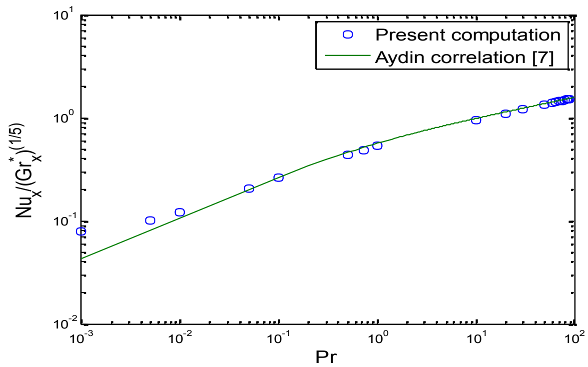


Fig 7. Comparison of the present computation with semi-empirical formula.

4.6 Comparison with experiments

Laminar free-convection flows of air over a vertical flat with constant heat flux were made by Dotson [3]. The temperature and heat transfer data reported by Dotson have been used to calculate local Nusselt numbers defined acocording to eq (25). Figure 8 illustrates the comparison of the present numerical line given by eq (26) with experimental data.

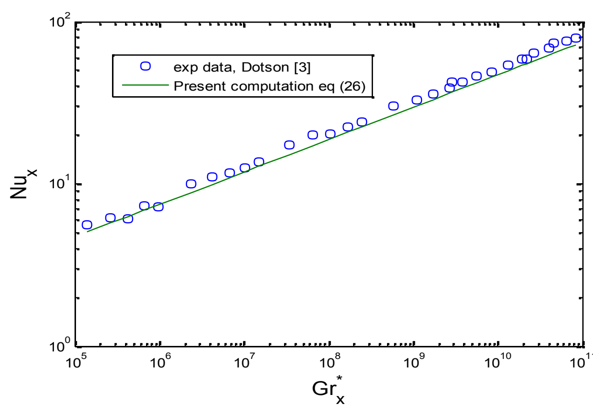


Fig 8. Comparison of the computed Nusselt numbers with experimental data.

5. Conclusion

In the present numerical simulation, laminar free-convection flow and heat transfer over a vertical plate with constant heat flux is presented. Details of the solution procedure of the nonlinear coupled partial differential equations of flow are discussed. The computer codes are developed for this numerical analysis in Matlab environment. Velocity and temperature profiles for Prandtl numbers of 0.001, 0.005, 0.01, 0.05, 0.1, 0.5, 0.72, 1, 10, 20, 30, 50, 60, 70, 80, 90, 100 and 1000 are computed using these codes. The computed and experimental velocity and temperature distributions are in good agreement with results published in literatures. Flow and heat transfer parameters (giving physically important quantities such as shear stress, drag, heat-transfer-rate) are derived as functions of Prandtl number alone. The computed results of the present simulation are compared with frequently used semi empirical heat transfer correlation and find a good agreement. A good agreement between the present results and the past indicates that the present numerical simulation may be an efficient and stable numerical scheme in natural convection.

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